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A minimizing problem of a quadratic functional for the systems described by right invertible operators in Hilbert space

by

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In this paper we shall consider a control problem inwhich the cost is a quadratic functional and the system is linear and described by a right invertible operator in Hilbert space H.

A necessary and sufficient condition for the optimal control is proved in the case when there is no constraint imposed on the control and when the set of all admissible controls is closed and convex.

1. Introduction

1. Suppose that H is a real Hilbert space and D is a right invertible operator from H into H with a right inverse R of the operator D, i.e. R satisfies relation

$$DR=I$$

where I is the identify operator (see [1]).

DEFINITION 1.1 (cf [1]).

$$Z_D:=\{x \in H: Dx=0\}$$

i.e. $Z_D=\ker D$ is the set of the constants.

In the space H we consider the linear system

$$Dx=Ax,\ A \in L_0(H) \tag{1.1}$$

where $L_0(H)$ is the set of all linear operator with domain H, and an initial condition

$$Fx=x_0 \tag{1.2}$$

where $x_0 \in Z_D$ and F is an initial operator for the operator D corresponding to the operator R (see [1]).

Note that, such a system always has a unique solution being of the form:

$$x_a=(I-RA)^{-1}x_0 \quad \text{for each } x_0\in Z_D \tag{1.3}$$

provided that the operator $I-RA$ is invertible (cf [1]).

Moreover we assume that

a — the operator $A\in L_0(H)$ is closed and one-to-one.

b — the operator R is continuous and one-to-one.

Since the Hilbert space is a complete linear metric space it follows that the operator RA is continuous and one-to-one. Since the operator $I-RA$ is closed and one-to-one we conclude that the operator $(I-RA)^{-1}$ is closed and one-to-one and then $(I-RA)^{-1}$ is continuous because $D=_{I-RA}=H$. (cf [2]).

2. Now suppose that U_{ad} is a closed convex subset of a Hilbert space $U\subset H$ and $B\in L(U,H)$ is bounded and satisfies the condition

$$BU_{ad}\supset(D-A)H. \tag{1.4}$$

Remark 1.1. Since a superposition of right invertible operators and some polynomials in a right invertible operator with operator coefficients (in partcular, with scalar coefficients) are right invertible, without loss of generality we can consider a system in a simple form (1.1).

Now we consider a control system

$$Dx=Ax+Bu, \tag{1.5}$$

$$Fx=x_0, \quad x_0\in Z_D,$$

where $u\in U_{ad}$ is the set of all admissible controls.

Remark 1.2. If $U_{ad}=U$, we have the control system without constraint imposed on the control u.

For each $u\in U_{ad}$ the system (1.5) has a unique solution x_u defined by the formula (see [1])

$$x_u=(I-RA)^{-1}(x_0+RBu). \tag{1.6}$$

Proposition 1.1. The mapping $u\to x_u$ from U into H is continuous.

It is the corollary of the assumption a), b) and the condition imposed on the operator B.

3. Let a quadratic functional $J(x_0,u)$ be given:

$$J(x_0,u)=\langle x_u, Qx_u\rangle_H+\langle u, Pu\rangle_U, \tag{1.7}$$

where $\langle.\rangle_H$ and $\langle.\rangle_U$ denote the scalar products in the spaces H and U respectively.

We assume that the operators P and Q have the following properties:

(i) Q is bounded, selfadjoint positive semidefinite operator from H into H, i.e. $Q^*=Q$ and there exists a positive number $a>0$ such that

$$0\leq\langle x, Qx\rangle_H\leq a\,||x||^2 \quad \text{for all} \quad x\in H,$$

where $||x||^2=\langle x,x\rangle_H$.

(ii) P is a bounded selfadjoint positive definite operator from U into U, i.e. $P^*=P$ and there exists positive number b_1, b_2 such that

$$b_1 \|u\|^2 \leq \langle u, Pu\rangle_U \leq b_2 \|u\|^2 \quad \text{for all} \quad u \in U,$$

where $\|u\|^2 = \langle u, u\rangle_U$.

REMARK 1.3. From the condition (ii) imposed on P, follows that P is invertible.

Now we consider the problem of finding

$$\inf_{u \in U_{ad}} J(x_0, u)$$

where $J(x_0, u)$ is defined by (1.7), and x_u is a solution of the system (1.5).

