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Semicontinuity in constrained optimization. Part Ib. Norm of spaces

by

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4. Nearly convex multifunctions

We introduce a class of multifunctions for which the above approximation results yield especially important outcomes.

Let X and Y be normed spaces. We say that a multifunction $\Gamma\colon Y\to 2^X$ is nearly convex that at (x_0, y_0), if firstly there is a ball $B=B(x_0, \varepsilon)\times B(y_0, \eta)$ and a family of convex, closed multifunction $\{\Lambda_{(x,y)}\}$, $(x, y)\in B\cap G(\Gamma)$, $\Lambda_{(x,y)}: Y\to 2^X$. $(x, y)\in G(\Lambda_{(x,y)})$ such that there is $p_0>0$ with $\Lambda_{(x_0,z_0)}x\neq\emptyset$ for $x\in B(x_0, p_0)$ and such that $p<p_0$ for each $\xi>0$ there is a neighbourhood W of (x_0, y_0) so that for $(x, y)\in W$

$$B(\Lambda^{-1}_{(x,y)} B(x_0, p), \xi)\supset \Lambda B^{-1}_{(x_0,y_0)}(x_0, p) \tag{4.1}$$

and secondly for each $\vartheta>0$ there are numbers $r_0>0$, $\alpha>\vartheta$, $\varepsilon_1>0$, $\eta_1>0$, such that for all $(x, y)\in G(\Gamma)\cap B_1$ $(=B(x_0, \varepsilon_1)\times B(y_0, \eta_1))$ and all $r<r_0$

$$B(\Gamma^{-1} B(x, r), \vartheta r)\supset \Lambda^{-1}_{(x,y)} B(x, r)\cap B(y, \alpha r). \tag{4.2}$$

Γ is called almost convex, if the second condition is replaced by the weaker one: there are numbers $r_0>0$, $\alpha\vartheta>0$ such that for all $(x, y)\in G(\Gamma)\cap B_p$ (4.2) holds.

In both the above cases we shall call $\Lambda_{(x,z)}$ a derivative of Γ.

4.1. Remark. Rolewicz [29] introduces a notion of the image continuity. Let $\{C_t\}_{t\in T}$ be a family of continuous linear operators, $C_t\colon X\to Y$ (X, Y are Banach spaces, T is a metric space). We say that $\{C_t\}_{t\in T}$ is image continuous at t_0 if there is a closed ball U such that for each $\xi>0$ there is a neighbourhood W of t_0 such that for $t\in W$

$$B(C_t U, \xi)\supset C_{t_0} U, B(C_{t_0} U, \xi)\supset C_t U. \tag{4.3}$$

It is then natural to talk about the image lower (Hausdorff) semicontinuity if the first relation holds.

We observe that formula (4.1) generalizes this definition to arbitrary multifunctions (C_t in (4.3) defines the multifunction $\Delta_t y=\{x: C_t x=y\}$).

Certainly, if $\{C_t\}$ is continuous in the operator norm topology, then (4.3) is verified.

In fact, on assuming the operator norm continuity

$$\operatorname{dist}\left(C_t \overline{B(0,1)}, C_{t_0} \overline{B(0,1)}\right) \leqslant \sup_{\|x\| \leqslant 1} \|C_t x - C_{t_0} x\| = \|C_t - C_{t_0}\| \tag{4.4}$$

and setting $U=\overline{B(0,1)}$ we obtain (4.3).

4.2. *Example.* A closed convex multifunction is nearly convex at each $(x_0, y_0) \in G(\Gamma)$. The family $\{\Gamma\}_{(x,z) \in G(\Gamma)}$ is its derivative.

4.3. *Example.* Nearly convex multifunctions generalize a notion of continuously Fréchet defferentiable mappings $F: X \to Y$, X, Y Banach spaces. To show this define $\Gamma y = F^{-1}(y)$ and set $\Lambda_{(x, F(x))} y = \{v: F(x) + F'(x_0)(v-x) = y\}$. Since the graph $G(\Lambda_{(x, F(x))})$ is equal to $(0, -F(x) + F'(x_0)x) + G(F'(x_0))$ the continuity of F around x_0 (4.1) around $(x_0, F(x_0))$.

