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A sufficient condition for evasion in a nonlinear game Part 1

by

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We consider a nonlinear differential game of evasion and discuss a condition under which for every initial state of the game there can be constructed a strategy of evasion, that is, a strategy which ensures that the trajectory of the game remains all the time outside a fixed terminal subspace. The sufficient condition for evasion given here is a generalization of the condition from [2].

1. Introduction

We consider the differential game of evasion given by an equation

$$\dot{z}=P_0(z)+f(z, u, v),\ z \in R^n,\ u \in U \subset R^p,\ v \in V \subset R^q \tag{1.1}$$

a control set U for the pursuer, a control set V for the evader and a linear subspace M of R^n. Assume that codiom $M \geqslant 2$, the sets U, V are compact and the right-hand side $P(z, u, v)=P_0(z)+f(z, u, v)$ of the equation satisfies the following conditions:

(a) $P(z, u, v)$ is a continuous function on $R^n \times U \times V$,

(b) there exist constants A, B such that

$$|z \cdot P(z, u, v)| \leqslant A\, |z|^2+B \text{ for all } u \in U, v \in V, z \in R^n,$$

(c) for every $r>0$ there exists a constant C_r such that if $|z| \leqslant r$, $|\bar{z}| \leqslant r$ then:

$$|P(z, u, v)-P(\bar{z}, u, v)| \leqslant C_r\, |z-\bar{z}| \text{ for all } u \in U, v \in V.$$

The conditions (a)—(c) imply that for any measurable functions $u(t), v(t)$ taking values in U and V, respectively, and for any initial condition $z_0 \in R^n$ there exists for $t \in [0, +\infty)$ a unique solution of the problem:

$$\begin{cases} \dot{z}(t)=P\big(z(t), u(t), v(t)\big). \\ z(0)=z_0 \end{cases} \tag{1.2}$$

The aim of the evader is to avoid the subspace M; that is, to ensure that $z(t) \notin M$ for $t \in [0, +\infty)$ whenever $z_0 \notin M$. We assume, forllowing for example [1, 2], that at any instant of the time the evader does not know the future behaviour of the opposer and knows its past and present behaviour. More precisely, we use the following concept of strategy for the evader. A mapping $v^u(z_0; t)$ which for a fixed initial condition z_0 assigns to each pursuer's control function $u(t), u(t) \in U$, $t \in [0, +\infty)$ an evader's control function $v(t) = v^u(z_0, t)$, $v(t) \in V$, $t \in [0, +\infty)$ is called as strategy if for any two control functions $u^1(t), u^2(t)$ and any $T > 0$ the equality $u^1(t) = u^2(t)$ a.e. in $[0, T]$ implies that $v^{u_1}(z_0, t) = v^{u_2}(z_0, t)$ a.e. in $[0, T]$. We can say now that the aim of the evader in to find a strategy $v^u(z_0, t)$ defined for all $z_0 \notin M$ such that any corresponding trajectory satisfies $z(t) \notin M$ for $t \in [0, +\infty)$.

We formulate a condition under which a strategy of evasion exists and state a theorem of evasion. The condition which we call condition (F) when applied to the linear game coincides with the condition given in [2] by R. V. Gamkrelidze and K. L. Kcharatishvili although we formulate it in a different way. Then we discuss condition (F). We give its equivalent form in Proposition 2.1, compare with the condition from [2] and give an example for which condition (F) holds. In the next paper we shall prove the theorem of evasion. Our construction of an evasion strategy much differs than that in [2]. Moreover, we construct a strategy of evasion while in [2] only the existence of a relaxed strategy is shown, where the evader chooses at any moment t a collection $(\mu_1(t), \ldots, \mu_r(t), v_1(t), \ldots, v_r(t))$, $\sum_{i=1}^{r} \mu_i(t) = 1$, $\mu_i \geq 0$, $v_i(t) \in V$, $i = 1, \ldots, r$, instead a point $v(t) \in V$.

2. The sufficient condition of evasion

In this section we formulate a condition under which an evasion strategy exists and state an evasion theorem.

We assume further that the mapping $P_0(z)$ is continuously differentiable as many times as it will be differentiated. Let $DP_0(z)$ denotes the differential of $P_0(z)$ at a point z. $DP_0 \cdot P_0$ is again a mapping from R^n into R^n where $DP_0 \cdot P_0(z) = DP_0(z) \times P_0(z)$ and we can take its differential $D(DP_0 \cdot P_0)(z)$. Denote

$$C_0(z) = I,\ C_1(z) = DP_0(z) \text{ and inductively } C_k(z) = D\big(C_{k-1}(z) \cdot P_0(\dot{z})\big)$$

where I is the identity matrix.

Integrating p-times by parts the integral form of the equation (1.2) we obtain the following formula:

$$z(t) = s_p(t; z_0) + \int_0^t \sum_{i=0}^{p-1} C_i\big(z(\tau)\big) f\big(z(\tau), u(\tau)\, v(\tau)\big) \frac{(t-\tau)^i}{i!}\, d\tau + R(t^{p+1}) \qquad (2.1)$$

where $s_p(t; z_0) = z_0 + P_0(z_0)\, t + \ldots + C_{p-1}(z_0) P_0(z_0) \frac{t^p}{p!}$ is a polynomial in t of degree at most p and the rest is of the form

$$R(t^{p+1})=\int_0^t C_p(z(\tau))\left(P_0(z(\tau))+f(z(\tau),u(\tau),v(\tau))\right)\frac{(t-\tau)^p}{p!}\,d\tau$$

The conditions (a)—(c) imply that for every $r>0$, $T>0$ there exists a ball $K(0, h_{r,T})$ around the origin of radius $h_{r,T}$ such that if z_0 is from the ball $K(0,r)$ of radius r around the origin then for every trajectory $z(t)$ of the equation (1.2) $z(t)\in \notin K(0,h_{r,T})$ for $t\in[0,T]$. Therefore there exists a constant $N_{r,T}$ such that for any $z_0\in K(0,r)$ and any control functions $u(t), v(t)$ the following inequality holds:

$$|R(t^{p+1})|\leqslant N_{r,T}\,t^{p+1} \text{ for } t\in[0,T]. \tag{2.2}$$

In which follows $R(t^m)$ will always denote a term such that $|R(t^m)/t^m|$ is bounded uniformly with respect to all variables on which it may depend.

Take a point $z_*\in M$ an integer p and a k-dimensional subspace L orthogonal to M. Denote

$$F_{p-1}(t,z,u,v)=f(z,u,v)+C_1(z)f(z,u,v)\,t+\ldots+C_{p-1}(z)f(z,u,v)\,t^{p-1}.$$

Take a linear mapping π_L of the form $\pi_L=AP_L$ where P_L is the orthogonal projection of R^n onto L, A is an isometric mapping of R^n which maps L onto $R^k= =\{x\in R^n|x_{k+1}=\ldots=x_n=0\}$. We shall consider $\pi_L F_{p-1}(t,z,u,v)$ for z from a neighboorhood $\mathscr{U}_{z_*}$ of z and small t. At first recall some facts concerning analytical matrix-functions (see [1] also [2]).

Let $H(t)$ be a function defined for t from a neighbourhood of zero whose values are $k\times k$-matrices. Assume that $H(t)$ is analytical and such that the matrices $H(t)$ are non-singular for positive t. Then $H(t)$ may be written in the following form:

$$H(t)=A(t)\begin{vmatrix} t^{l_1} & & 0\\ & t^{l_2} & \\ & & \ddots \\ 0 & & t^{l_k}\end{vmatrix}B(t) \tag{2.3}$$

where $l_1,\ldots,l_k$ are integers, $0\leqslant l_1\leqslant l_2\leqslant\ldots\leqslant l_k$ which depend only on the function $H(t)$ and are called indices of the function $H(t)$, the matrix functions $A(t), B(t)$ are analytical and such that $\det A(0)\neq 0$, $\det B(0)\neq 0$. The latter implies that the functions $A^{-1}(t), B^{-1}(t)$ are analytical in a neighbourhood of zero, therefore for any $m\geqslant l_k$ the function $t^m H^{-1}(t)$ is analytical in a neighbourhood of zero.