DEFINITION 1.2. A control $u_* \in U_{ad}$ satisfying the condition

$$J(x_0, u_*) = \inf_{u \in U_{ad}} J(x_0, u)$$

is called the optimal control for the functional $J(x_0, u)$.

2. The necessary and sufficient condition for the optimal control in the case $U_{ad}=U$

To begin with, we define for all $u, v \in U_{ad}$ the form

$$\pi(u, v) = \langle Q(x_u - x_d), x_v - x_d\rangle_H + \langle Pu, v\rangle_U, \tag{2.1}$$

where x_d, x_u are defined by (1.3) and (1.6) respectively; P and Q satisfy the condition (i) and (ii).

Proposition 2.1. The mapping $(u, v) \to \pi(u, v)$ defined by (2.1) is a symmetric, bilinear and continuous form.

Moreover there is a positive real c such that

$$\pi(u, u) \geq c \|u\|^2$$

for all $u \in U_{ad}$.

Proof. By our assumption we have

$$\pi(u, v) = \pi(v, u)$$

because the operators Q and P are selfadjoint.

By the proposition 1.1 and the properties of P and Q, the function $\pi(u, v)$ is continuous.

Moreover, from (2.1) we have

$$\pi(u, u) = \langle Q(x_u - x_d), x_u - x_d\rangle_H + \langle Pu, u\rangle.$$

Since

$$\langle Q(x_u - x_d), x_u - x_d\rangle_H \geq 0 \quad \text{for all} \quad u \in U_{ad},$$

we have

$$\pi(u, u) \geq \langle Pu, u \rangle \geq c\, ||u||^2$$

for all $u \in U_{ad}$, and c is some positive real number.

Proposition 2.2 (see [3]). There exists a unique optimal control $u_* \in U_{ad}$ for the functional (1.7), i.e. there exists a unique $u_* \in U_{ad}$ such that

$$J(x_0, u_*) = \inf_{u \in U_{ad}} J(x_0, u).$$

Proof. Observe that by the definition of the $\pi(u, v)$ and $J(x_0, u)$, it is not difficult to verify that

$$J(x_0, u) = \pi(u, u) - 2L(u) - \langle x_d, Qx_d \rangle_H \tag{2.2}$$

where

$$L(u) = -\langle x_u - x_d, Qx_d \rangle_H.$$

The uniqueness of u_* follows from the fact that the functional $J(x_0, u)$ is strictly convex. Indeed if u_1 and u_2 minimize $J(x_0, u)$, then we have

$$\frac{u_1}{2} + \frac{u_2}{2} \in U_{ad} \text{ and } J\left(x_0, \frac{1}{2} u_1 + \frac{1}{2} u_2\right) < \inf_{u \in U_{ad}} J(x_0, u),$$

unless $u_1 = u_2$.

Now suppose that $\{u_n\}$, $u_n \in U_{ad}$ is a sequence such that

$$\lim_{i \to \infty} J(x_0, u_z) = \inf_{u \in U_{ad}} J(x_0, u). \tag{2.3}$$

By (2.2) we have

$$J(x_0, u) \geq c\, ||u||^2 - c_1\, ||u||, \; c_1 \text{ is a constant.}$$

This equality and (2.3) imply that

$$||u_n|| \leq M, \; M = \text{const.}$$

and then there exists a subsequence $\{u_k\}$ such that

$$u_k \to w \text{ weakly in } U.$$

Since U_{ad} is closed convex, U_{ad} is weakly closed and therefore we have

$$w \in U_{ad}. \tag{2.4}$$

The function $u \to \pi(u, u)$ is lower semicontinuous in the weak topology of U and the function $u \to L(u)$ is continuous in the weak topology. Thus the functional $J(x_0, u)$ is weakly lower semicontinuous and hence

$$\lim \inf J(x_0, u_k) \geqslant J(x_0, w).$$

By (2.3) and (2.4) it follows that

$$w \in U_{ad} \text{ and } J(x_0, w) \leq \inf_{u \in U_{ad}} J(x_0, u).$$

Then we have

$$J(x_0, w)= \inf_{u \in U_{ad}} J(x_0, u).$$

THEOREM 2.1. A control $u_* \in U_{ad}=U$ is optimal for the functional $J(x_0, u)= =\langle x_n, Qx_u\rangle_H+\langle u, Pu\rangle_U$ where x_u is the solution of the system

