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On the existence of Nash equilibrium points for differential games with linear and non-linear dynamics

by

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In the paper an effort is made to give a possibly complete account of the results concerned with the existence of open-loop Nash equilibrium solutions for non-zero-sum differential games. The Fan-Glicksberg fixed-point theorem is used to prove, first, the existence of the equilibrium in the class of ordinary controls for games with linear dynamics and convex scot functionals, and, next, the existence of the equilibrium in the class of relaxed controls for general, non-linear game problems. The first part of the paper is mostly a recapitilation of the results known from the literature. The second part, however, contains a theorem which is apparently new.

1. Introduction

Consider a game with N players, where for $i \in I=\{1, ..., N\}$ the i-th player chooses a strategy u_i from a convex, compact subset U_i of a linear topological space Z_i, trying to minimize a cost functional (minus payoff) $J_i(u_1, ..., u_i, ..., u_N)$, defined on the set $U=U_1 \times ... \times U_N$. According to Nash [7], a point $u^*=(u_1^*, ..., u_N^*)$ is an equilibrium solution of the game $\Gamma=\langle I, \{U_i\}, \{J_i\}\rangle$ if

$$\forall i \in I, \forall u_i \in U_i, J_i(u^*) \leqslant J_i(u_1^*, ..., u_{i-1}^*, u_i, u_{i+1}^*, ..., u_N^*). \tag{1}$$

Suppose now, that for $i \in I$ the strategy u_i is a measurable function from a given time interval $T=[t_0, t_f]$ to m_i — dimensional space R^{m_i}, U_i is closed, bounded, convex subset of the Hilbert space $L_2^{m_i}(T)$ (it follows, that U_i is weakly compact) and cost functional J_i is defined as

$$J_i(u_1, ..., u_N)=\int_{t_0}^{t_f} L_i\big(x(t), u_1(t), ..., u_N(t), t\big)\, dt+K_i\big(x(t_f)\big), \tag{2}$$

where

$$\dot{x}(t)=f\big(x(t), u_1(t), ..., u_N(t), t\big), \; x(t_0)=x_0 \tag{3}$$

and for $t \in T$ $x(t)$ is the n-dimensional vector. This game is known in the literature as the open-loop, N-person, non-zero-som differential game. If the system of differential equations (3) is linear, i.e.

$$\dot{x}(t) = A(t)\, x(t) + \sum_{i=1}^{N} B_i(t)\, u_i(t),\ x(t_0) = x_0 \tag{4}$$

then the game is referred to as the differential game with linear dynamics, or sometimes simply as the linear differential game. In the following the game (2)—(3) will be denoted by Γ^d and the game (2)—(4) by Γ^l.

The problem of the existence of equilibrium solutions for games defined in general linear, topological spaces was investigated by several authors in early fifties [2, 4, 8]. For example, Nikaido and Isoda [8], analysing the function

$$\Phi(u, v) = \sum_{i=1}^{N} J_i(v_1, \ldots, v_{i-1}, u_i, v_{i+1}, \ldots, v_N) \tag{5}$$

where $u = (u_1, \ldots, u_N)$, $v = (v_1, \ldots, v_N)$, and using the Brouwer's fixed point theorem, proved the existence of the equilibrium solution for the game Γ under assumptions, that J_i are convex and, that they satisfy some continuity conditions. Fifteen years later, Varaiya [12], using the same method, proved the existence of the equilibrium point for the game Γ^l, provided functions L_i have some special form and satisfy, as well as functions K_i, some regularity and convexity assumptions. An approach to the problem, which is much simpler than the method of Nikaido and Isoda, although it is slightly less general, is based on the generalization of the Kakutani's fixed point theorem [5] for the infinite dimensional linear spaces. This approach was suggested by Bohnenblast and Karlin [2], who extended the Kakutani's theorem for the case of an arbitrary Banach space. This result was further extended on the general, locally convex, linear, topological spaces by Fan [3] and Glicksberg [4]. In the context of differential games the Fan-Glicksberg theorem was first used by Skérus and Jačiauskas [11], and then, under more general assumptions about the game, by Vidyasagar [13]. For the game Γ^l the latter author obtained in a very simple way an existence, theorem, which was more general than this of Varaiya (Varaiya, however, obtained also a result for games, where sets U_i were not bounded — see Section 5). It is worth to note, that the problem of the existence of the equilibrium solution for the games Γ^l was also investigated by Scalzo [10], who obtained the result similar to that of Vidyasagar but in a considerably more complicated way.

