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On a generalization of Morrison's method

by

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The paper presents a generalization of the Morrison's method, i.e. one-parameter shifted penalty function technique for equality constrained optimization problems, which can be applied to minimize a functional on a topological space under constraints of various types. A modification of the penalty function, which depends in this case on three parameters, has been performed and the influence of parameters p and r on the rate of convergence of the method has been investigated.

1. Introduction

The penalty function techniques is one of the most general tools for the numerical solution of constrained minimization problems. A method, which avoids most of the difficulties associated with unconstrained methods described so far, was introduced by Rockafellar in 1970 [7] under the name of augmented Lagrangean method. The original idea, presented independently by Hestenes and Powell in 1969 ([3], [6]) for equality constrained problems, leads to the development of several algorithms for inequality-constrained minimization problems, for example: the shifted penalty function — Wierzbicki in 1971 [8]. A comprehensive survey of works dealing with this problem has been given by Mangasarian [4]. He presents optimality conditions, without inequalities, in the form of $n+m$ equations, for following problem:

Problem 1.

$$\text{minimize } f(x) \text{ subject to } g(x) \leqslant 0$$

$$\text{where } f\colon R^n \to R \text{ and } g\colon R^n \to R^m.$$

Then, he offers an interpretation of the m-parameters augmented Lagrangean method as a means of finding a solution of the system considered.

For the equality constrained problem, Morrison gives in 1968 [5] a one-parameter exact penalty function, which is always non-negative, reaching the zero value at the solution of the problem. This property of the function is in practice a convenient termination criterion.

It appears that the Morrison's method may be applied to Problem 1. Let

$$Q=[x\in X\colon g(x)\leqslant 0].$$

We assume that a number M_0 is known, such that

$$f(x)\geqslant M_0>-\infty \text{ for } x\in Q.$$

Define

$$I_k(x)=(f(x)-M_k)^2+G(x) \text{ for } x\in R^n$$

and

$$M_{k+1}=M_k+\{I_k(x_k)\}^{1/2},\ k=0,1,2,\dots,$$

where

$$I_k(x_k)\leqslant (f(x)-M_k)^2+G(x) \text{ for } x\in R^n$$

$$G(x)=\sum_{i=1}^{m}[g_i(x)_+]^2,\ g_i(x)_+=\max\{0,g_i(x)\}.$$

Thus, it follows that for Problem 1, particularly if m is large, the Morrison's method have the advantage of being a one-parameter exact penalty method.

In the paper we will discuss a generalization of this method to the case of the minimization of a functional on a topological space under constraints of various types. Besides, we will modify the penalty function so that it depends now on three parameters, only one of them being a variable. In this case the penalty function for Problem 1 has the following form:

$$I_k(x)=p\,(f(x)-M_k)_+^r+G(x),\ k=0,1,2,\dots$$

where

$$p>0,\ r>0,\quad p,r \text{ — fixed},$$

$$G(x)\geqslant 0 \text{ for } x\in R^n,$$

$$G(x)=0\Leftrightarrow x\in Q.$$

For example

$$G(x)=\sum_{i=1}^{m}[g_i(x)_+]^r.$$

Observe that $I_k(x)$ retains some properties of $f(x)$ and $g(x)$ (convexity, lower semicontinuity), which the Morrison's penalty function lacks. In this paper we will give very general sufficient conditions for the numerical convergence of the method and explain the influence of p and r on the rate of convergence of the method. The material presented is based on the doctoral dissertation ([2] Pt. I), written under the supervision of Dr. Ś. Ząbek.