Take an arbitrary $\vartheta > 0$. Since the derivative $F'(\cdot)$ is continuous in a neighbourhood of x_0 (in the operator norm topology) there is $\varepsilon_1 > 0$ and $r_0 > 0$ such that for $x \in B(x_0, \varepsilon_1)$ and for $v \in B(x, r_0)$ we have — in virtue of the mean value theorem (see Ioffe-Tikhomirov [17] p. 38)

$$\|F(v) - F(x) - F'(x_0)(v-x)\| \leqslant \vartheta \|v-x\| \tag{4.5}$$

or in terms of the Hausdorff distance

$$B(\Gamma^{-1} v, \vartheta \|v-x\|) \supset \Lambda^{-1}_{(x, F(x))} v \tag{4.6}$$

which implies (4.2).

4.4. *Example.* A nearly convex multifunction may admit several derivatives. Consider again a continuously Fréchet differentiable mapping F and its associated multifunction $\Gamma y = F^{-1}(y)$. Now let $\Lambda_{(x, F(x))} y = \{v: F(x) + F'(x)(v-x) = y\}$. From the assumptions that $F'(\cdot)$ is continuous in the operator norm topology it follows that (Kato [45] p. 258) it is continuous in the sense of the following metric $\hat{d}$ (defined on the set of subspaces of $X \times Y$). Let M, N be subspaces of $X \times Y$. Put $d(M, N) = \sup_{u \in S_M} \operatorname{dist}(u, S_N)$, where S_M and S_N are unit spheres in M and N respectively. Set $\hat{d}(M, N) = \max(d(M, N), d(N, M))$. This is equivalent to Hausdorff metric restricted to the unit ball provided that the sets involved are subspaces. Thus (4.1) holds.

The mean value theorem gives now a formula similar to (4.5): for each $\vartheta > 0$ there are $\varepsilon_1 > 0$ and $r_0 > 0$ so that for $x \in B(x_0, \varepsilon_1)$ and $w \in B(x, r_0)$

$$\|F(w) - F(x) - F'(x)(w-x)\| \leqslant \vartheta \|w-x\| \tag{4.7}$$

which implies (4.6) and (4.2).

4.5. *Example.* Let F be continuously Fréchet differentiable at x_0 and let $G \subset X$ and $K \subset Y$ be closed convex sets. The multifunction Γ, $\Gamma y = \{x \in G: F(x) \in y+K\}$ is nearly convex at (x_0, y_0) for $y_0 \in \Gamma^{-1} x_0 = F(x_0) - K$.

For any $y \in F(x)-K$ define $\Lambda_{(x,y)} z = \{v \in G: F(x)+F'(x_0)(v-x) \in z+K\}$. (4.1) follows as in Example 4.3.

The checking of (4.2) goes very much the same as in Evample 4.3 starting by formula (4.5). For $v \in B(x, r_0) \cap G$, $\Lambda_{(x,y)}^{-1} v = F(x)+F'(x_0)(v-x)+K$ and $\Gamma v = F(v)+K$, hence (4.6) and consequently (4.2) are valid.

5. Semicontinuity of nearly convex multifunctions

Let us begin by an auxiliary result

5.1. Lemma. Let X and Y be normed spaces and let T be a topological space. We are given a family $\{\Lambda(t)\}_{t \in T}$ of closed convex multifunctions: $\Lambda(t): Y \to 2^X$, such that there is p_0 so that for $0<p<p_0$ for each $\xi>0$ we can find a neighbourhood W of t_0 so that $B(\Lambda(t)^{-1} B(x_0, p), \zeta) \supset \Lambda(t_0)^{-1} B(x_0, p)$. Suppose that there are $t_0 \in T$, $x_0 \in X$, $y_0 \in Y$ and numbers $r_0>0$, $s_0>0$ such that

$$\overline{\Lambda(t_0)^{-1} B(x_0, r_0)} \supset B(y_0, s_0). \tag{5.1}$$

Then for any $s_1<s_0$, $r_1>r_0$ there are numbers $\varepsilon>0$, $\eta>0$ and a neighbourhood W of t_0, such that for $t \in <$ and for $r<r_1$

$$\overline{\Lambda(t)^{-1} B(x, r)} \supset B\left(y, \frac{r s_1}{r_1}\right) \tag{5.2}$$

for $x \in B(x_0, \varepsilon)$ and $y \in \Lambda(t)^{-1} x \cap B(y_0, \eta)$.