Consider for z in a neighbourhood $\mathscr{U}_{z_*}$ of z_* and t in some interval $[0,T]$ the following representations of the mappings $\pi_L F_{p-1}(t,z,u,v)$, $p=1,2,\ldots$:

$$\pi_L F_{p-1}(t,z,u,v)=H(t)\left(\psi_0(z,u,v)+\ldots+\psi_{p-1}(z\,u,v)\,t^{p-1}\right)+$$
$$+\sum_{i=0}^{p-1}\alpha_i(z,u,v)\,t^i+\sum_{i=0}^{p-1}\beta_i\,t^i+R(t^p) \text{ for } t\in[0,T],\ z\in\mathscr{U}_{z_*}, u\in U,\ v\in V. \tag{2.4}$$

where: (r) $H(t)$ is a $k\times k$-matrix-function analytical in a neighbourhood of zero which contains the interval $[0, T]$, non-singular for $t\in[0, T]$ and such that all indices of $H(t)$ are at most $p-1$; the functions $\psi_i(z, u, v)$, $\psi_i(z, u, v)\in R^k$, $i=0, \ldots, p-1$, are continuous; $\beta_i\in R^k$, $i=0, \ldots, p-1$, are constant vectors; $R(t^p)=R(t, z, u, v)$ is such that $|R(t^p)/t^p|$ is bounded uniformly with respect to all variables, the functions $\alpha_i(z, u, v)$, $\alpha_i(z, u, v)\in R^k$, $i=0, \ldots, p-1$, satisfy for some constant D the following estimation:

$$|\alpha_i(z, u, v)|\leqslant D\rho^{p-i}(z, M) \text{ for } z\in\mathcal{U}_{z_*},\ u\in U,\ v\in V. \tag{2.5}$$

We can formulate now the condition of evasion (F):

(F) For every point $z_*\in M$ there exist a compact neighbourhood $\mathcal{U}_{z_*}$ of z_*, a two-dimensional subspace $L=L(z_*)$ of R^n orthogonal to M an integer $p=p(z_*)$ and $T=T(z_*)$, $T>0$ such that the mapping $\pi_L F_{p-1}(t, z, u, v)$ has a representation:

$$\pi_L F_{p-1}(t, z, u, v)=H(t)\big(\psi_0(z, u, v)+\psi_1(z, u, v)\,t+\ldots+$$

$$+\psi_{p-1}(z, u, v)\,t^{p-1}\big)+\sum_{i=0}^{p-1}\alpha_i(z, u, v)\,t^i+\sum_{i=0}^{p-1}\beta_i\,t^i+R(t^p)$$

which satisfies (r) and such that

(i) the set $\bigcap\limits_{u\in U}\operatorname{co}\psi_0(z_*, u, V)$ contains an interior point with respect to R^2.

Under the condition (F) a strategy of evasion can be constructed. The following theorem holds:

THEOREM 2.1. If for the game (1.1) the condition (F) is satisfied then there exist closed sets W, W_1, a strategy of evasion $v^u(z_0; t)$ defined for all $z_0\notin M$, $t\in[0, +\infty)$ and positive functions $T(\xi)$, $\xi\in(0, +\infty)$, $T(\xi)<1$ and $\gamma(\xi_1, \xi_2)$, $\xi_1, \xi_2\in(0, +\infty)$ such that $M\subset\operatorname{int} W_1\subset\operatorname{int} W$ and any trajectory $z(t)$ corresponding to the strategy $v^u(z_0; t)$ satisfies:

if $z_0\in W$ then $\rho(z(t), M)\geqslant\gamma\big(\rho(z_0, M), |z_0|\big)$ for
$t\in[0, T(|z_0|)]$ and $z\big(T(|z_0|)\big)\notin W$,

if for some t_1 $z(t_1)\notin W$, then $z(t)\notin W_1$ for all $t\geqslant t_1$,

if $z(t_1)\in W$ then for some $t_2\in[t_1, t_1+T\big(|z(t_1)|\big)]$, $z(t_2)\notin W$.

Conditions of evasion of such type as the condition (F) and a division of the right-hand side into a sum $P(z, u, v)=P_0(z)+f(z, u, v)$ appear in a natural way when one considers the game of evasion between two objects x, y in R^m whose motions are described by equations of different orders. Take $m=2$ and consider two objects a pursuer $x=(x_1, x_2)$ and an evader $y=(y_1, y_2)$:

$$\begin{cases} x_1^{(p_1)}=F_1(x, u)\\ x_2^{(p_2)}=F_2(x, u),\ u\in U\end{cases}\qquad\begin{cases} y_1^{(q_1)}=G_1(y, v)\\ y_2^{(q_2)}=G_2(y, v),\ v\in V\end{cases}$$

Assume that $p_1 \leqslant p_2$, $q_1 \leqslant q_2$. Consider the corresponding game in R^s where $s = p_1 + p_2 + q_1 + q_2$:

$$z = (z_1, z_2, \ldots, z_s) = (x_1, \dot{x}_1, \ldots, x^{(p_1-1)}, x_2, \dot{x}_2, \ldots, x_2^{(p_2-1)}, y_1, \dot{y}_1, \ldots, y_1^{(q_1-1)},$$

$$y_2, \dot{y}_2, \ldots, y_2^{(q_2-1)},$$

$$M = \{z \in R^s | z_1 = z_{p_1+p_2+1}, z_{p_1+1} = z_{p_1+p_2+q_1+1}\}$$

$$\dot{z} = P_0(z) + f(z, u, v) \text{ where}$$

$$P_0(z) = (z_2, z_3, \ldots, z_{p_1}, 0, z_{p_1+2}, \ldots, z_{p_1+p_2}, 0, z_{p_1+p_2+2}, \ldots,$$

$$z_{p_1+p_2+q_1}, 0, z_{p_1+p_2+2}, \ldots, z_s, 0)$$

$$f(z, u, v) = (0, \ldots, 0, F_1(z, u), 0, \ldots, 0, F_2(z, u), 0, \ldots, 0,$$

$$G_1(z, v), 0, \ldots, 0, G_2(z, v).$$

Computing $\pi_L C_r(z) f(z, u, v)$, $r = 0, \ldots, p_2$ where $L = M^\perp$ one can check that the condition (F) takes after extracting $H(t) = \begin{vmatrix} t^{q_1-1}, & 0 \\ 0 & , t^{q_2-1} \end{vmatrix}$ the following forms depending on the orders p_1, p_2, q_1, q_2. In the case when $q_1 < p_1$, $q_2 < p_2$ it takes form:

for every $y \in R^2$ int co $G(y, V) \neq \emptyset$ where

$$G(y, v) = (G_1(y, v), G_2(y, v));$$

in the case when $q_1 < p_1$, $q_2 = p_2$:

for every $y \in R^2$ there exists $w_0 \in R^2$ such that $w_0 + \tilde{F}(y, U) \subset$ int co $G(y, V)$ where $\tilde{F}(y, u) = (0, F_2(y, u))$; and in the case $q_1 = p_1$, $q_2 = p_2$ it takes form:

for every $y \in R^2$ there exists w_0 such that

$$w_0 + F(y, U) \subset \text{int co } G(y, V) \text{ where } F(y, u) = (F_1(y, u), F_2(y, u)).$$

The condition (F) is of a rather complicated form. There naturally arises the question of having some criterion which allows for a given mapping $\pi F_{p-1}(t, z, u, v)$ to conclude wheather or not it has a representation of the form (2.4) that satisfies (r) and (i). We give certain sufficient and necessary condition for the existence of such representation in the following Proposition 2.1.

Consider a mapping $G(t, z, u, v)$ of the form $G(t, z, u, v) = \sum_{i=0}^{p-1} q_i(z, u, v) t^i$ where $t \in [0, T]$, $g_i(z, u, v)$, $i = 0, \ldots, p-1$, are continuous functions defined for $z \in \mathscr{U}_{z_*}$, $u \in U$, $v \in V$, taking values in R^2. Having chosen some basis in R^2 we shall denote for any function $g(z, u, v)$, $g(z, u, v) \in R^2$, by $\begin{vmatrix} g^1(z, u, v) \\ g^2(z, u, v) \end{vmatrix}$ or for simplicity by $\begin{vmatrix} g^1 \\ g^2 \end{vmatrix}$ its components.

Proposition 2.1. There exists a representation of the mapping $G(t, z, u, v)$ of the form

$$G(t, z, u, v) = H(t) \sum_{i=0}^{p-1} \psi_i(z, u, v) t^i + \sum_{i=0}^{p-1} \alpha_i(z, u, v) t^i + \sum_{i=0}^{p-1} \beta_i t^i + R(t^p) \quad (2.6)$$

satisfying (r) and (i) iff there exists a basis in R^2 such that for some integers l, m, $0\leqslant l\leqslant p-1$, $0\leqslant m\leqslant p-l-1$, and some numbers $a_1, \ldots, a_m$ the mapping $G(t, z, u, v)$ takes the following form:

$$G(t, z, u, v)=t^l \sum_{i=0}^{p-1-l} f_i(z, u, v)\, t^i+\sum_{i=0}^{p-1} \alpha_i(z, u, v)\, t^i+\sum_{i=0}^{p-1} \beta_i\, t^i \tag{2.7}$$

where $f_i(z,u,v)$, $i=0, \ldots, p-1-l$, are continuous functions such that $f_0^2=0$, $f_i^2=a_i f_0^1+a_{i-1} f_1^1+\ldots+a_1 f_{i-1}^1$ for $i=1, \ldots, m-1$, $f_m^2=f^*+a_m f_0^1+\ldots+a_1 f_{m-1}^1$, $\alpha_i(z, u, v)$, β_i, $i=0, \ldots, p-1$ are as it is required in (r) and $\psi_0(z, u, v)=\begin{vmatrix} f_0^1(z, u, v) \\ f^*(z, u, v)\end{vmatrix}$ satisfies (i).