2. One-parameter shifted penalty function

Let X denote a topological space

$$F\colon X\to R \text{ — a real functional},$$

$$Q\subset X,\ Q \text{ — a non-empty set.}$$

Problem 2. Calculate $D=\inf_{x\in Q} F(x)$ assuming that

$$D\geqslant M>-\infty,\ M\text{ — a known number.} \tag{1}$$

Let there be two numbers $p>0$ and $r>0$. We take a functional $G: X\to R$, such that

$$G(x)\geqslant 0 \text{ for } x\in X, \tag{2}$$

$$G(x)=0\Leftrightarrow x\in Q. \tag{3}$$

Define a functional I as

$$I(x)=p\,(F(x)-M)_+^r+G(x),\ x\in X \tag{4}$$

where

$$(F(x)-M)_+=\max\{0, F(x)-M\}$$

(we can also take $(F(x)-M)_+=|F(x)-M|$) and establish $0<\varepsilon<1$.

Problem 3. Find $x_0\in X$ such that

$$\inf_{x\in X} I(x)\leqslant I(x_0)\leqslant(1+\varepsilon)\inf_{x\in X} I(x_0). \tag{5}$$

We assume that the forms of functional G and $r>0$ are such that we known an efficient method of solving Problem 3. From the method presented below it follows that Problem 2 is to be replaced by an equivalent sequence of problems of the Problem 3 type. The introduction of ε assures the existence of $x\in X$ satisfying (5), and also assures the convergence of the method even if the minimization of $I(x)$ is not too exact. Moreover, one can apply this method to find $\bar{x}\in Q$ if it exists such that

$$F(\bar{x})=\inf_{x\in Q} F(x)=D.$$

This results from the following fact. If we know the number D and there exists $\bar{x}\in Q$, such that $F(\bar{x})=D$, then

$$I_D(\bar{x})=0 \text{ and } I_D(x)\geqslant 0 \text{ for all } x\in X,$$

where

$$I_D(x)=p\,(F(x)-D)_+^r+G(x),\ p>0,\ r>0,$$

and, conversely, if there exists $\tilde{x}\in X$ such that

$$I_D(\tilde{x})=0 \text{ then } F(\tilde{x})=D \text{ and } \tilde{x}\in Q.$$

To solve Problem 2 we define the following iteration:

Let $M_0=0$. We find a sequence $x_k\in X$, $k=0, 1, 2, \ldots$, such that

$$d_k\leqslant I_k(x_k)\leqslant(1+\varepsilon)\,d_k, \tag{6}$$

and a sequence M_k

$$M_{k+1}=M_k+\{(1-\varepsilon)\,I_k(x_k)/p\}^{1/r},\ k=0, 1, 2, \ldots$$

where

$$I_k(x) = p\,(F(x) - M_k)^r_+ + G(x),\ x \in X \tag{8}$$

$$d_k = \inf_{x \in X} I_k(x). \tag{9}$$

Now we shall prove the following

THEOREM 1. If $M_0 \leqslant D$, the sequence x_k satisfies (6) and M_k is defined by (7), then

$$M_{k+1} \leqslant D,\ k=0, 1, 2, \ldots, \tag{10}$$

$$\lim_{k\to\infty} F(x_k) = \lim_{k\to\infty} M_k \leqslant D, \tag{11}$$

$$\lim_{k\to\infty} I_k(x_k) = \lim_{k\to\infty} G(x_k) = 0. \tag{12}$$

Proof. Since

$$(1-\varepsilon)\, I_0(x_0) \leqslant d_0 = \inf_{x \in X} I_0(x) \leqslant \inf_{x \in Q} I_0(x) \leqslant p\,(D - M_0)^r_+,$$

then

$$D \dot{-} M_0 \geqslant \{(1-\varepsilon)\, I_0(x_0)/p\}^{1/r}$$

and

$$D \geqslant M_1.$$

Similarly, one can prove that $M_k \leqslant D$ for $k=1, 2, \ldots$ From this and the fact that the sequence M_k is increasing it follows that there exists $\lim_{k\to\infty} M_k \leqslant D$. Hence

$$0 = \lim_{k\to\infty} p\,(M_{k+1} - M_k)^r = \lim_{k\to\infty} (1-\varepsilon)\, I_k(x_k)$$

which implies that

$$\lim_{k\to\infty} G(x_k) = 0 \text{ and } \lim_{k\to\infty} F(x_k) = \lim_{k\to\infty} M_k. \qquad \text{Q.E.D.}$$

The next theorem provides a condition for

$$\lim_{k\to\infty} M_k = D.$$

Let

$$Q_t = [x \in X:\ G(x) \leqslant t],\ t \geqslant 0, \tag{13}$$

$$h(t) = \inf_{x \in Q_t} F(x). \tag{14}$$

Since $Q_0 = Q$, then $h(0) = D$.