The existence of equilibrium solutions in pure strategies is assured only for the games with convex cost functionals. In the context of the differential games it means, that one can hardly hope to obtain a meaningfull result for the games with nonlinear dynamics, if only classical controlfunctions are considered. According to Glicksberg [4], the general continuous games of the type Γ have equilibrium solutions in the dass of mixed strategies δ_i, $i \in I$, defined as the Radon probability measures concentrated on U_i. In the case of the differential games it seems rather fruitless ot consider the strategies defined as measures on the Σ-fields of the spaces $L_2^m(T)$

Much more suitable approach is to assume, that the players use the so-called relaxed controls [14,,15], i.e. measurable functions $\sigma_i(\cdot), i \in I$, from T to the spaces of Radon probability measures, concentrated on some compact subsets of Euclidean spaces R^{m_i}.

The aim of this paper is to recapitulate the known results concerned with the existence of the Nash open-loop equilibrium solution in the class of ordinary controls, for the differential games with linear dynamics, and to present a new theorem, which states the existence of the equilibrium point in the class of relaxed controls, for the games with nonlinear state equations. In the next Section a simple conclusion for the Fan-Glicksberg theorem is given. This conclusion is basic for the proofs of theorems from Sections 3 and 4, which deal with games Γ^l and Γ^d, respectively. The last Section contains some remarks on differential games, where sets U_i of admissible controls are unbounded.

2. The basic lemma

The following theorem is fundamental for the greater part of this paper.

Theorem 1. Let U be a compact, convex set in a real, linear, locally convex, topological space Z; and let ψ be a mapping that assigns, to each $u \in U$, a compact convex subset $\psi(u)$ of U, such that, for any sequence $\{u_n\}$ in U,

$$u_n \to u,\ y_n \in \psi(u_n),\ y_n \to y \Rightarrow y \in \psi(u). \tag{6}$$

Then, there exists an $u^* \in U$ with the property that $u^* \in \psi(u^*)$.

This theorem implies the following lemma, which is a slight modification of the theorem 2.1 of [13].

Lemma 1. Consider the game $\Gamma = \langle I, \{U_i\}, \{J_i\}\rangle$ and assume

A1. $\forall i \in I\ U_i$ is a compact, convex subset of locally convex, topological space Z_i.

A2. $\forall i \in I\ J_i$ is continuous in $u_j, j \neq i$, for fixed u_i.

A3. $\forall i \in I\ J_i$ is lower semi-continuous in u_i.

A4. $\forall i \in I$

$$\psi_i(u) = \{w_i \in U_i \mid J_i(u_1, \ldots, u_{i-1}, w_i, u_{i+1}, \ldots, u_N) = \inf_{v_i \in U_i} J_i(u_1, \ldots, u_{i-1}, v_i, u_{i+1}, \ldots, u_N)\}, \tag{7}$$

is a convex subset of U_i.

Then, there exists an equilibrium solution of the game Γ.