Proof. We prove the necessity first. We can assume that $\alpha_i(z, u, v)=0$, $\beta_i=0$, otherwise we consider $\tilde{G}(t, z, u, v)=G(t, z, u, v)-\sum_{i=0}^{p-1} \alpha_i(z, u, v)\, t^i-\sum_{i=0}^{p-1} \beta_i\, t^i$. Thus we have:

$$G(t,z,u,v)=H(t) \sum_{i=0}^{p-1} \psi_i(z, u, v)+R(t^p).$$

Denote the indices of $H(t)$ by $l, l+m$. Then for any basis in R^2 we have

$$H(t)=A(t)\begin{vmatrix} t^l, 0 \\ 0, t^{l+m}\end{vmatrix} B(t)$$

for some analytical, non-singular matrix-functions $A(t)=\sum_{i=0}^{\infty} A_i\, t^i$, $B(t)=\sum_{i=0}^{\infty} B_i\, t^i$. Choose such a basis that $A^{-1}(t)$ which is also analytical is of the form: $A^{-1}(t)=I+C_1\, t+C_2\, t^2+\ldots$. Next notice that we can assume that $B(t)\equiv I$ otherwise we replace $\psi_i(z, u, v)$ by $\tilde{\psi}_i(z, u, v)=\sum_{j=0}^{i} B_j\, \psi_{i-j}(z, u, v)$, $i=0, \ldots, p-1$, which have the same properties, that is, they are continuous and $\tilde{\psi}_0$ satisfies (i) since B_0 is non--singuler. We have then:

$$\left(I+\sum_{i=0}^{p-1} C_i\, t^i\right) \sum_{i=0}^{p-1} g_i(z, u, v)\, t^i=t^l \begin{vmatrix} 1, 0 \\ 0, t^m\end{vmatrix} \sum_{i=0}^{p-1} \psi_i(z, u, v)\, t^i+R(t^p). \tag{2.8}$$

Clearly $g_i\equiv 0$ for $i=0, \ldots, l-1$. Assume for simplicity that $l=0$. Denote by c_i^2 the second row of the matrix C_i, $i=1, \ldots, p-1$. Let s_i, w_i be such that $c_i^2=(-s_i, -w_i)$ for $i=1, \ldots, p-1$. The equality (2.8) gives that:

$$g_0^1=\psi_0^1$$
$$g_0^2=0$$
$$g_i^2=s_1\, g_{i-1}^1+s_2\, g_{i-2}^1+\ldots+s_i\, g_0^1+w_1\, g_{i-1}^2+\ldots+w_{i-1}\, g_1^2,$$
$$i=1, \ldots, m-1,$$
$$g_m^2=\psi_0^2+s_1\, g_{m-1}^1+\ldots+s_m\, g_0^1+w_1\, g_{m-1}^2+\ldots+w_{m-1}\, g_1^2. \tag{2.9}$$

Denote

$$a_1=s_1,\; a_i=s_i+w_1\, a_{i-1}+w_2\, a_{i-2}+\ldots+w_{i-1}\, a_1,\; i=2, \ldots, m. \tag{2.10}$$

From (2.9) and (2.10) we obtain by an induction argument that:

$$g_0^2=0$$

$$g_i^2=a_1\, g_{i-1}^1+a_2\, g_{i-2}^1+\ldots+a_i\, g_0^1,\ i=1, \ldots, m-1$$

$$g_m^2=\psi_0^2+a_1\, g_{m-1}^1+a_2\, g_{m-2}^1+\ldots+a_m\, g_0.$$

Therefore (2.7) holds for $f_i(z, u, v)=g_i(z, u, v),\ i=0, \ldots, p-1, f^*(z, u, v)=$ $=\psi_0^2(z, u, v),\ \alpha_i(z, u, v)=0,\ \beta_i=0,\ i=0, \ldots, p-1.$

In order to prove sufficiency assume (2.7) and define:

$$C_0=I,\ C_i=\begin{vmatrix} 0, & 0 \\ -a_i, & 0 \end{vmatrix} \text{ for } i=1, \ldots, m,\ C(t)=\sum_{i=0}^{m} C_i\, t^i.$$

Then:

$$C(t)\cdot t^l \sum_{i=0}^{p-l-1} f_i(z, u, v)\, t^i=t^l \sum_{i=0}^{p-l-1} \tilde{\psi}_i(z, u, v)\, t^i+R(t^p)$$

where

$$\tilde{\psi}_0=\begin{vmatrix} f_0^1 \\ 0 \end{vmatrix},\ \tilde{\psi}_i=\begin{vmatrix} f_i^1 \\ 0 \end{vmatrix},\ i=0, \ldots, m-1,\ \tilde{\psi}_m=\begin{vmatrix} f_m^1 \\ f^* \end{vmatrix}$$

$$\tilde{\psi}_i=\begin{vmatrix} f_i^1 \\ f_1^2-\sum_{j=1}^{m} a_j\, f_{i-j} \end{vmatrix}\quad i=m+1, \ldots, p-l-1,$$

Therefore (2.7) gives that

$$G(t, z, u, v)=C^{-1}(t)\begin{vmatrix} t^l, & 0 \\ 0, & t^{l+m} \end{vmatrix} \sum_{i=0}^{p-1} \psi_i(z, u, v)\, t^i+R(t^p)+$$

$$+\sum_{i=0}^{p-1} \alpha_i(z, u, v)\, t^i+\sum_{i=0}^{p-1} \beta_i\, t^i$$

where $\psi_0(z, u, v)=\begin{vmatrix} f_0^1(z, u, v) \\ f^*(z, u, v) \end{vmatrix}$; that is $G(t, z, u, v)$ is of the form (2.6) if we put $H(t)=C^{-1}(t)\begin{vmatrix} t^l, & 0 \\ 0, & t^{l+m} \end{vmatrix}.$

Following [2] we might take in the condition (F) a k-dimensional subspace $L(z_*)$, $k\geqslant 2$, and consider representations of the mapping $\pi_{\tilde{L}} F_{p-1}(t, z, u, v)$ of the form (2.4) such that (i) holds with respect to R^k; that is the set $\bigcap_{u\in U} \operatorname{co} \psi_0(z_*, u, V)$ contains an interior point with respect to R^k. This however would not make the condition (F) more general since the following:

Proposition 2.2. If there exists a k-dimensional subspace $\tilde{L}$, $k\geqslant 2$, orthogonal to M such that the mapping $\pi_{\tilde{L}} F_{p-1}(t, z, u, v)$ can be represented in the form (2.4) in such a way that (r) and (i) hold then there exists a two-dimensional subspace L orthogonal to M such that the mapping $\pi_L F_{p-1}(t, z, u, v)$ has a representation of the form (2.4) which satisfies (r) and (i).

Proof. Take the representation of the mapping $\pi_L F_{p-1}(t, z, u, v)$ and write $H(t)$ in the form

$$H(t)=A(t)\begin{vmatrix} t^{l_1} & & 0 \\ & \ddots & \\ 0 & & t^{l_k} \end{vmatrix} B(t).$$

We assume as before that $B(t)\equiv I$. Denote $A^{-1}(t)=\sum_{i=0}^{\infty} D_i t^i$, $C(t)=A^{-1}(t)\times \times D_0^{-1}=I+\sum_{i=1}^{\infty} C_i t^i$. We have:

$$C(t) D_0 \pi_L F_{p-1}(t, z, u, v)=\begin{vmatrix} t^{l_1} & & 0 \\ & \ddots & \\ 0 & & t^{l_k} \end{vmatrix} \sum_{i=0}^{p-1} \psi_i(z, u, v) t^i+$$

$$+C(t) D_0 \sum_{i=0}^{p-1} \alpha_i(z, u, v) t^i+C(t) D_0 \sum_{i=0}^{p-1} \beta_i t^i+R(t^p)$$

where $\psi_0(z, u, v)$ satisfies (i) with respect to R^k. Let π_2 denotes the orthogonal projection onto the subspace $R^2=\{x \in R^n | x_3=\ldots=x=0\}$. We shall show that $\pi_2 D_0 \pi_L F_{p-1}(t, z, u, v)$ satisfies the condition (2.7) and hence has a representation of the form (2.6) in R^2 for which (i) and (r) hold. We can assume that $\alpha_i(z, u, v)=0$, $\beta_i=0$, $i=0, \ldots, p-1$, otherwise we consider $\pi_2 (D_0 \pi_L F_{p-1}(t, z, u, v)-D_0 \sum_{i=0}^{p-1} \alpha_i \times \times(z, u, v) t^i D_0 \sum_{i=0}^{p-1} \beta_i t^i)$. Therefore we have

$$\left(I+\sum_{i=0}^{p-1} C_i t^i\right) D_0 \pi_L F_{p-1}(t, z, u, v)=\begin{vmatrix} t^{l_1} & & 0 \\ & \ddots & \\ 0 & & t^{l_k} \end{vmatrix} \sum_{i=0}^{p-1} \psi_i(z, u, v) t^i+R(t^p). \tag{2.11}$$