THEOREM 2. If $h(t)$ is right continuous at $t=0$, then

$$\lim_{k\to\infty} M_k = D,$$

where M_k is defined by (7).

Proof. Let $t_k = G(x_k)$, $k=0, 1, 2, \ldots$. Clearly

$$F(x_k) \geqslant h(t_k) \text{ and } \lim_{k\to\infty} t_k = 0.$$

Hence

$$\underline{\lim}_{k\to\infty} M_k = \underline{\lim}_{k\to\infty} F(x_k) \geqslant \underline{\lim}_{k\to\infty} h(t_k) = h(0) = D \geqslant \underline{\lim}_{k\to\infty} M_k \text{ and } \underline{\lim}_{k\to\infty} M_k = D. \qquad \text{Q.E.D.}$$

The result presented below refers to the weak convergence of sequence x_k.

THEOREM 3. Let X be a reflexive Banach space, F, G be weak lower semicontinuous on X and let there exists $R>0$ and $\delta>0$, such that

$$(F(x)-D)_+ + G(x) \geqslant \delta \quad \text{if} \quad \|x\| \geqslant R \tag{15}$$

then

$$\text{there exists } \bar{x} \in Q, \text{ such that } F(\bar{x}) = D = \inf_{x\in Q} F(x) \tag{16}$$

$$\lim_{k\to\infty} M_k = \lim_{k\to\infty} F(x_k) = D \tag{17}$$

the sequence x_k has a weak cluster point $x^* \in X$ and,

$$\text{moreover, } x^* \in Q,\ F(x^*) = D \tag{18}$$

and, if there exists exactly one point $\bar{x} \in Q$ such

$$\text{that } F(\bar{x}) = D, \text{ then } x_k \to \bar{x} \text{ (weakly).} \tag{19}$$

Proof. Since the set $Q \cap [x \in X\colon (F(x)-D)_+ + G(x) \leqslant \delta]$ is non-empty, bounded and weakly closed, then the Generalized Theorem of Weierstrass implies (16).

If $\|x_{k_i}\| \to \infty$, then by (12) and (15) we have

$$F(x_{k_i}) \geqslant D + \frac{\delta}{2} \text{ for } i \geqslant n_0 \in N.$$

From (11) it follows that the sequence x_k is bounded and includes a subsequence convergeding weakly to $x^* \in x$

The assumptions of the Theorem under discussion imply that

$$0 = \lim_{k\to\infty} G(x_k) \geqslant G(x^*) \geqslant 0 \to x^* \in Q,\ F(x^*) \geqslant D$$

and

$$D \geqslant \lim_{k\to\infty} F(x_k) \geqslant F(x^*) \geqslant D.$$

Thus

$$x^* \in Q,\ F(x^*) = D,\ \lim_{k\to\infty} F(x_k) = \lim_{k\to\infty} M_k = D,$$

which implies also (19). Q.E.D.

In the space $X = R^n$ a weak convergence is, at the same time, a strong convergence.