Proof [13]. Because of the continuity of J_i and the compactness of U_i J_i attains its infimum on U_i, what implies, that $\psi_i(u)$ are nonempty. Next, by (A3) $\psi_i(u)$ are closed and by (A4) — convex. Let $U = U_1 \times \ldots \times U_N$ and $\tilde{\psi}(u) = \{\psi_1(u), \ldots, \psi_N(u)\}$. We want to show, that ψ satisfies the hypothesis of Theorem 1. Obviously, for all $u \in U$ $\psi(u)$ is a nonempty, compact, convex subset of U. Now, let $\{u^j\}$ be a sequence in U, $u^j \to u^o$, $w^j \in \psi(u^j)$, and $w^j \to w^o$; we must show, that $w^o \in \psi(u^o)$. Suppose p is any element of $\psi(u^o)$. Then, for $i \in I$,

$$J_i(u_1^o, \ldots, u_{i-1}^o, p_i, u_{i+1}^o, \ldots, u_N^o) = \lim_{j\to\infty} J_i(u_1^j, \ldots, u_{i-1}^j, p_i, u_{i+1}^j, \ldots, u_N^j). \tag{8}$$

From the definition of w^j, for any j we have

$$J_i(u_1^j, \ldots, u_{i-1}^j, p_i, u_{i+1}^j, \ldots, u_N^j) \geq J_i(u_1^j, \ldots, u_{i-1}^j, w_i^j, u_{i+1}^j, \ldots, u_N^j). \tag{9}$$

As $j \to \infty$ (9) becomes

$$J_i(u_1^o, \ldots, u_{i-1}^o, p_i, u_{i+1}^o, \ldots, u_N^o) \geq \limsup_{j\to\infty} J_i(u_1^j, \ldots, u_{i-1}^j, w_i^j, u_{i+1}^j, \ldots, u_N^o). \tag{10}$$

From (A3) we have

$$\liminf_{j\to\infty} J_i(u_1^j, \ldots, u_{i-1}^j, w_i^j, u_{i+1}^j, \ldots, u_N^j) \geq J_i(u_1^o, \ldots, u_{i-1}^o, w_i^o, u_{i+1}^o, \ldots, u_N^o). \tag{11}$$

Combining (10) and (11) gives

$$J_i(u_1^o, \ldots, u_{i-1}^o, p_i, u_{i+1}^o, \ldots, u_N^o) \geq J_i(u_1^o, \ldots, u_{i-1}^o, w_i^o, u_{i+1}^o, \ldots, u_N^o). \tag{12}$$

Since $p \in \psi(u^o)$, what means $p_i \in \psi_i(u^o)$ for $i \in I$, inequality (12) must in fact be equality, and $w_i^o \in \psi_i(u^o)$. Therefore, the map ψ satisfies the hypothesis of Theorem 1. It follows, that there exists a point $u^* \in U$ such that $u^* \in \psi(u^*)$ Obviously u^* is an equilibrium solution of the game Γ.

3. Differential games with nonlinear dynamics

In this Section we consider differential games of the type Γ^l, with dynamics described by the equation (4), and cost functionals of the form

$$J_i(u_1, \ldots, u_N) = \int_{t_0}^{t_f} L_i(x(t), u_i(t), t)\, dt + K_i(x(t_f)). \tag{13}$$

The theorem below gives sufficient conditions for the existence of the equilibrium solution of the game (4), (13).

THEOREM 2. Assume the following conditions are satisfied

B1. $\forall i \in I$, U_i is closed, bounded, convex subset of $L_2^{m_i}(T)$.

B2. The materices $A, B_i, i \in I$, have appropriate dimension and their coefficients are bounded, measurable functions of t.

B3. $\forall i \in I$ $L_i(x, u_i, t)$ is convex in (x, u_i) and $K_i(x)$ is convex in x.

B4. $\forall i \in I$ $L_i(x, u_i, t)$ is measurable in t, Lipschitz continuous in x on every bounded subset of R^n, continuous and bounded from below in u_i on R^{m_i}; $K_i(x)$ is continuous on R^n.

Then the game (4), (13) has an equilibrium solution.