Denote

$$D_0 \pi_L F_{p-1}(t, z, u, v)=\sum_{i=0}^{p-1} g_i(z, u, v) t^i,$$

$$g_i=\begin{vmatrix} g_i^1 \\ \tilde{g}_i^2 \end{vmatrix}, \quad \tilde{g}_i^2=\begin{vmatrix} g_i^2 \\ \vdots \\ g_i^k \end{vmatrix}, \quad i=0, \ldots, p-1,$$

$$C_i=\begin{vmatrix} c_{1,1}^i, \ldots, c_{1,k}^i \\ \vdots \\ c_{k,1}^i, \ldots, c_{k,k}^i \end{vmatrix}, \quad -\tilde{s}_i=\begin{vmatrix} c_{2,1}^i \\ \vdots \\ c_{k,1}^i \end{vmatrix};$$

$$\tilde{w}_i=\begin{vmatrix} c_{2,2}^i, \ldots, c_{2,k}^i \\ \vdots \\ c_{k,2}^i, \ldots, c_{k,k}^i \end{vmatrix}, \quad i=1, \ldots, p-1.$$

Then (2.11) gives for $l=l_1, m=l_2-l_1$ the following equalities

$$g_i=0,\ i=0, \ldots, l-1,\ g_l^1=\psi_0^1,\ \tilde{g}_l^2=0,$$

$$\tilde{g}_{l+i}^2=\sum_{j=1}^{i} \tilde{s}_j\, g_{l+i-j}^1+\sum_{j=1}^{i-1} \tilde{w}_j\, \tilde{g}_{l+i-j}^2,\ i=1, \ldots, m-1,$$

$$\tilde{g}_m^2=\tilde{g}^*+\sum_{j=1}^{m} \tilde{s}_j\, g_{l+m-j}^1+\sum_{j=1}^{m-1} \tilde{w}_j\, \tilde{g}_{l+m-j}^2 \tag{2.12}$$

where $\tilde{g}^*$ is such that its first component is equal to ψ_0^2.

Similarily as in the proof of Proposition 2.1 we obtain from (2.12) that for some vectors $\tilde{a}_1, \ldots, \tilde{a}_m \in R^{k-1}$ the following equalities hold:

$$\tilde{g}_l^2=0,\ \tilde{g}_{l+i}^2=\sum_{j=1}^{i} \tilde{a}_j\, g_{l+i-j}^1,\ i=1, \ldots, m-1,$$

$$\tilde{g}_{l+m}^2=\tilde{g}^*+\sum_{j=1}^{m} \tilde{a}_j\, g_{l+m-j}.$$

This implies that $\pi_2 D_0 \pi_{\tilde{L}} F_{p-1}(t, z, u, v)$ satisfies the condition from Proposition 2.1, thus has a required representation. Denote $K=\pi_2 D_0 \pi_{\tilde{L}}$. Take a non--singular mapping B such that $B(\operatorname{Im} K)=(\ker K)^{\perp}$ where the latter denotes the orthogonal complement of the kernel of the mapping K. Next take a non-singular mapping $\bar{B}$ such that $\bar{B}|_{(\ker K)^{\perp}}=(BK|_{(\ker K)^{\perp}})^{-1}$ that is its restriction to $(\ker K)^{\perp}$ is inverse to the restriction of BK. Then $\bar{B}BK$ is the orthogonal projection of R^n onto the two-dimensional subspace $L=(\ker K)^{\perp}$ which is orthogonal to M. Let $\bar{\bar{B}}$ be an isometric mapping which maps L onto R^2. Take $\pi_L=\bar{\bar{B}}\bar{B}BK$. Then $\pi_L F_{p-1} \times (t, z, u, v)$ has a required representation as $KF_{p-1}(t, z, u, v)$ has it and $\bar{\bar{B}}\bar{B}B$ is a non-singular mapping from R^2 into R^2. This completes the proof of Proposition 2.2.

Remark. The condition of evasion (F) is a generalization of the condition of evasion given in [2] for a linear game. It may not be immediately seen however, since they are formulated in different ways. There is considered in [2] the commutative ring of all locally integrable functions on the interval $[0, +\infty)$ with the multiplication defined as the convolution; that is

$$x(t)*y(t)=\int_0^t x(t-\tau)\, y(\tau)\, d\tau,$$

Let S denotes the function identically equal to one; that is, $S*x(t)=\int_0^t x(\tau)\, d\tau$. The ring is extended to a ring in which the element S has an inverse element $S^{-1}=D$. There are defined next entire elements over the extended ring; that is, the elements of the form

$$x=\hat{\lambda}_0+S*\hat{\lambda}_1+S^2*\hat{\lambda}_2+\ldots,$$

where $S*\hat{\lambda}_1+S^2*\hat{\lambda}_2+...=\lambda_1+t\lambda_2+\frac{t^2}{2!}\lambda_3+...$ is an entire function of t. All powers here are understood in the sens of the convolution, $\hat{\lambda}_i$ are so called constants of the ring and they are defined by $\hat{\lambda}=D*\lambda$ where λ is a real constant. Similarily there are defined entire matrices as $H(S)=\hat{H}_0+\hat{H}_1*S+\hat{H}_2*S^2+...$, where $\hat{H}_i$ are matrices of constant elements of the ring and then the determinant $\overset{*}{\det} H(S)$ which is defined in the usual formal way where multiplication is understood as the convolution. The part of the trajectory of the linear game $\dot{z}=Cz+f(u,v)$ which depends on controls takes in terms of the entire elements the following form:

$$S*R(S)*f(u,v) \text{ where } R(S)=\hat{I}+\hat{C}*S+\hat{C}^2*S^2+\dots.$$

It is projected into a k-dimensional subspace L orthogonal to M and the following representations are considered

$$\hat{\pi}_L*R(S)*f(u,v)=H(S)\left(\psi_0(u,v)+\psi_1(u,v)*S+\psi_2(u,v)*S^2+...\right)+\beta(t) \tag{2.13}$$

where $\beta(t)=\sum_{i=0}^{\infty}\beta_i\frac{t^i}{i!}$ is an entire function, $H(S)$ is an entire $k\times k$-matrix such that $\overset{*}{\det} H(S)\neq 0$. The latter implies that

$$H(S)=(\hat{A}_0+\hat{A}_1*S+...)\begin{vmatrix} S^{l_1} & & 0\\ & \ddots & \\ 0 & & S^{l_k}\end{vmatrix}(\hat{B}_0+\hat{B}_1*S+...)$$

where $\overset{*}{\det}\hat{A}_0\neq 0$, $\overset{*}{\det}\hat{B}_0\neq 0$, $l_1, ..., l_k$ are integers such that $0\leqslant l_1\leqslant...\leqslant l_k$. Denote $l_k=p-1$. The representation (2.13) gives a representation of the form (2.4) in the following way. (2.13) implies that

$$\hat{\pi}_L*f(u,v)+\widehat{\pi_L C}*f(u,v)*S+...+\widehat{\pi_L C^{p-1}}*S^{p-1}*f(u,v)=$$

$$=(\hat{A}_0+\hat{A}_1*S+...\hat{A}_{p-1}*S^{p-1}*\begin{vmatrix} S^{l_1} & & 0\\ & \ddots & \\ 0 & & S^{p-1}\end{vmatrix}*(\hat{B}_0+\hat{B}_1*S+...+$$

$$+\hat{B}_{p-1}*S^{p-1})\left(\psi_0(u,v)+...+\psi_{p-1}(u,v)*S^{p-1}\right)+\beta_1+\beta_2*S+...+$$

$$+\beta_{p-1}*S^{p-1}+R(S^p) \tag{2.14}$$

where $R(S^p)$ is of the form $R(S^p)=S^p*\left(\varphi(u,v)+...+S^r_*\,\varphi_r(u,v)\right)$ for some r and some functions $\varphi_i(u,v)$. Two entire elements are equal if their corresponding coefficients are equal. We compare the coefficients of the both sides of (2.14) and deduce that:

$$\pi_L F_{p-1}(t,z,u,v)=\pi_L f(u,v)+\pi_L Cf(u,v)\,t+...+\pi_L C^{p-1}f(u,v)\,t^{p-1}=$$

$$=(A_0+A_1 t+\ldots+A_{p-1} t^{p-1}) \begin{vmatrix} t^{l_1} & & 0 \\ & \ddots & \\ 0 & & t^{p-1} \end{vmatrix} (B_0+B_1 t+\ldots+B_{p-1} t^{p-1}) \times$$

$$\times (\psi_0(u,v)+\psi_1(u,v) t+\ldots+\psi_{p-1}(u,v) t^{p-1}+\sum_{i=0}^{p-1} \beta_i t^i+R(t^p),$$

det $A_0 \neq 0$, det $B_0 \neq 0$ thus

$$H(t)=(A_0+A_1 t+\ldots+A_{p-1} t^{p-1}) \begin{vmatrix} t^{l_1} & & 0 \\ & \ddots & \\ 0 & & t^{p-1} \end{vmatrix} (B_0+B_1 t+\ldots+B_{p-1} t^{p-1})$$

is as it is required in (r). Therefore we have obtained a representation of the form (2.4). The game considered in [2] is described by a linear equation of higher order. However, it may by transformed to an equation of the first order in a higher dimension and if the condition of evasion holds for the equation of higher order than it holds also for the corresponding equation of the first order. Therefore our sufficient condition of evasion is a generalization of that from [2].