$$\begin{cases} Dx=Ax+Bu \\ Fx=x_0, \; x_0 \in Z_D \end{cases} \tag{2.5}$$

for each $u \in U_{ad}=U$, if and only if

$$u_*=-P^{-1} A_{U^{-1}} B^* R^* (I-RA)^{*-1} Q(x_*)$$

where x_* is a solution of the system

$$\begin{cases} Dx=Ax+Bu_* \\ Fx=x_0. \end{cases} \tag{2.6}$$

and A_U is a canonical isomorphism of U onto U^*.

Proof. It is well-known that a control $u_* \in U_{ad}$ is optimal if and only if

$$\pi(u_*, u-u_*) \geq L(u-u_*) \quad \text{for every} \quad u \in U_{ad}$$

and in the case $U_{ad}=U$ the inequality becomes (equality)

$$\pi(u_*, u-u_*)=L(u-u_*) \quad \text{for every} \quad u \in U$$

(see [3]).

Therefore we have

$$\pi(u_*, u-u_*)-L(u-u_*)=0 \quad \text{for} \quad \forall u \in U$$

where

$$L(u)=-\langle x_u-x_d, Qx_d\rangle,$$

or

$$\langle Q(x_{u_*}-x_d), x_{u-u_*}-x_d\rangle_H+\langle Pu_*, u-u_*\rangle_U+\langle x_{u-n_*}-x_d, Qx_d\rangle_H=0.$$

Then we obtain

$$\langle Q(x_{u_*}-x_d)+Qx_d, x_{u-u_*}-x_d\rangle_H+\langle Pu_*, u-u_*\rangle_U=$$
$$=\langle Qx_{u_*}, x_{u-u_*}-x_d\rangle_H+\langle Pu_*, u-u_*\rangle_U=0. \tag{*}$$

Since

$$x_{u-u_*}-x_d=(I-RA)^{-1}\{x_0+RB(u-u_*)\}-(I-RA)^{-1} x_0=$$
$$=(I-RA)^{-1} RB(u-u_*)$$

$$x_{u_*}=(I-RA)^{-1}(RB \cdot u_*+x_0)$$

we may rewrite the formula $(*)$ as follows:

$$\langle Q(I-RA)^{-1}(RBu_*+x_0), (I-RA)^{-1}RB(u-u_*)\rangle_H+\langle Pu_*, u-u_*\rangle_U=$$

$$=\langle B^* R^*(I-RA)^{*-1} Q(I-RA)^{-1}(RBu_*+x_0), u-u_*\rangle_U+$$

$$+\langle Pu_*, u-u_*\rangle_U=0 \qquad (**)$$

for all $u\in U$.

Observe that $(I-RA)^{-1}(RBu_*+2x_0)=x_*$ is the solution of the equation (2.6). If we define A_U as a canonical isomorphism of U onto U^*

$$A_U: U\to U^*$$

then from $(**)$ we obtain

$$\langle A_U^{-1} B^* R^*(I-RA)^{*-1} Q(x_*)+Pu_*, u-u_*\rangle_U=0$$

for all $u\in U$.

Hence

$$Pu_*+A_U^{-1} B^* R^*(I-RA)^{*-1} Q(x_*)=0$$

i.e.

$$u_*=-P^{-1} A_U^{-1} B^* R^*(I-RA)^{*-1} Q(x_*).$$

3. The necessary and sufficient condition for the optimal control in the case $U_{ad}\subset U$.

We again consider the system described by a right invertible operator in a Hilbert space H:

$$Dx=Ax+Bu \qquad (3.1)$$

$$Fx=x_0, \quad x_0\in Z_D,$$

for each u in the set of admissible controls $U_{ad}\subset U$.

As in §2 we consider the problem of minimizing of the functional:

$$J(x_0, u)=\langle Qx_u, x_u\rangle_H+\langle Pu, u\rangle_U \qquad (3.2)$$

subject to the system (3.1).

We assume that the set U_{ad} is closed convex in U and all the assumptions imposed on the system (3.1) and on the functional $J(x_0, \cdot)$ are the same as before.