Proof. To prove the theorem we have to show, that the game under consideration satisfies the hypothesis of Lemma 1. From (B1) it follows, that for any $i \in I$ U_i is a weakly compact, convex subset of $L_2^{m_i}(T)$. Now observe, that for any $t \in T$ we have

$$x(t)=\Phi(t, t_0)\, x_0+\sum_{i=1}^{N} \int_{t_0}^{t} \Phi(t, \tau)\, B_i(\tau)\, u_i(\tau)\, d\tau \tag{14}$$

where Φ is the transition matrix function associated with (4). The operator

$$h_t^i(u_i)=\int_{t_0}^{t} \Phi(t, \tau)\, B_i(\tau)\, u_i(\tau)\, d\tau=\int_{t_0}^{t_f} \chi_{[t_0, t]}(\tau)\, \Phi(t, \tau)\, B_i(\tau)\, u_i(\tau)\, d\tau \tag{15}$$

where

$$\chi_{[t_0, t]}(\tau)=\begin{cases}1 & \text{if} \quad \tau \in [t_0, t] \\ 0 & \text{if} \quad \tau \in [t, t_f]\end{cases} \tag{16}$$

is a strongly continuous linear operator of $L_2^{m_i}(T)$ into the finite-dimensional space R^n. Hence, the operator h_t^i is weakly continuous and, in consequence, the function

$$(u_1, \ldots, u_N) \rightarrow x(t, u_1, \ldots, u_N): U_1 \times \ldots \times U_N \rightarrow R^n \tag{17}$$

is weakly continuous for any $t \in T$. Next, we see, that by assumption, for any $i \in I$, the set of admissible controls u_i is bounded in $L_2^{m_i}(T)$ norm. Combining this with [14] gives

$$\|x(t, u)\|_{\sup}=\sup_{t \in T} |x(t, u)| \leqslant M \tag{18}$$

for some $M<\infty$, uniformly for all admissible $u=(u_1, \ldots, u_N)$. This implies, that $x(\cdot)$ takes values only in a bounded subset of R^n. Now, consider a sequence $\{\hat{u}_i^k\}=$ $=\{(u_1^k, \ldots, u_{i-1}^k, u_i, u_{i+1}^k, \ldots, u_N^k)\}$, where u_i is fixed, and assume, that it converges weakly to $\hat{u}_i^o$. Let $\hat{x}_i^k=x(\cdot, u_i^k)$. By (17) we have

$$\hat{x}_i^k(t) \rightarrow \hat{x}_i^o(t) \text{ for any } t \in T. \tag{19}$$

Combining (B4) with (19) gives

$$\left|K_i(\hat{x}_i^k(t_f))-K_i(\hat{x}_i^o(t_f))\right| \xrightarrow[k \rightarrow \infty]{} 0 \tag{20}$$

$$\int_{t_0}^{t_f} \left|L_i(\hat{x}_i^k(t), u_i(t), t)-L_i(\hat{x}_i^o(t), u_i(t), t)\right| dt \leqslant$$

$$\leqslant \int_{t_0}^{t_f} M' |\hat{x}_i^k(t)-\hat{x}_i^o(t)|\, dt \leqslant M \sqrt{(t_f-t_0)}\, \|\hat{x}_i^k-\hat{x}_i^o\|\, L_2^n(T) \xrightarrow[k \rightarrow \infty]{} 0 \tag{21}$$

and finally

$$|J_i(\hat{u}_i^k)-J_i(\hat{u}_i^o)| \xrightarrow[k \rightarrow \infty]{} 0. \tag{22}$$

Thus, we have shown, that J_i is weakly continuous in $u_j, j \neq i$, for any fixed u_i. The next step is to show, that J_i is strongly lower semi-continuous in u_i. To do this, it is enough to prove, that the set

$$V_i^\alpha=\left\{u_i \in L_2^{m_i}(T) \,\middle|\, \int_{t_0}^{t_f} L_i(x(t, u), u_i(t), t)\, dt \leqslant \alpha\right\} \tag{23}$$