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References

1. PONTRYAGIN L. S.: A linear differential game of escape. *Trudy Mat. Inst. Steklov* **112** (1971) 30–63.
2. GAMKRELIDZE R. V., KCHATATSHVILI G. L.: A differential game of evasion with nonlinear control. *SIAM J. Contr.* **12** (1974) 332–349.
3. KAŚKOSZ B.: On a nonlinear evasion problem. *SIAM J. Contr.* **15** (1977) 661–673.

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Warunek wystarczający ucieczki w grze nieliniowej. Cz. 1.

Rozpatrzono nieliniową grę różniczkową ucieczki. Podano warunek, przy którym dla każdego stanu początkowego gry można skonstruować strategię ucieczki, tzn. strategię zapewniającą, że trajektoria gry pozostanie przez cały czas poza pewną ustaloną podprzestrzenią skończoną. Podany tu warunek wystarczający dla ucieczki jest uogólnieniem warunku z pracy [2].

Достатичное условие убегания в нелинейной игре (Часть I)

Рассматривается нелинейная дифференциальная игра убегания. Представлено условие при котором для каждого начального состояния игры можно построить стратегию убегания, т.е. стратегию обеспечивающую то, что траектория игры останется в течение всего времени вне некоторого терминального подпространства. Представленное здесь достаточное условие убегания является обобщением условия из [2].

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Semicontinuity in constrained optimization. Part Ib. Norm of spaces

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4. Nearly convex multifunctions

We introduce a class of multifunctions for which the above approximation results yield especially important outcomes.

Let X and Y be normed spaces. We say that a multifunction $\Gamma\colon Y \to 2^X$ is nearly convex that at (x_0, y_0), if firstly there is a ball $B = B(x_0, \varepsilon) \times B(y_0, \eta)$ and a family of convex, closed multifunction $\{\Lambda_{(x,y)}\}$, $(x, y) \in B \cap G(\Gamma)$, $\Lambda_{(x,y)}\colon Y \to 2^X$. $(x, y) \in G(\Lambda_{(x,y)})$ such that there is $p_0 > 0$ with $\Lambda_{(x_0, z_0)} x \neq \emptyset$ for $x \in B(x_0, p_0)$ and such that $p < p_0$ for each $\xi > 0$ there is a neighbourhood W of (x_0, y_0) so that for $(x, y) \in W$

$$B(\Lambda_{(x,y)}^{-1} B(x_0, p), \xi) \supset \Lambda B_{(x_0, y_0)}^{-1}(x_0, p) \tag{4.1}$$

and secondly for each $\vartheta > 0$ there are numbers $r_0 > 0$, $\alpha > \vartheta$, $\varepsilon_1 > 0$, $\eta_1 > 0$, such that for all $(x, y) \in G(\Gamma) \cap B_1$ $(= B(x_0, \varepsilon_1) \times B(y_0, \eta_1))$ and all $r < r_0$

$$B(\Gamma^{-1} B(x, r), \vartheta r) \supset \Lambda_{(x,y)}^{-1} B(x, r) \cap B(y, \alpha r). \tag{4.2}$$

Γ is called almost convex, if the second condition is replaced by the weaker one: there are numbers $r_0 > 0$, $\alpha\vartheta > 0$ such that for all $(x, y) \in G(\Gamma) \cap B_p$ (4.2) holds.

In both the above cases we shall call $\Lambda_{(x,z)}$ a derivative of Γ.

4.1. Remark. Rolewicz [29] introduces a notion of the image continuity. Let $\{C_t\}_{t \in T}$ be a family of continuous linear operators, $C_t\colon X \to Y$ (X, Y are Banach spaces, T is a metric space). We say that $\{C_t\}_{t \in T}$ is image continuous at t_0 if there is a closed ball U such that for each $\xi > 0$ there is a neighbourhood W of t_0 such that for $t \in W$

$$B(C_t U, \xi) \supset C_{t_0} U, \; B(C_{t_0} U, \xi) \supset C_t U. \tag{4.3}$$

It is then natural to talk about the image lower (Hausdorff) semicontinuity if the first relation holds.

We observe that formula (4.1) generalizes this definition to arbitrary multifunctions (C_t in (4.3) defines the multifunction $\Delta_t y=\{x: C_t x=y\}$).

Certainly, if $\{C_t\}$ is continuous in the operator norm topology, then (4.3) is verified.

In fact, on assuming the operator norm continuity

$$\operatorname{dist}\left(C_t \overline{B(0,1)}, C_{t_0} \overline{B(0,1)}\right) \leqslant \sup_{\|x\| \leqslant 1} \|C_t x - C_{t_0} x\| = \|C_t - C_{t_0}\| \tag{4.4}$$

and setting $U=\overline{B(0,1)}$ we obtain (4.3).

4.2. *Example.* A closed convex multifunction is nearly convex at each $(x_0, y_0) \in G(\Gamma)$. The family $\{\Gamma\}_{(x,z)\in G(\Gamma)}$ is its derivative.

4.3. *Example.* Nearly convex multifunctions generalize a notion of continuously Fréchet defferentiable mappings $F: X \to Y$, X, Y Banach spaces. To show this define $\Gamma y=F^{-1}(y)$ and set $\Lambda_{(x, F(x))} y=\{v: F(x)+F'(x_0)(v-x)=y\}$. Since the graph $G(\Lambda_{(x,F(x))})$ is equal to $\left(0, -F(x)+F'(x_0)x\right)+G\left(F'(x_0)\right)$ the continuity of F around x_0 (4.1) around $\left(x_0, F(x_0)\right)$.

Take an arbitrary $\vartheta>0$. Since the derivative $F'(\cdot)$ is continuous in a neighbourhood of x_0 (in the operator norm topology) there is $\varepsilon_1>0$ and $r_0>0$ such that for $x \in B(x_0, \varepsilon_1)$ and for $v \in B(x, r_0)$ we have — in virtue of the mean value theorem (see Ioffe-Tikhomirov [17] p. 38)

$$\|F(v)-F(x)-F'(x_0)(v-x)\| \leqslant \vartheta \|v-x\| \tag{4.5}$$

or in terms of the Hausdorff distance

$$B(\Gamma^{-1} v, \vartheta \|v-x\|) \supset \Lambda^{-1}_{(x, F(x))} v \tag{4.6}$$

which implies (4.2).

4.4. *Example.* A nearly convex multifunction may admit several derivatives. Consider again a continuously Fréchet differentiable mapping F and its associated multifunction $\Gamma y=F^{-1}(y)$. Now let $\Lambda_{(x,F(x))} y=\{v: F(x)+F'(x)(v-x)=y\}$. From the assumptions that $F'(\cdot)$ is continuous in the operator norm topology it follows that (Kato [45] p. 258) it is continuous in the sense of the following metric $\hat{d}$ (defined on the set of subspaces of $X \times Y$). Let M, N be subspaces of $X \times Y$. Put $d(M,N)= \sup_{u \in S_M} \operatorname{dist}(u, S_N)$, where S_M and S_N are unit spheres in M and N respectively. Set $\hat{d}(M,N)=\max\left(d(M,N), d(N,M)\right)$. This is equivalent to Hausdorff metric restricted to the unit ball provided that the sets involved are subspaces. Thus (4.1) holds.

The mean value theorem gives now a formula similar to (4.5): for each $\vartheta>0$ there are $\varepsilon_1>0$ and $r_0>0$ so that for $x \in B(x_0, \varepsilon_1)$ and $w \in B(x, r_0)$

$$\|F(w)-F(x)-F'(x)(w-x)\| \leqslant \vartheta \|w-x\| \tag{4.7}$$

which implies (4.6) and (4.2).

4.5. *Example*. Let F be continuously Fréchet differentiable at x_0 and let $G \subset X$ and $K \subset Y$ be closed convex sets. The multifunction Γ, $\Gamma y = \{x \in G: F(x) \in y+K\}$ is nearly convex at (x_0, y_0) for $y_0 \in \Gamma^{-1} x_0 = F(x_0) - K$.

For any $y \in F(x) - K$ define $\Lambda_{(x,y)} z = \{v \in G: F(x) + F'(x_0)(v-x) \in z+K\}$. (4.1) follows as in Example 4.3.