Let $F\colon X \to R$, X — a reflexive Banach space, F — Fréchet-differentiable on X, and let there exists $c>0$ such that

$$F(x) - F(y) \geqslant (F'(y), x-y) + c\|x-y\|^2 \text{ for all } x, y \in X \tag{20}$$

then one can verify that

$$\left.\begin{array}{l} x_k \to x^* \text{ (weakly)} \\ F(x_k) \to F(x^*) \end{array}\right\} \Rightarrow \{\|x_k - x^*\| \to 0\}.$$

Example. Let

$$Q=[(x, u) \in H_n^1 [0, T] \times L_m^1 [0, T] : \dot{x}(t)=A(t)\, x(t)+B(t)\, u(t)$$

$$x(0)=x_0 \in R^n,\ 0 \leqslant t \leqslant T < +\infty,\ T \text{— fixed}],$$

where $H_n^1 [0, T]$ — a Sobolev space, and let $A[n\times n]$, $B[n\times n]$ be continuous.

Let

$$F(x, u)=||x-q||^2+||u||^2,\ q \in L_n^2 [0, T],\ q \text{— fixed}.$$

Problem S. Minimize $F(x, u)$: $(x, u) \in Q$.

Here we take

$$M_0=0$$

$$G(x, u)=||x'-Ax-Bu||^2+||x(0)-x_0||^2$$

$$I_k(x, u)=p\,(F(x, u)-M_k)_+^r+G(x, u),\ p>0,\ r>0,$$

By Theorem 3 there exists $(\bar{x}, \bar{u}) \in Q$ such that

$$F(\bar{x}, \bar{u}) < F(x, u) \text{ for all } (x, u) \in Q.$$

We assume that

$$0 < F(x_1, u_1) \leqslant F(x, u) \text{ for } (x, u) \in Q,$$

$$0 < F(x_2, u_2) \leqslant F(x, u) \text{ for } (x, u) \in Q,$$

then we take $\lambda \in (0, 1)$, and denoting

$$x_\lambda=\lambda x_1+(1-\lambda)\, x_2,\ u_\lambda=\lambda u_1+(1-\lambda)\, u_2$$

we get $G(x_\lambda, u_\lambda)=0$,

$$F(x_\lambda, u_\lambda) \leqslant \lambda^2 F(x_1, u_1)+(1-\lambda)^2 F(x_1, u_2) < F(x_1, u_1).$$

Consequently, $x_1=x_2=\bar{x}$ and $u_1=u_2=\bar{u}$.

Since $F(x, u)$ satisfies the condition (20) with $c=1$, then Theorem 3 implies

$$\left.\begin{array}{l} ||x_k-\bar{x}|| \to 0 \\ ||u_k-\bar{u}|| \to 0 \end{array}\right\} \text{ as } k \to \infty$$

where the sequence (x_k, u_k) is defined as $\{x_k\}$ in (6).

3. Influence of p and r on the rate of convergence of the Morrison's method

In this section we will consider the influence of p and r on the rate of convergence of the Morrison's method in the limit case $\varepsilon=0$. However, as previously, we fix $M>-\infty$ such that

$$M \leqslant D=\inf_{x \in Q} F(x),$$

where X is a topological space.

We denote

$$d(p,r)=\inf_{x\in X}\{p\,(F(x)-M)^r_+ + G(x)\},\ p>0,\ r>0, \tag{21}$$

$$M(p,r)=M+\{d(p,r)/p\}^{1/r}. \tag{22}$$

Applying Theorem 1 we see that $M(p,r)\leq D$.

Here we are interested in choosing the values p and r, so that $M(p,r)$ may attain the greatest possible value. First, we shall introduce some additional notations. For a fixed $p>0$ and $r>0$ we take a sequence $x_k\in X$ such that

$$\lim_{k\to\infty}\{p\,(F(x_k)-M)^r_+ + G(x_k)\}=d(p,r).$$

The sequence $F(x_k)$, $G(x_k)$ are bounded, so that one can select a subsequence x_{k_i} such that there exist the limits

$$\lim_{i\to\infty} F(x_{k_i}) \text{ and } \lim_{i\to\infty} G(x_{k_i}).$$

Then we define

$$f(p,r)=\lim_{i\to\infty} F(x_{k_i}) \tag{23}$$

$$g(p,r)=\lim_{i\to\infty} G(x_{k_i}) \tag{24}$$

and we have

$$d(p,r)=p\,(f(p,r)-M)^r_+ + g(p,r) \text{ for } p>0,\ r>0. \tag{25}$$

Now we can discuss the influence of r on $M(p,r)$.