Now we define the adjoint system:

$$(I-RA)^*\psi_u=Qx_u \quad \text{for each} \quad u\in U_{ad}. \qquad (3.3)$$

Since for each $u\in U_{ad}$, x_u is uniquely determined, then the solution ψ_u of the system (3.3) is dependent on u.

THEOREM 3.1. A control $u_*\in U_{ad}$ is optimal for the functional

$$J(x_0, u)=\langle Qx_u, x_u\rangle_H+\langle Pu, u\rangle_U,$$

where x_u obeys the system

$$Dx=Ax+Bu, \quad u\in U_{ad}$$
$$Fx=x_0$$

if and only if

$$\langle \Lambda_U^{-1} B^* \psi_{u_*}+Pu_*, u_*\rangle_U \leqslant \langle \Lambda_U^{-1} B^* \psi_{u_*}+Pu_*, u\rangle_U$$

for all $u\in U_{ad}$, where ψ_{n_*} is a solution of the adjoint system

$$(I-RA)^* \psi_u=Qx_u \quad \text{for} \quad u=u_*$$

and Λ_U is a canonical isomorphism of U onto U^*.

Proof. By the theorem (1.3) of [3], the necessary and sufficient condition for the control u_* minimizing the functional $J(x_0, \cdot)$ is

$$J'(x_0, u_*)(u-u_*)\geqslant 0 \quad \text{for all} \quad u\in U_{ad}.$$

Since

$$J'(x_0, u_*)(u-u_*)=2\langle Qx_{u_*}, x'_{u_*}(u-u_*)\rangle_H+2\langle Pu_*, u-u_*\rangle_U,$$

(where F' denotes the Fréchet derivative of a mapping F) it follows that u_* is optimal if and only if:

$$\langle Qx_{u_*}, x'_{n_*}(u-u_*)\rangle_H+\langle Pu_*, u-u_*\rangle_U\geqslant 0, \quad \text{for all} \quad u\in U_{ad}. \qquad (*)$$

But $x'_{u_*}(u-u_*)=(I-RA)^{-1}RB(u-u_*)$. Hence we obtain from $(*)$:

$$\langle Qx_{u_*}, (I-RA)^{-1}RB(u-u_*)\rangle_H+\langle Pu_*, u-u_*\rangle_U=$$
$$=\langle Qx_{u_*}, x_u-x_{*u}\rangle_H+\langle Pu_*, u-u_*\rangle_U$$
$$=\langle (I-RA)^*\psi_{u_*}, x_u-x_{u_*}\rangle_H+\langle Pu_*, u-u_*\rangle_U \quad \text{(by equation (3.3))}$$
$$=\langle \psi_{u_*}, (I-RA)(x_u-x_{*n})\rangle_H+\langle Pu_*, u-u_*\rangle_U$$
$$=\langle \psi_{u_*}, RB(u-u_*)\rangle_H+\langle Pu_*, u-u_*\rangle_U\geqslant 0.$$

Let Λ_U be a canonical isomorphism of U onto U^*

$$\Lambda_U: U\to U^*.$$

We have

$$\langle \psi_{u_*}, RB(u-u_*)\rangle_H+\langle Pu, u-u_*\rangle=$$
$$=\langle \Lambda_U^{-1} B^* R^* \psi_{U_*}, u-u_*\rangle_U+\langle Pu_*, u-u_*\rangle_U=$$
$$=\langle \Lambda_U^{-1} B^* R^* \psi_{u_*}+Pu_*), u-u_*\rangle_U\geqslant 0 \quad \text{for all} \quad u\in U_{ad}.$$

Finally we obtain the inequality

$$\langle \Lambda_U^{-1} B^* R^* \psi_{u_*}+Pu_*, u_*\rangle_U\leqslant\langle \Lambda_U^{-1} B^* R^* \psi_{n_*}+Pu_*, u\rangle_U$$

for all $u=U_{ad}$.

Example. Let A be an one-to-one linear bounded operator a real Hilbert space H, The dynamical equation considered here is of the form

$$x(t)=Ax(t)+Bu(t),$$
$$x(0)=x_0. \qquad \text{(E)}$$

Moreover we assume that

$$\|A\|<1/T$$

and B is a linear bounded transformation mapping a Hilbert space U into H, $u(\cdot)$ is an element of $W_u=L_2((0,T);U)$.