Proof. Choose positive numbers ε, η, ξ such that $s_1+\eta+\xi<s_0$ and $r_0+\varepsilon<r_1$.

Let $x \in B(x_0, \varepsilon)$, $y \in \Lambda(t)^{-1} x \cap B(y_0, \eta)$. Then

$$\Lambda(t)^{-1} B(x, r_0+\varepsilon) \supset \Lambda(t)^{-1} B(x_0, r_0).$$

On the other hand, $B(\Lambda(t)^{-1} B(x_0, r_0), \xi) \supset \Lambda(t_0)^{-1} B(x_0, r_0)$, if $r_0<p_0$ and for $t \in W$, where W corresponds to ξ. Therefore, on using (5.1) we obtain

$$\overline{\Lambda(t)^{-1} B(x, r_0+\varepsilon)+B(0, \zeta)} \supset y_0+\overline{B(0, s_0)} \supset y+\overline{B(0, s_0-\eta)}.$$

By the Rådström cancellation theorem [53]

$$\overline{\Lambda(t)^{-1} B(x, r_0+\varepsilon)} \supset \overline{B(y, s_0-\eta-\zeta)}.$$

Now let $0 \leqslant \lambda \leqslant 1$. Because $G(\Lambda(t))$ is convex we have (see Robinson [26])

$$\overline{\Lambda(t)^{-1} B(x, \lambda(r_0+\varepsilon))} = \overline{\Lambda(t)^{-1} (\lambda(B(x, r_0+\varepsilon)+(1-\lambda)x))} \supset$$
$$\supset \lambda \overline{\Lambda(t)^{-1} B(x, r_0+\varepsilon)}+(1-\lambda) y \supset$$
$$\supset \lambda \overline{B(y, s_0-\eta-\zeta)}+(1-\lambda) y = \overline{B(y, \lambda(s_0-\eta-\zeta))} \tag{5.3}$$

that is, for $r<r_1$, $\overline{\Lambda(t)^{-1} B(x, r)} \supset B\left(y, r \dfrac{s_0-\eta-\zeta}{r_0+\varepsilon}\right)$.

We say that a multifunction $\Gamma\colon Y\to 2^X$ is (locally) controllable at y_0, whenever y_0 is an interior point of $\Gamma^{-1}X$.

5.2. THEOREM. Let Y be Banach space, let $\Gamma\colon Y\to 2^X$ be nearly convex at (x_0, y_0) and let $\Lambda_{(x,y)}$ be one of its derivatives.

If $\Lambda_{(x_0,y_0)}$ is locally controllable at y_0, then Γ is δ-u.H.s.c. linearly and unformly at (x_0, y_0).

5.3. THEOREM. Let Y be Banach space and let Γ be a closed convex multifunction.

If Γ is locally controllable at y_0, then it is locally u.H.s.c. at (x_0, y_0) for each $x_0\in\Gamma y_0$ (linearly, uniformly with arbitrarily small balls taken for Q).

5.4. REMARK. The former theorem shows how a relatively weak property of an approximating family $\Lambda_{(x,y)}$ induces a much stronger property of the approximated multifunction. In the latter, the weak property of a multifunction implies another stronger property of that very same multifunction. Although the conclusion of the latter theorem is stronger than that of the former, Theorem 5.3. should be viewed as that special case, when a multifunction constitutes its own approximation.

Proof of Theorems 5.2 and 5.3. If a convex multifunction Λ is locally controllable at $y_0\colon \Lambda^{-1}X\supset B(y_0, t)$, then in view of the Baire theorem (see for instance [41]) for each $x_0\in\Lambda y_0$ there are numbers $r_1>0$, $s_1>0$ and $B(y_1, s_1)\subset B(y_0, t)$ such that $\overline{\Lambda^{-1}B(x_0, r_1)}\supset B(y_1, s_1)$. $B(2y_0-y_1, s_1)$ is a subset of $\Lambda^{-1}X$ and the Baire theorem gives $\overline{\Lambda^{-1}B(x_0, r_2)}\supset B(y_2, s_2)$ for some $B(y_2, s_2)\subset B(2y_0-y_1, s_1)$. Using now the convexity of Λ we obtain that $\overline{\Lambda^{-1}B(x_0, r_0)}\supset B(y_0, s_0)$ for some $r_0>0$, $s_0>0$.