THEOREM 4. Apart from (21)—(25), $M\leq D$, assume that

$$r>s>0 \text{ and } (f(p,s)-M)^s_+\geq 1/e \tag{26a}$$

or

$$s>r>0 \text{ and } (f(p,s)-M)^s_+ + g(p,s)/p\leq 1/e \tag{26b}$$

then

$$M(p,r)\leq M(p,s). \tag{27}$$

Proof. Let

$$y(t)=(a^t+b)^{1/t},\ a>0,\ b>0,\ t>0,$$

then

$$y'(t)<0 \Leftrightarrow (a^t)^{a^t}<(a^t+b)^{a^t+b}.$$

Let $u=a^t$. The function $\varphi(u)=u^u$ is increasing, if $u\geq 1/e$, and decreasing if $0<u\leq 1/e$. Hence, by hypothesis (26) we have

$$M(p,s)=M+\{(f(p,s)-M)^s_+ + g(p,s)/p\}^{1/s}\geq$$
$$\geq M+\{(f(p,s)-M)^r_+ + g(p,s)/p\}^{1/r}.$$

Now (21), (22) imply (26). Q.E.D.

Thus, we see that in constructing the functional

$$I_k(x)=p\,(F(x)-M_k)^r_+ + G(x),\ x\in X$$

we usually take small r's for small k's.

However, this situation will change in the iteration, in which

$$(f(p,r)-M)_+^r+g(p,r)/p\leqslant 1/e.$$

In practice, we most often choose $r=1, 2, 3$. We are still less restricted in the selection of the value p, whose role will be explained by Theorems 5 and 6.

THEOREM 5. Assuming (21)—(24), let $M\leqslant D$, $q>p>0$, then

$$M(q,r)\leqslant M(p,r) \tag{28}$$

$$g(q,r)\geqslant g(p,r) \tag{29}$$

$$\lim_{p\to 0+} M(p,r)\leqslant D \tag{30}$$

$$\lim_{p\to 0+} g(p,r)=0. \tag{31}$$

Proof. Since

$$(f(p,r)-M)_+^r+g(p,r)/p\geqslant (f(p,r)-M)_+^r+g(p,r)/q\geqslant$$
$$\geqslant (f(q,r)-M)_+^r+g(q,r)/q, \text{ then (28).}$$

Adding the inequalities

$$\begin{aligned} d(p,r)&=p\,(f(p,r)-M)_+^r+g(p,r)\leqslant p\,(f(q,r)-M)_+^r+g(q,r)\\ d(q,r)&=q\,(f(q,r)-M)_+^r+g(q,r)\leqslant q\,(f(p,r)-M)_+^r+g(p,r)\end{aligned} \tag{32}$$

we have

$$(f(q,r)-M)_+\leqslant (f(p,r)-M)_+$$

and, consequently, due to (32)

$$d(p,r)=p\,(f(p,r)-M)_+^r+g(p,r)\leqslant p\,(f(p,r)-M)_+^r+g(p,r)$$

which implies (29).