The checking of (4.2) goes very much the same as in Evample 4.3 starting by formula (4.5). For $v \in B(x, r_0) \cap G$, $\Lambda_{(x,y)}^{-1} v = F(x) + F'(x_0)(v-x) + K$ and $\Gamma v = F(v) + K$, hence (4.6) and consequently (4.2) are valid.

5. Semicontinuity of nearly convex multifunctions

Let us begin by an auxiliary result

5.1. Lemma. Let X and Y be normed spaces and let T be a topological space. We are given a family $\{\Lambda(t)\}_{t \in T}$ of closed convex multifunctions: $\Lambda(t): Y \to 2^X$, such that there is p_0 so that for $0 < p < p_0$ for each $\xi > 0$ we can find a neighbourhood W of t_0 so that $B(\Lambda(t)^{-1} B(x_0, p), \zeta) \supset \Lambda(t_0)^{-1} B(x_0, p)$. Suppose that there are $t_0 \in T$, $x_0 \in X$, $y_0 \in Y$ and numbers $r_0 > 0$, $s_0 > 0$ such that

$$\overline{\Lambda(t_0)^{-1} B(x_0, r_0)} \supset B(y_0, s_0). \tag{5.1}$$

Then for any $s_1 < s_0$, $r_1 > r_0$ there are numbers $\varepsilon > 0$, $\eta > 0$ and a neighbourhood W of t_0, such that for $t \in <$ and for $r < r_1$

$$\overline{\Lambda(t)^{-1} B(x, r)} \supset B\left(y, \frac{r s_1}{r_1}\right) \tag{5.2}$$

for $x \in B(x_0, \varepsilon)$ and $y \in \Lambda(t)^{-1} x \cap B(y_0, \eta)$.

P r o o f. Choose positive numbers ε, η, ξ such that $s_1 + \eta + \xi < s_0$ and $r_0 + \varepsilon < r_1$.

Let $x \in B(x_0, \varepsilon)$, $y \in \Lambda(t)^{-1} x \cap B(y_0, \eta)$. Then

$$\Lambda(t)^{-1} B(x, r_0 + \varepsilon) \supset \Lambda(t)^{-1} B(x_0, r_0).$$

On the other hand, $B(\Lambda(t)^{-1} B(x_0, r_0), \xi) \supset \Lambda(t_0)^{-1} B(x_0, r_0)$, if $r_0 < p_0$ and for $t \in W$, where W corresponds to ξ. Therefore, on using (5.1) we obtain

$$\overline{\Lambda(t)^{-1} B(x, r_0 + \varepsilon) + B(0, \zeta)} \supset y_0 + \overline{B(0, s_0)} \supset y + \overline{B(0, s_0 - \eta)}.$$

By the Rådström cancellation theorem [53]

$$\overline{\Lambda(t)^{-1} B(x, r_0 + \varepsilon)} \supset \overline{B(y, s_0 - \eta - \zeta)}.$$

Now let $0 \leqslant \lambda \leqslant 1$. Because $G(\Lambda(t))$ is convex we have (see Robinson [26])

$$\begin{aligned}\overline{\Lambda(t)^{-1} B(x, \lambda(r_0 + \varepsilon))} &= \overline{\Lambda(t)^{-1}(\lambda(B(x, r_0 + \varepsilon) + (1-\lambda)x))} \supset \\ &\supset \lambda \overline{\Lambda(t)^{-1} B(x, r_0 + \varepsilon)} + (1-\lambda) y \supset \\ &\supset \lambda \overline{B(y, s_0 - \eta - \zeta)} + (1-\lambda) y = \overline{B(y, \lambda(s_0 - \eta - \zeta))}\end{aligned} \tag{5.3}$$

that is, for $r < r_1$, $\overline{\Lambda(t)^{-1} B(x, r)} \supset B\left(y, r \frac{s_0 - \eta - \zeta}{r_0 + \varepsilon}\right)$.

We say that a multifunction $\Gamma\colon Y\to 2^X$ is (locally) controllable at y_0, whenever y_0 is an interior point of $\Gamma^{-1}X$.

5.2. THEOREM. Let Y be Banach space, let $\Gamma\colon Y\to 2^X$ be nearly convex at (x_0, y_0) and let $\Lambda_{(x,y)}$ be one of its derivatives.

If $\Lambda_{(x_0,y_0)}$ is locally controllable at y_0, then Γ is δ-u.H.s.c. linearly and unformly at (x_0, y_0).

5.3. THEOREM. Let Y be Banach space and let Γ be a closed convex multifunction.

If Γ is locally controllable at y_0, then it is locally u.H.s.c. at (x_0, y_0) for each $x_0\in\Gamma y_0$ (linearly, uniformly with arbitrarily small balls taken for Q).

5.4. REMARK. The former theorem shows how a relatively weak property of an approximating family $\Lambda_{(x,y)}$ induces a much stronger property of the approximated multifunction. In the latter, the weak property of a multifunction implies another stronger property of that very same multifunction. Although the conclusion of the latter theorem is stronger than that of the former, Theorem 5.3. should be viewed as that special case, when a multifunction constitutes its own approximation.

Proof of Theorems 5.2 and 5.3. If a convex multifunction Λ is locally controllable at $y_0\colon \Lambda^{-1}X\supset B(y_0, t)$, then in view of the Baire theorem (see for instance [41]) for each $x_0\in\Lambda y_0$ there are numbers $r_1>0$, $s_1>0$ and $B(y_1, s_1)\subset B(y_0, t)$ such that $\overline{\Lambda^{-1}B(x_0, r_1)}\supset B(y_1, s_1)$. $B(2y_0-y_1, s_1)$ is a subset of $\Lambda^{-1}X$ and the Baire theorem gives $\overline{\Lambda^{-1}B(x_0, r_2)}\supset B(y_2, s_2)$ for some $B(y_2, s_2)\subset B(2y_0-y_1, s_1)$. Using now the convexity of Λ we obtain that $\overline{\Lambda^{-1}B(x_0, r_0)}\supset B(y_0, s_0)$ for some $r_0>0, s_0>0$.

Therefore according to Lemma 5.1 the local controllability at y_0 implies

$$\overline{\Lambda^{-1}_{(x,y)}B(x, r)}\supset B(y, \alpha_1 r) \tag{5.4}$$

where $\alpha_1>0$ is a universal constant for all $x\in B(x_0, \varepsilon_1)$ and $y\in\Gamma^{-1}x\cap B(y_0, \eta_1)$ for some $0<\varepsilon_1\leqslant\varepsilon$ and $0<\eta_1\leqslant\eta$.

Take a number $\vartheta<\alpha_1$. Since Γ is nearly convex there are $r_0, \varepsilon_1, \eta_1$ and $\alpha>\vartheta$ so that (4.2) is fulfilled and thus

$$B\big(\Gamma B(x, r), \vartheta_1 r\big)\supset B\big(y, (\alpha\vee\alpha_1) r\big) \tag{5.5}$$

for any $\vartheta<\vartheta_1<\alpha\vee\alpha_1$.

The assumptions of Theorem 3.2 are now satisfied with $q(r)=(\alpha\vee\alpha_1)r$ and $\omega(r)=\dfrac{\vartheta_1}{\alpha\vee\alpha_1}r$, so Γ is δ-u.H.s.c. uniformly at (x_0, y_0) at $p(r)\leqslant(\alpha\vee\alpha_1-\vartheta_1)r$ and the proof of Theorem 5.2 is complete.

Since Γy_0 of Theorem 5.3 is convex, Γ is locally u.H.s.c. uniformly at (x_0, y_0) according to Theorem 2.13 and a rate $p(r)$ may be taken less or equal to $r\cdot(\alpha\vee\alpha_1-\vartheta_1)/2$.

5.5. THEOREM. Let Y and X be Banach space. If the multifunction Γ is almost convex at (x_0, y_0) and if one of its derivatives $A_{(x,z)}$ verifies (5.4) so that $\vartheta<\alpha_1$ (ϑ occurs in formula (4.2)), then Γ is δ-u.H.s.c. at (x_0, y_0) at linear rate.

Proof is identical as the previous proof started from (5.4).

5.6. *Example*. Theorem 5.3 generalizes Theorem 1 of Robinson [26]. The assumption that y_0 be a internal point of $\Gamma^{-1} X$ (i.e. such that for each $h \in Y$ there is $\lambda_0>0$ so that $y_0+\lambda h \in \Gamma^{-1} X$ for $0\leqslant\lambda\leqslant\lambda_0$) easily implies that y_0 is an interior point, for $\Gamma^{-1} X$ is convex and Y is a Banach space. The conclusion of that theorem follows from Theorem 5.3 in view of Corollary 3.3.

Theorem 2 of [26] is a special case of Corollary 3.3. formulated for convex closed multifunctions.