Therefore according to Lemma 5.1 the local controllability at y_0 implies

$$\overline{\Lambda_{(x,y)}^{-1}B(x, r)}\supset B(y, \alpha_1 r) \tag{5.4}$$

where $\alpha_1>0$ is a universal constant for all $x\in B(x_0, \varepsilon_1)$ and $y\in\Gamma^{-1}x\cap B(y_0, \eta_1)$ for some $0<\varepsilon_1\leqslant\varepsilon$ and $0<\eta_1\leqslant\eta$.

Take a number $\vartheta<\alpha_1$. Since Γ is nearly convex there are $r_0, \varepsilon_1, \eta_1$ and $\alpha>\vartheta$ so that (4.2) is fulfilled and thus

$$B\big(\Gamma B(x, r), \vartheta_1 r\big)\supset B\big(y, (\alpha\vee\alpha_1) r\big) \tag{5.5}$$

for any $\vartheta<\vartheta_1<\alpha\vee\alpha_1$.

The assumptions of Theorem 3.2 are now satisfied with $q(r)=(\alpha\vee\alpha_1)r$ and $\omega(r)=\dfrac{\vartheta_1}{\alpha\vee\alpha_1}r$, so Γ is δ-u.H.s.c. uniformly at (x_0, y_0) at $p(r)\leqslant(\alpha\vee\alpha_1-\vartheta_1)r$ and the proof of Theorem 5.2 is complete.

Since Γy_0 of Theorem 5.3 is convex, Γ is locally u.H.s.c. uniformly at (x_0, y_0) according to Theorem 2.13 and a rate $p(r)$ may be taken less or equal to $r\cdot(\alpha\vee\alpha_1-\vartheta_1)/2$.

5.5. THEOREM. Let Y and X be Banach space. If the multifunction Γ is almost convex at (x_0, y_0) and if one of its derivatives $\Lambda_{(x,z)}$ verifies (5.4) so that $\vartheta<\alpha_1$ (ϑ occurs in formula (4.2)), then Γ is δ-u.H.s.c. at (x_0, y_0) at linear rate.

Proof is identical as the previous proof started from (5.4).

5.6. *Example.* Theorem 5.3 generalizes Theorem 1 of Robinson [26]. The assumption that y_0 be a internal point of $\Gamma^{-1} X$ (i.e. such that for each $h \in Y$ there is $\lambda_0>0$ so that $y_0+\lambda h \in \Gamma^{-1} X$ for $0\leqslant\lambda\leqslant\lambda_0$) easily implies that y_0 is an interior point, for $\Gamma^{-1} X$ is convex and Y is a Banach space. The conclusion of that theorem follows from Theorem 5.3 in view of Corollary 3.3.

Theorem 2 of [26] is a special case of Corollary 3.3. formulated for convex closed multifunctions.

The Robinson formula (used also in his other works)

$$\operatorname{dist}(x, \Gamma y)\leqslant k \operatorname{dist}(y, \Gamma^{-1} x) \tag{5.6}$$

has the following meaning: $B(\Gamma y, kr)\supset \Gamma B(y, r)$ for each r. Indeed $\operatorname{dist}(y, \Gamma^{-1} x)<r$ is equivalent to $B(y, r)\cap \Gamma^{-1} x\neq\emptyset$ or to $x\in\Gamma B(y, r)$ and $\operatorname{dist}(x, \Gamma y)<kr$ means that $x\in B(\Gamma y, kr)$. Adding the condition of Theorem 2 of [26] that $y\in B(y_0, \eta)$ we get the δ-upper Hausdorff semicontinuity. We obtain the stronger local upper Hausdorff semicontinuity. A corollary of the Robinson theorems is the Banach open mapping theorem.

5.7. COROLLARY (Banach open mapping theorem)

Let be F a linear continuous map of a Banach space X onto a Banach space Y. Then the multifunction $f^{-1}\colon Y\to 2^X$ is Hausdorff continuous.