The condition (10) implies

$$M\leqslant M(p,r)\leqslant D \text{ for } p>0, r>0.$$

Thus, due to (28), there exists a limit

$$\lim_{p\to 0+} M(p,r)\leqslant D.$$

Since (29) and $g(p,r)\geqslant 0$ and $\lim_{p\to 0+} M(p,r)\leqslant D$, then there exist the limits

$$\lim_{p\to 0+} g(p,r)/p \text{ and } \lim_{p\to 0+} g(p,r)=0. \qquad \text{Q.E.D.}$$

In Theorem 6 we give a condition for

$$\lim_{p\to 0+} M(p,r)=D.$$

THEOREM 6. Let $h(t)$ be defined by (14) and let $h(t)$ be right continuous at $t=0$, $M \leqslant D$, then

$$\lim_{p\to 0+} M(p, r) = \lim_{p\to 0+} f(p, r) = D, \tag{33}$$

$$\lim_{p\to 0+} g(p, r)/p = 0. \tag{34}$$

Proof. Since $\lim\limits_{p\to 0+} g(p, r)=0$, then we have

$$D \geqslant \lim_{p\to 0+} M(p, r) \geqslant \varliminf_{p\to 0+} \{M+(f(p, r)-M)\} =$$
$$= \varliminf_{p\to 0+} f(p, r) \geqslant \lim_{p\to 0+} h(g(p, r)) = D.$$

Thus

$$\lim_{p\to 0+} M(p, r) = \lim_{p\to 0+} f(p, r) = D$$

and also

$$\lim_{p\to 0+} \{(f(p, r)-M)^r + g(p, r)/p\}^{1/r} = D-M$$

which implies (34). Q.E.D.

REMARKS: 1. Theorem 6 states that the sequence M_k defined by (7) is parametrically superlinearly convergent.

2. For $0<\varepsilon<1$ one can prove that

$$\lim_{p\to 0+} M(p, r) \geqslant D-[1-(1-\varepsilon)^{1/r}](D-M).$$

Below we present an example of the influence of parameter p on the costs of the method.

Example. Let

$$F(x)=x_1^2+x_2^2+x_3^2+x_4^2-x_1 x_2-x_2 x_3-x_3 x_4,\ x \in R^4,$$

$$g_1(x)=x_1-x_3-x_4+3,$$

$$g_2(x)=x_2-x_3-x_4,$$

$$Q=[x \in R^4\colon g_1(x)=0,\ g_2(x)=0].$$

Find $\bar{x} \in Q$, such that $F(\bar{x}) \leqslant F(x)$ for $x \in Q$.

The solution of this problem is $\bar{x}=(-1.5, -0.5, 0.5, -1.0)$. For this problem we apply the Generalized Morrison's Method with r (er) $=2$.

Let $M_0=0$ (because $F(x) \geqslant 0$ for $x \in R^4$), $G(x)=g_1^2(x)+g_2^2(x)$, $x \in R^4$.

We have performed the computation on a computer, using the procedure "qnm-der" [1], the initial point $x_0=(0, 0, 0, 0)$ and the identical termination criterion

$$\|\nabla I_{k+1}(x_k)\| \leqslant 10^{-8}$$

and have obtained

p	itr	nf
1	45	1088
0.1	14	344
0.01	6	142
0.001	4	81

where: itr — the number of iterations (the sequence M_k), nf — the number of computed values of $I_k(x)$.

Thus, we see that the insertion of parameter p to the Morrison's penalty function can, in practice, radically reduce the costs of the method.

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Uogólnienie metody Morrisona

Przedstawiono uogólnienie metody Morrisona, to jest metody jednoparametrowej przesuwanej funkcji kary dla zadań optymalizacji z ograniczeniami równościowymi, która może być stosowana do minimalizacji funkcjonału określonego na przestrzeni topologicznej przy różnego typu ograniczeniach. Przedstawiono modyfikację funkcji kary, zależnej w tym przypadku od trzech parametrów, i zbadano wpływ parametrów p oraz r na szybkość zbieżności metody.

Обобщение метода Моррисона

В работе представлено обобщение метода Моррисона, т.е. метода сдвигаемой штрафной функции с одним параметром для задач оптимизации с ограничениями типа равенств, который может быть применен для минимизации функционала определенного на топологическом пространстве при разного рода ограничениях. Представлена модификация штрафной функции, которая зависит в этом случае от трех параметров и исследовано влияние параметров p и r на скорость сходимости метода.