The Robinson formula (used also in his other works)

$$\operatorname{dist}(x, \Gamma y)\leqslant k \operatorname{dist}(y, \Gamma^{-1} x) \tag{5.6}$$

has the following meaning: $B(\Gamma y, kr)\supset \Gamma B(y, r)$ for each r. Indeed $\operatorname{dist}(y, \Gamma^{-1} x)<r$ is equivalent to $B(y, r)\cap\Gamma^{-1} x\neq\emptyset$ or to $x\in\Gamma B(y, r)$ and $\operatorname{dist}(x, \Gamma y)<kr$ means that $x\in B(\Gamma y, kr)$. Adding the condition of Theorem 2 of [26] that $y\in B(y_0, \eta)$ we get the δ-upper Hausdorff semicontinuity. We obtain the stronger local upper Hausdorff semicontinuity. A corollary of the Robinson theorems is the Banach open mapping theorem.

5.7. COROLLARY (Banach open mapping theorem)

Let be F a linear continuous map of a Banach space X onto a Banach space Y. Then the multifunction $f^{-1}\colon Y\to 2^X$ is Hausdorff continuous.

5.8. COROLLARY (Robinson [27])

Ler F be a continuously Fréchet differentiable (at x_0) mapping from a Banach space X to a Banach space Y, let C be closed and convex subset of X and let K be a closed convex cone of Y. If

$$0\in \operatorname{Int}\{F(x_0)\, F'(x_0)(C-x_0)+K\} \tag{5.7}$$

then is δ-u.H.s.c. at y_0 ($\in Fx_0+K$).

Proof. See Example 4.5 and note that the condition (5.7) means that A is locally controllable at (x_0, y_0).

5.9. COROLLARY (Lusternik theorem [49], [17])

Let X, Y be Banach spaces, V a neighbourhood of x_0 and let F be a continuously Fréchet differentiable at x_0 and $F'(x_0)X=Y$. Then there are a neighbourhood $V_1\subset V$, a number k and a mapping of U_1 to X: $\xi\to x(\xi)$ such that for all $\xi\in U_1$

$$\begin{aligned} &F(\xi+x(\xi))=F(x_0),\\ &\|x(\xi)\|\leqslant k\,\|F(\xi)-F(x_0)\|. \end{aligned} \tag{5.8}$$

(Consequent'y the tangent space to $\{x: F(x)=F(x_0)\}$ is equal to the kernel of $F'(x_0)$).

Proof. The thesis may be reformulated: for any $x \in U_1$,

$$\operatorname{dist}\left(x, F^{-1}\left(F(x_0)\right)\right) \leqslant k \|F(x)-F(x_0)\|$$

which means the δ-upper Hausdorff semicontinuity of Γ at $(x_0, F(x_0))$ (Example 2.14). In view of Example 4.3 the multifunction F^{-1} is nearly convex at $(x_0, F(x_0))$. The condition $F'(x_0) X=Y$ means that $(F'(x_0))^{-1}$ is controllable, hence locally controllable and we are in the assumptions of Theorem 5.2.

5.10. Corollary. (Ioffe-Tikhomirov extension of the Lusternik theorem [17] p. 45) Let X and Y be Banach spaces, L a linear continuous operator: $X \to Y$, F a mapping of a neighbourhood U of x_0 to Y.

Suppose that $LX=Y$ and denote $C(L)=\|\tilde{L}^{-1}\|$, where $\tilde{L}$ is the quotient mapping of L. Assume that there is a number $\delta>0$ such that $\delta C(L)<1/2$ and

$$\|F(x)-F(v)-L(x-v)\| \leqslant \delta \|x-v\| \tag{5.9}$$

for all x, v from U. Then there is a neighbourhood U_1 and $k>0$ and a mapping of U_1 to X: $\xi \to x(\xi)$ such that (5.8) holds.

Proof. We set $\Gamma y=F^{-1}(y)$ and $\Lambda_{(x, F(x))} y=\{v: F(x)+L(v-x)=y\}$. By the assumptions

$$\Lambda^{-1}_{(x_0, F(x_0))} B(x_0, r) \supset B\left(F(x_0), \frac{r}{C(L)}\right)$$

and (5.9) yields (4.2) with $\vartheta=\delta$. If $\delta C(L)<1$ then by Theorem 5.5 Γ is δ-u.H.s.c. at (x_0, y_0) and a linear rate may be chosen. This gives (5.8) in view of Example 2.14.

5.11. Theorem. Let X and Y be Banach spaces. Consider a (Fréchet) differentiable mapping F of X to Y such that the derivative $F'(\cdot)$ is locally Lipschitz continuous at x_0 (in the operator norm) and assume that $F'(x_0) X=Y$. Then the multifunction F^{-1} is locally u.H.s.c. at $(x_0, F(x_0))$ (uniformly, linearly and for arbitrarily small balls taken for Q).

Proof. It is enough to show that $M \stackrel{\mathrm{df}}{=} F^{-1}(F(x_0))$ fulfils (2.4) of Theorem 2.12, because Theorem 5.2 may be used to obtain the δ-upper Hausdorff semicontinuity.

Let c be the Lipschitz constant of $F'(\cdot)$ and let k be the constant of $q(r)=kr$ (in Theorem 5.2).

Take $0<\varepsilon<\frac{1}{16kc}$, such that $F'(\cdot)$ is Lipschitz continuous in $B(x_0, 2\varepsilon)$ and such that F^{-1} is δ-u.H.s.c. at a rate kr at $(x, F(x))$, $x \in B(x_0, 2\varepsilon)$.

Suppose that $x_1 \in \overline{B(x_0, \varepsilon)}$ and also $x_1 \in B(M, r)$, where $r \leqslant \frac{1}{8kc+1}$. Let $x_2 \in \tilde{M} \stackrel{\mathrm{df}}{=} M \cap \overline{B(x_0, \varepsilon)}$ satisfy $\|x_1-x_2\| \leqslant \operatorname{dist}(x_1, M)+r^2$ and such that there is $0<\zeta \leqslant r$ such that $\overline{B(x_2, \zeta)} \subset B(x_0, \varepsilon)$.

The tengent space $x_2+\ker F'(x_2)$ to M at x_0 is denoted by L and by definition $\tilde{L}=L\cap \overline{B(x_2,\zeta)}$.

Let $x_3\in\tilde{L}$ be such that $\|x_1-x_3\|\leqslant \operatorname{dist}(x_1,\tilde{L})+r^2$. Let $x_4\in M$ fulfil $\|x_1-x_4\|\leqslant r$. By the assumptions there is $x_5\in L$ such that

$$\begin{aligned}\|x_4-x_5\|\leqslant k\,\|F'(x_2)\,x_4-F'(x_2)\,x_5\|=k\,\|F'(x_2)(x_4-x_2)\|=\\ =k\,\|F(x_4)-F(x_2)-F'(x_2)(x_4-x_2)\|\leqslant\\ \leqslant k\sup_{z\in[x_4,\,x_2]}\|F'(x_2)-F'(z)\|\,\|x_4-x_2\|\leqslant\\ \leqslant k\cdot c\,\|x_4-x_2\|^2.\end{aligned}\tag{5.10}$$

We have used the mean value theorem and the assumption of Lipschitz continuity of $F'(\cdot)$.

Let x_6 denote the point of intersection of $[x_1,x_3]$ with $\{x: \|x-x_2\|-\zeta\}$ and let $x_7\in L$ fulfil

$$\frac{\|x_1-x_5\|}{\|x_1-x_3\|}=\frac{\|x_6-x_7\|}{\|x_6-x_3\|}.\tag{5.11}$$

Observe that $\|x_6-x_3\|\leqslant \operatorname{dist}(x_6,\tilde{L})+r^2$. Therefore, in view of Theorem 2.13 applied to the convex set L and to the ball $\overline{B(x_2,\zeta)}$, the ratio (5.11) is greater or equal to 2.

In virtue of the Lusternik theorem

$$\begin{aligned}\operatorname{dist}(x_3,M)\leqslant k\,\|F(x_3)-F(x_2)\|=k\,\|F(x_3)-F(x_2)-F'(x_2)(x_3-x_2)\|\leqslant\\ \leqslant kc\cdot\|x_3-x_2\|^2.\end{aligned}\tag{5.12}$$

If each $x_8\in M$ realizing these estimates lies outside $\overline{B(x_2,\zeta)}$, we apply estimates of the type (5.12) to the point $x_3+kc\,\|x_3-x_2\|\,(x_2-x_3)$ to be sure that there is $x_8\in\tilde{M}$ such that

$$\|x_8-x_3\|\leqslant 2kc\,\|x_3-x_2\|^2\leqslant 8kc\,\zeta^2\leqslant 8kcr^2.\tag{5.13}$$

For brevity we introduce the notation $s=\|x_1-x_2\|$. In accordance to (5.10)—(5.13) we have

$$\begin{aligned}s\leqslant\|x_1-x_3\|+\|x_3-x_8\|+r^2\leqslant 2\,\|x_1-x_5\|+(8kc+1)\,r^2\leqslant\\ \leqslant 2\,(\|x_1-x_4\|+\|x_4-x_5\|)+(8kc+1)\,r^2\leqslant\\ \leqslant 2\,(r+kc\,\|x_4-x_2\|^2)+(8kc+1)\,r^2.\end{aligned}\tag{5.14}$$

But $\|x_4-x_2\|\leqslant\|x_4-x_1\|+\|x_2-x_1\|\leqslant r+s$. Hence (5.14) becomes

$$s\leqslant 2r+2kc\,(r+s)^2+(8kc+1)\,r^2\leqslant 2r+4kcs^2+(8kc+1)\,r^2\leqslant 3r+1/2\,s,\tag{5.15}$$

... $s\leqslant 6r$.