5.8. COROLLARY (Robinson [27])

Ler F be a continuously Fréchet differentiable (at x_0) mapping from a Banach space X to a Banach space Y, let C be closed and convex subset of X and let K be a closed convex cone of Y. If

$$0\in \operatorname{Int}\{F(x_0)\, F'(x_0)\dot{}(C-x_0)+K\} \tag{5.7}$$

then is δ-u.H.s.c. at y_0 $(\in Fx_0+K)$.

Proof. See Example 4.5 and note that the condition (5.7) means that Λ is locally controllable at (x_0, y_0).

5.9. COROLLARY (Lusternik theorem [49], [17])

Let X, Y be Banach spaces, V a neighbourhood of x_0 and let F be a continuously Fréchet differentiable at x_0 and $F'(x_0)\,X=Y$. Then there are a neighbourhood $V_1\subset V$, a number k and a mapping of U_1 to X: $\xi\to x(\xi)$ such that for all $\xi\in U_1$

$$\begin{gathered} F(\xi+x(\xi))=F(x_0), \\ \|x(\xi)\|\leqslant k\,\|F(\xi)-F(x_0)\|. \end{gathered} \tag{5.8}$$

(Consequent'y the tangent space to $\{x: F(x)=F(x_0)\}$ is equal to the kernel of $F'(x_0)$).

Proof. The thesis may be reformulated: for any $x \in U_1$,

$$\operatorname{dist}\left(x, F^{-1}\left(F(x_0)\right)\right) \leqslant k\,\|F(x)-F(x_0)\|$$

which means the δ-upper Hausdorff semicontinuity of Γ at $(x_0, F(x_0))$ (Example 2.14). In view of Example 4.3 the multifunction F^{-1} is nearly convex at $(x_0, F(x_0))$. The condition $F'(x_0)X=Y$ means that $(F'(x_0))^{-1}$ is controllable, hence locally controllable and we are in the assumptions of Theorem 5.2.

5.10. Corollary. (Ioffe-Tikhomirov extension of the Lusternik theorem [17] p. 45) Let X and Y be Banach spaces, L a linear continuous operator: $X \to Y$, F a mapping of a neighbourhood U of x_0 to Y.

Suppose that $LX=Y$ and denote $C(L)=\|\tilde{L}^{-1}\|$, where $\tilde{L}$ is the quotient mapping of L. Assume that there is a number $\delta>0$ such that $\delta C(L)<1/2$ and

$$\|F(x)-F(v)-L(x-v)\| \leqslant \delta\,\|x-v\| \tag{5.9}$$

for all x, v from U. Then there is a neighbourhood U_1 and $k>0$ and a mapping of U_1 to X: $\xi \to x(\xi)$ such that (5.8) holds.

Proof. We set $\Gamma y=F^{-1}(y)$ and $\Lambda_{(x, F(x))}\, y=\{v: F(x)+L(v-x)=y\}$. By the assumptions

$$\Lambda^{-1}_{(x_0, F(x_0))} B(x_0, r) \supset B\left(F(x_0), \frac{r}{C(L)}\right)$$

and (5.9) yields (4.2) with $\vartheta=\delta$. If $\delta C(L)<1$ then by Theorem 5.5 Γ is δ-u.H.s.c. at (x_0, y_0) and a linear rate may be chosen. This gives (5.8) in view of Example 2.14.

5.11. Theorem. Let X and Y be Banach spaces. Consider a (Fréchet) differentiable mapping F of X to Y such that the derivative $F'(\cdot)$ is locally Lipschitz continuous at x_0 (in the operator norm) and assume that $F'(x_0)X=Y$. Then the multifunction F^{-1} is locally u.H.s.c. at $(x_0, F(x_0))$ (uniformly, linearly and for arbitrarily small balls taken for Q).

Proof. It is enough to show that $M \overset{\text{df}}{=} F^{-1}(F(x_0))$ fulfils (2.4) of Theorem 2.12, because Theorem 5.2 may be used to obtain the δ-upper Hausdorff semicontinuity.

Let c be the Lipschitz constant of $F'(\cdot)$ and let k be the constant of $q(r)=kr$ (in Theorem 5.2).

Take $0<\varepsilon<\frac{1}{16kc}$, such that $F'(\cdot)$ is Lipschitz continuous in $B(x_0, 2\varepsilon)$ and such that F^{-1} is δ-u.H.s.c. at a rate kr at $(x, F(x))$, $x \in B(x_0, 2\varepsilon)$.