The equation (E) has a unique solution continuous on $(0,T)$ for each $u\in W_u$ (see [2]).

Let the functional

$$J(u)=\int_0^T [Qx(t),x(t)]_H\,dt+\int_0^T [u(t),u(t)]_U\,dt$$

where Q is a linear bounded nonnegative definite operator mapping H into itself be given.

Our problem is to define the control $u_*(\cdot)\in W_u$ minimizing the functional $J(u)$.

To begin with, we express the equation (E) and the functional $J(u)$ in language of the right invertible operator.

If we define $D=\frac{d}{dt}$, then the operator R is of the form (see [1])

$$(Rx)(t)=\int_0^t x(\tau)\,d\tau$$

and R is a continuous and one-to-one operator.

The equation (E) can be rewriten as follows

$$Dx=Ax+Bu$$

$$F(x)=x(0)=x_0.$$

Denote $W=L_2((0,T);H)$ and $\Phi_1=(I-RA)^{-1}RB$ as a transformation mapping W_u into W

$$\Phi_0\,x_0=(I-RA)^{-1}\,x_0$$

$$x_0\in Z_D.$$

Then the solution of (E) for every $u\in W_u$ is given by the formula

$$x_u=\Phi_1\,u+\Phi_0\,x_0$$

and

$$J(u)=[Qx_u,x_u]_W+[u,u]_{W_u}.$$

Therefore the optimal control u_* can be determined by the formula

$$u_*(t)=(-\Lambda_{W_u}^{-1}B^*R^*(I-RA)^{-1*}Qx_*)(t)$$

where $x_*(t)=(\Phi_0\,x_0+\Phi_1\,u)(t)$.

Note that if

$$Rx(t)=\int_0^t x(\tau)\,d\tau \text{ for } x\in L_2((0,T);H)$$

then

$$(R^* y)(t) = \int_t^T y(s)\, ds \text{ for } y \in L_2^*((0, T); H).$$

Moreover by assumption

$$||A|| < 1/T$$

we have

$$||RA|| < 1$$

and then the operator $\Phi_0 = (I - RA)^{-1}$ exists, and can be expressed by a power series

$$\Phi_0 = (I - RA)^{-1} = \sum_{n=0}^{\infty} (RA)^n.$$

We have

$$(RAx_0)(t) = \int_0^t Ax_0\, dt = Ax_0\, t$$

$$[(RA)^2 x_0](t) = [(RA)\, Ax_0\, s](t) = \int_0^t A^2 s\, ds = \frac{A^2 t^2}{2!} x_0.$$

By introduction we have

$$[(RA)^n x_0](t) = \frac{A^n t^n}{n!} x_0$$

and then

$$[\Phi_0 x_0](t) = [(I - RA)^{-1} x_0](t) = \sum_{n=0}^{\infty} [(RA)^n x_0](t) =$$

$$= \sum_{n=0}^{\infty} \frac{A^n t^n}{n!} x_0 = \exp(At)\, x_0;$$

hence

$$Ax_0 = \lim_{t \to 0} \frac{\Phi_0(t) - I}{t} x_0.$$

An analogous result can be obtained, but in another way (see [4]).

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Problem minimalizacji funkcjonału kwadratowego dla układów opisanych przez operatory prawostronnie odwracalne w przestrzeni Hilberta

Rozpatrzono problem sterowania, w którym koszt jest funkcjonałem kwadratowym a układ jest liniowy i opisany przez prawostronnie odwracalny operator w przestrzeni Hilberta H.

Warunek konieczny i wystarczający istnienia sterowania optymalnego udowodniono dla przypadku, gdy nie ma ograniczenia na przestrzeni sterowań U i gdy ubiów sterowań dopuszczalnych jest domknięty i wypukły.

Задача минимизации квадратичного функционала для систем описываемых посредством правосторонне обратимых опереторов в гильбертовом пространстве

В работе рассматривается задача управления в которой стоимость является квадратичным функционалом, а система линейна и описывается посредством правосторонне обратимых операторов в гильбертовом пространстве H.

Необходимое и достаточное условие существования оптимального управления доказано для случая, когда отсутствуют ограничения на пространство управлений U и когда множество допустимых управлений является замкнутым и выпуклым.