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References

1. C. Berge, Espaces topologiques, Dunod, Paris 1966.
2. F. H. Clarke, A new approach to Lagrange multipliers. *Math. Opns. Res.* **1** (1976) 165–174.
3. L. Cesari, Seminormality and upper semicontinuity in optimal control. *J. Opt. Theory Appl.* **6** (1970), 114–137.
4. G. B. Dantzig, J. Folkman, N. Shaplro; On the continuity of the minimum set of a continuous function. *J. Math: Anal. Appl.* **17** (1967) 519–548.
5. S. Dolecki, A classification of controllability concepts for infinite dimensional linear systems. *Control a. Cyber.* **5** (1976) 33–44.
6. S. Dolecki, Bounded controlling sequences, lower stability and certain penalty procedures. *Appl. Math. Opt.*, to appear.
7. S. Dolecki, Constraints stability and moduli of semicontinuity. 2nd IFAC Symp. on Distributed Parameter Systems, Warwick, June 1977 (preprint).
8. S. Dolecki, Semi-continuité des contraintes aux problèmes extremaux, to appear.
9. S. Dolecki, Remarks on semicontinuity, Bull. Acad. Pol. Sci., to appear.
10. S. Dolecki, S. Kurcyusz, On fi-convexity in extremal problems, to appear in *SIAM J. Contr. Optimization.*
11. S. Dolecki, S. Kurcyusz, Convexitć généralisé et optimisation. *C.R.Ac.Sc.* Paris **283** (1976) A-91-94.
12. S. Dolecki, S. Rolewicz, A characterization of semicontinuity — preserving multifunctions, to appear in *J. Math. Anal. Appl.*
13. S. Dolecki, D. L. Russell, A general theory of observation and control. *SIAM J. Contr. Optimization* **15** (1977).
14. J. P. Evans, F. J. Gould, Stability in nonlinear programming. *Ppns. Res.* **11** (1963) 399–417.
15. K. Fan, On the Krein-Milman theorem, in Convexity, Proc. Symp. Pure Math. VII. A.M.S. (1963) 211–220.
16. H. J. Greenberg, W. P. Pierskalla, Extensions of the Evans-Gould stability theorems for mathematical problems. *Opns. Res.* **20** (1972) 143–153.
17. A. D. Ooffe, V. M. Tikhomirov, Theory of extremal problems (in Russian). Izd. Nauka, Moskva 1974.
18. K. Kuratowski, Topology (English transl. from French). I, II. Academic Press and Polish Scientific Publishers, 1966.
19. S. Kurcyusz, Some remarks on generalized Lagrangians. Proc. 7th IFIP Conf. Nice 1975.
20. P. Levine, J.-C. Pomerol, C-closed mappings and Kuhn-Tucker vector in convex programming. CORE 7620. Universitć Catholique de Louvain.
21. P. O. Lindberg, A generalization of Fenchel conjugation giving generalized Lagrangians and symmetric nonconvex duality. IX Symp. Math. Programming, Budapest 1976.
22. P. Michel, Problème des inégalités. Application à la programmation et au contrôle optimal. *Bull. Soc. Math. France* **101** (1973) 413–439.
23. C. Olech, Existence theory in optimal control problems, in Internat. Conf., Diff. Eqns., Academic Press, New York, San Francisco, London 1975.
24. V. Pták, A quantitative refinement of the closed graph theorem. *Czech. Math. J.* **24** (99) (1974) 503–506.
25. V. Pták, Nondiscrete mathematical induction and iterative existence proofs. *Lin. Algebra* **1?** (1976) 223–239.
26. S. M. Robinson, Regularity and stability for convex multivalued functions. *Math* **1** (1976) 136–143.
27. S. M. Robinson, Stability theory for systems of inequalities. II. Differen

29. S. Rolewicz, Funktionalanalysis und Steuerungstheorie. Berlin, Springer Verlag 1976.
30. S. Rolewicz, On universal time for controllability of time depending linear control systems. *Studia Math.* **49** (1976) 27–32.
31. H. H. Schaefer, Topological vector spaces. Springer Verlag, New York—Heidelberg—Berlin 1971.
32. I. Singer, The theory of best approximation and functional analysis. SIAM, Philadelphia 1974.
33. I. A. Vainstein, On closed mappings of metric spaces. *DAN, S.U.* 57 (1947) 419–321 (in Russian).
34. A. P. Wierzbicki, S. Kurcyusz, Projection on a cone, penalty functionals and duality theory for problems with inequality constraints in Hilbert space. *SIAM J. Contr. Opt.* **15** (1977) 25–56.
35. J. Zabczyk, Remarks on the algebraic Riccati equation in Hilbert space. *Appl. Math. Opt.* **2** (1975/76) 251–258.
36. A. V. Balakrishnan, A computational approach to the maximum principle. *J. Comp. Syst. Sc.* **5** (1971) 163–191.
37. E. J. Balder, An extension of duality-stability relations to nonconvex optimization problems. *SIAM J. Contr. Opt.* **15** (1977) 329–343.
38. D. P. Bertsekas, Combined primal-dual and penalty method for constrained minimization. *SIAM J. Contr.* **13** (1975) 521–544.
39. S. Dolecki, On construction of constraints multifunctions, to appear.
40. H. Halkin, Mathematical programming without differentiability in Calculus of Variations and Control Theory. D. L. Russell ed. Academic Press. New York 1976 p. 279–288.
41. J. Horváth, Topological vector spaces and distributions. Addison Wesley Publ. 1966.
42. S. M. Howe, New conditions for exactness of a simple penalty function. *SIAM J. Contr.* **11** (1973) 378–381.
43. A. D. Ioffe, Regular points of Lipschitz mappings, to appear.
44. A. D. Ioffe, Necessary and sufficient conditions for a local minimum II, Conditions of Levitin–Miljutin–Osmolovskii type, to appear.
45. T. Kato, Perturbation theory for linear operators, Springer Verlag, Berlin 1966.
46. M. A. Krasnoselski, Continuity of the operator $Fu(x)=f(x, u(x))$. *DAN S.U.* 77 (1951) (in Russian).
47. E. S. Levitin, On conditions for the local minimum and extremal constrained problems. *DAN S.U.* **221** (1975) 1031–1034 (in Russian).
48. E. S. Levitin, A. A. Miljutin, N. P. Osmolovskii, On conditions of local extremum in constrained problems, in Mathematical Economics and Functional Analysis. B. S. Mitjagin ed. Nauk, Moscow 1974 (in Russian).
49. L. A. Lusternik, On conditional extrema of functionals. Matem. sb. 41 (1934) 390–401 (in Russian).
50. O. L. Mangasarian, Unconstrained Lagrangians in nonlinear programming. *SIAM J. Contr.* **13** (1975) 772–791.
51. T. Pietrzykowski, The potential method for conditional maxima in locally compact metric spaces. *Numer. Math.* **14** (1970) 325–329.
52. B. T. Poljak, Semicontinuity of integral functionals and existence theorems. Mat. sb. 78 (1969) 65–84.
53. H. Radström, An embedding theorem for spaces of convex sets. *Proc. Amer. Marth. Soc.* **3** (1952) 165–169.
54. R. T. Rockafellar, Dual problems of Lagrange for arcs of bounded variation, in Calculus of Variations and Control Theory, D. L. Russell ed. Academic Press, New York 1976 p. 155–192.
55. A. B. Schwarzkopf, Controllability and tenability of nonlinear systems with state equality constraints. *SIAM J. Contr.* **13** (1975) 695–705.

56. J. Warga, Derivate containers, inverse functions and controllability, in Calculus of Variations and Control Theory. D. L. Russell ed. Academic Press, New York 1976, p. 13–46.
57. A. P. Wierzbicki, A penalty function shifting method in constrained static optimization and its convergence properties. *Arch. Autom. Telemech.* **16** (1971) 395–416.
58. J. Zabczyk, Comments on duality theorems, to appear.
59. W. I. Zangwill, Nonlinear programming. Prentice Hall, Englewood Cliffs, N. J. 1969.
60. S. Dolecki, S. Rolewicz, Metric characterizations of the upper semicontinuity, to appear.
61. S. Dolecki, S. Rolewicz, Exact penalty for local minima, to appear.