Suppose that $x_1 \in \overline{B(x_0, \varepsilon)}$ and also $x_1 \in B(M, r)$, where $r \leqslant \frac{1}{8kc+1}$. Let $x_2 \in \tilde{M} \overset{\text{df}}{=} M \cap \overline{B(x_0, \varepsilon)}$ satisfy $\|x_1-x_2\| \leqslant \operatorname{dist}(x_1, M)+r^2$ and such that there is $0<\zeta \leqslant r$ such that $\overline{B(x_2, \zeta)} \subset B(x_0, \varepsilon)$.

The tengent space $x_2+\ker F'(x_2)$ to M at x_0 is denoted by L and by definition $\tilde{L}=L\cap \overline{B(x_2,\zeta)}$.

Let $x_3\in\tilde{L}$ be such that $\|x_1-x_3\|\leqslant \operatorname{dist}(x_1,\tilde{L})+r^2$. Let $x_4\in M$ fulfil $\|x_1-x_4\|\leqslant r$. By the assumptions there is $x_5\in L$ such that

$$\begin{aligned}\|x_4-x_5\|\leqslant k\,\|F'(x_2)\,x_4-F'(x_2)\,x_5\|=k\,\|F'(x_2)(x_4-x_2)\|=\\ =k\,\|F(x_4)-F(x_2)-F'(x_2)(x_4-x_2)\|\leqslant\\ \leqslant k\sup_{z\in[x_4,x_2]}\|F'(x_2)-F'(z)\|\,\|x_4-x_2\|\leqslant\\ \leqslant k\cdot c\,\|x_4-x_2\|^2.\end{aligned}\tag{5.10}$$

We have used the mean value theorem and the assumption of Lipschitz continuity of $F'(\cdot)$.

Let x_6 denote the point of intersection of $[x_1,x_3]$ with $\{x\colon \|x-x_2\|-\zeta\}$ and let $x_7\in L$ fulfil

$$\frac{\|x_1-x_5\|}{\|x_1-x_3\|}=\frac{\|x_6-x_7\|}{\|x_6-x_3\|}.\tag{5.11}$$

Observe that $\|x_6-x_3\|\leqslant\operatorname{dist}(x_6,\tilde{L})+r^2$. Therefore, in view of Theorem 2.13 applied to the convex set L and to the ball $\overline{B(x_2,\zeta)}$, the ratio (5.11) is greater or equal to 2.

In virtue of the Lusternik theorem

$$\begin{aligned}\operatorname{dist}(x_3,M)\leqslant k\,\|F(x_3)-F(x_2)\|=k\,\|F(x_3)-F(x_2)-F'(x_2)(x_3-x_2)\|\leqslant\\ \leqslant kc\cdot\|x_3-x_2\|^2.\end{aligned}\tag{5.12}$$

If each $x_8\in M$ realizing these estimates lies outside $\overline{B(x_2,\zeta)}$, we apply estimates of the type (5.12) to the point $x_3+kc\,\|x_3-x_2\|\,(x_2-x_3)$ to be sure that there is $x_8\in\tilde{M}$ such that

$$\|x_8-x_3\|\leqslant 2kc\,\|x_3-x_2\|^2\leqslant 8kc\,\zeta^2\leqslant 8kcr^2.\tag{5.13}$$

For brevity we introduce the notation $s=\|x_1-x_2\|$. In accordance to (5.10)—(5.13) we have

$$\begin{aligned}s\leqslant\|x_1-x_3\|+\|x_3-x_8\|+r^2\leqslant 2\,\|x_1-x_5\|+(8kc+1)\,r^2\leqslant\\ \leqslant 2\,(\|x_1-x_4\|+\|x_4-x_5\|)+(8kc+1)\,r^2\leqslant\\ \leqslant 2\,(r+kc\,\|x_4-x_2\|^2)+(8kc+1)\,r^2.\end{aligned}\tag{5.14}$$

But $\|x_4-x_2\|\leqslant\|x_4-x_1\|+\|x_2-x_1\|\leqslant r+s$. Hence (5.14) becomes

$$s\leqslant 2r+2kc\,(r+s)^2+(8kc+1)\,r^2\leqslant 2r+4kcs^2+(8kc+1)\,r^2\leqslant 3r+1/2\,s,\tag{5.15}$$

$s\leqslant 6r$.

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References

1. C. Berge, Espaces topologiques, Dunod, Paris 1966.
2. F. H. Clarke, A new approach to Lagrange multipliers. *Math. Opns. Res.* **1** (1976) 165–174.
3. L. Cesari, Seminormality and upper semicontinuity in optimal control. *J. Opt. Theory Appl.* **6** (1970), 114–137.
4. G. B. Dantzig, J. Folkman, N. Shaplro; On the continuity of the minimum set of a continuous function. *J. Math: Anal. Appl.* **17** (1967) 519–548.
5. S. Dolecki, A classification of controllability concepts for infinite dimensional linear systems. *Control a. Cyber.* **5** (1976) 33–44.
6. S. Dolecki, Bounded controlling sequences, lower stability and certain penalty procedures. *Appl. Math. Opt.*, to appear.
7. S. Dolecki, Constraints stability and moduli of semicontinuity. 2[nd] IFAC Symp. on Distributed Parameter Systems, Warwick, June 1977 (preprint).
8. S. Dolecki, Semi-continuité des contraintes aux problèmes extremaux, to appear.
9. S. Dolecki, Remarks on semicontinuity, Bull. Acad. Pol. Sci., to appear.
10. S. Dolecki, S. Kurcyusz, On fi-convexity in extremal problems, to appear in *SIAM J. Contr. Optimization.*
11. S. Dolecki, S. Kurcyusz, Convexité généralisé et optimisation. *C.R.Ac.Sc.* Paris **283** (1976) A-91-94.
12. S. Dolecki, S. Rolewicz, A characterization of semicontinuity — preserving multifunctions, to appear in *J. Math. Anal. Appl.*
13. S. Dolecki, D. L. Russell, A general theory of observation and control. *SIAM J. Contr. Optimization* **15** (1977).
14. J. P. Evans, F. J. Gould, Stability in nonlinear programming. *Ppns. Res.* **11** (1963) 399–417.
15. K. Fan, On the Krein-Milman theorem, in Convexity, Proc. Symp. Pure Math. VII. A.M.S. (1963) 211–220.
16. H. J. Greenberg, W. P. Pierskalla, Extensions of the Evans-Gould stability theorems for mathematical problems. *Opns. Res.* **20** (1972) 143–153.
17. A. D. Ooffe, V. M. Tikhomirov, Theory of extremal problems (in Russian). Izd. Nauka, Moskva 1974.
18. K. Kuratowski, Topology (English transl. from French). I, II. Academic Press and Polish Scientific Publishers, 1966.
19. S. Kurcyusz, Some remarks on generalized Lagrangians. Proc. 7[th] IFIP Conf. Nice 1975.
20. P. Levine, J.-C. Pomerol, C-closed mappings and Kuhn-Tucker vector in convex programming. CORE 7620. Université Catholique de Louvain.
21. P. O. Lindberg, A generalization of Fenchel conjugation giving generalized Lagrangians and symmetric nonconvex duality. IX Symp. Math. Programming, Budapest 1976.
22. P. Michel, Problème des inégalités. Application à la programmation et au contrôle optimal. *Bull. Soc. Math. France* **101** (1973) 413–439.
23. C. Olech, Existence theory in optimal control problems, in Internat. Conf., Diff. Eqns., Academic Press, New York, San Francisco, London 1975.
24. V. Pták, A quantitative refinement of the closed graph theorem. *Czech. Math. J.* **24** (99) (1974) 503–506.
25. V. Pták, Nondiscrete mathematical induction and iterative existence proofs. *Lin. Algebra* 1? (1976) 223–239.
26. S. M. Robinson, Regularity and stability for convex multivalued functions. *Math* **1** (1976) 136–143.
27. S. M. Robinson, Stability theory for systems of inequalities. II. Differen

29. S. ROLEWICZ, Funktionalanalysis und Steuerungstheorie. Berlin, Springer Verlag 1976.
30. S. ROLEWICZ, On universal time for controllability of time depending linear control systems. *Studia Math.* **49** (1976) 27–32.
31. H. H. SCHAEFER, Topological vector spaces. Springer Verlag, New York—Heidelberg—Berlin 1971.
32. I. SINGER, The theory of best approximation and functional analysis. SIAM, Philadelphia 1974.
33. I. A. VAINSTEIN, On closed mappings of metric spaces. *DAN, S.U.* 57 (1947) 419–321 (in Russian).
34. A. P. WIERZBICKI, S. KURCYUSZ, Projection on a cone, penalty functionals and duality theory for problems with inequality constraints in Hilbert space. *SIAM J. Contr. Opt.* **15** (1977) 25–56.
35. J. ZABCZYK, Remarks on the algebraic Riccati equation in Hilbert space. *Appl. Math. Opt.* **2** (1975/76) 251–258.
36. A. V. BALAKRISHNAN, A computational approach to the maximum principle. *J. Comp. Syst. Sc.* **5** (1971) 163–191.
37. E. J. BALDER, An extension of duality-stability relations to nonconvex optimization problems. *SIAM J. Contr. Opt.* **15** (1977) 329–343.
38. D. P. BERTSEKAS, Combined primal-dual and penalty method for constrained minimization. *SIAM J. Contr.* **13** (1975) 521–544.
39. S. DOLECKI, On construction of constraints multifunctions, to appear.
40. H. HALKIN, Mathematical programming without differentiability in Calculus of Variations and Control Theory. D. L. Russell ed. Academic Press. New York 1976 p. 279–288.
41. J. HORVÁTH, Topological vector spaces and distributions. Addison Wesley Publ. 1966.
42. S. M. HOWE, New conditions for exactness of a simple penalty function. *SIAM J. Contr.* **11** (1973) 378–381.
43. A. D. IOFFE, Regular points of Lipschitz mappings, to appear.
44. A. D. IOFFE, Necessary and sufficient conditions for a local minimum II, Conditions of Levitin–Miljutin–Osmolovskii type, to appear.
45. T. KATO, Perturbation theory for linear operators, Springer Verlag, Berlin 1966.
46. M. A. KRASNOSELSKI, Continuity of the operator $Fu(x)=f(x, u(x))$. *DAN S.U.* 77 (1951) (in Russian).
47. E. S. LEVITIN, On conditions for the local minimum and extremal constrained problems. *DAN S.U.* **221** (1975) 1031–1034 (in Russian).
48. E. S. LEVITIN, A. A. MILJUTIN, N. P. OSMOLOVSKII, On conditions of local extremum in constrained problems, in Mathematical Economics and Functional Analysis. B. S. Mitjagin ed. Nauk, Moscow 1974 (in Russian).
49. L. A. LUSTERNIK, On conditional extrema of functionals. Matem. sb. 41 (1934) 390–401 (in Russian).
50. O. L. MANGASARIAN, Unconstrained Lagrangians in nonlinear programming. *SIAM J. Contr.* **13** (1975) 772–791.
51. T. PIETRZYKOWSKI, The potential method for conditional maxima in locally compact metric spaces. *Numer. Math.* **14** (1970) 325–329.
52. B. T. POLJAK, Semicontinuity of integral functionals and existence theorems. Mat. sb. 78 (1969) 65–84.
53. H. RADSTRÖM, An embedding theorem for spaces of convex sets. *Proc. Amer. Marth. Soc.* **3** (1952) 165–169.
54. R. T. ROCKAFELLAR, Dual problems of Lagrange for arcs of bounded variation, in Calculus of Variations and Control Theory, D. L. Russell ed. Academic Press, New York 1976 p. 155–192.
55. A. B. SCHWARZKOPF, Controllability and tenability of nonlinear systems with state equality constraints. *SIAM J. Contr.* **13** (1975) 695–705.

56. J. Warga, Derivate containers, inverse functions and controllability, in Calculus of Variations and Control Theory. D. L. Russell ed. Academic Press, New York 1976, p. 13–46.
57. A. P. Wierzbicki, A penalty function shifting method in constrained static optimization and its convergence properties. *Arch. Autom. Telemech.* **16** (1971) 395–416.
58. J. Zabczyk, Comments on duality theorems, to appear.
59. W. I. Zangwill, Nonlinear programming. Prentice Hall, Englewood Cliffs, N. J. 1969.
60. S. Dolecki, S. Rolewicz, Metric characterizations of the upper semicontinuity, to appear.
61. S. Dolecki, S. Rolewicz, Exact penalty for local minima, to appear.