for any real α and fixed $u_j, j\neq i$, is strongly closed. Let $\{u_i^k\}$ be a sequence of element from V_i converging to u_i^o in $L_2^{m_i}(T)$ norm and let $x^k=x(\cdot, u_1, ..., u_{i-1}, u_i^k, u_{i+1}, ..., u_N)$, where $u_j, j\neq i$, are arbitrary but fixed. Of course, $x^k \to x^o$ everywhere on T. Next, by Jegorov's theorem there exists a subsequence of $\{u_i^k\}$, which converges to u_i^o a.e. on T (from this moment $\{u_i^k\}$ will denote such a subsequence, if necessary). By (B4) and (18) L_i bounded from below. Thus, we have by Fatou's lemma

$$\alpha \geqslant \liminf \int_{t_0}^{t_f} L_i(x^k(t), u_i^k(t), t)\, dt \geqslant$$

$$\geqslant \int_{t_0}^{t_f} \liminf L_i(x^k(t), u_i^k(t), t)\, dt = \int_{t_0}^{t_f} L_i(x^o(t), u_i^o(t), t)\, dt. \tag{24}$$

Hence, $u_i^o \in V_i^\alpha$, what means, that V_i^α is closed, and J_i is strongly lower semi-continuous in u_i. Because of the linearity of the state equation and the assumption (B3) J_i is convex in u_i. So, $\psi_i(u)$ is convex and the strongly lower semi-continuous functional J_i is also weakly lower semi-continuous. This complete the proof.

Theorem 2 differs from the Vidyasagar's result only in details. The cost functionals J_i, as given by (13), are slightly more general than those in [13], and regularity assumptions about functions L_i are made more explicit.

It is worth to note, that in an important case, when the control functions take values in compact, convex subsets of R^{m_i}, the condition (B4) of Theorem 2 can be replaced by a weaker one. This fact can be stated in the following form.

COROLLARY 1. Suppose, for $i \in I$ and $t \in T$, $u_i(t) \in S_i$, where S_i is compact, convex subset of the Euclidean space R^{m_i}. Then, for the existence of the equilibrium solution of the game Γ^l defined by (4) and (13) it is sufficient to assume (B2), (B3) and

B4'. $\forall i \in I\ L_i(x, u_i, t)$ is measurable in t, continuous in x on R^n and continuous in u_i on S_i; $K_i(x)$ is continuous on R^n.

Proof. There exists a finite number M such that $|u_i(t)| \leqslant M$ for $i \in I$ and $t \in T$. Combining this with (14) gives

$$|x(t, u) - x(t', u)| \leqslant M' |t - t'| \tag{25}$$

for some finite M', any $t, t' \in T$, and all admissible controls u. Thus, the family of functions $x(\cdot, u)$ for all admissible u is equicontinuous and uniformly bounded. By a well known result this implies (see [15] theorem 1.5.3), that if x^k converges to x^o a.e. on T, then x^k converges to x^o uniformly on T. Reconsidering the proof of the Theorem 2, we see, that now the weak convergence of $\hat{u}_i^k$ to $\hat{u}_i^o$ implies the uniform convergence of $\hat{x}_i^k$ to $\hat{x}_i^o$. In consequence

$$\int_{t_0}^{t_f} L_i(\hat{x}_i^k(t), u_i(t), t) - L_i(\hat{x}_i^o(t), u_i(t), t)\, dt \xrightarrow[k\to\infty]{} 0. \tag{26}$$

The observation, that the continuous function L_i is bounded from below on the compact set S_i completes the proof.

4. Differential games with nonlinear dynamics

The linearity of the state equation has been the crucial assumption for the results of the previous Section. Without this assumption it is extremly difficult to say anything either of the continuity of the cost functionals (with respect to the weak topology of $L_2^{m_i}(T)$), or of the convexity of the sets $\psi(u)$ in the space of the ordinary controls. Thus, to deal with the differential games with nonlinear dynamics (as defined by (2) and (3)), one has to use a different approach. A convenient method is to consider the game in a new space of controls, i.e. in the space of the so-called relaxed controls, introduced to the calculus of variations and the optimal control theory by Young [15]. For the differential games this concept is even more meaningfull, as it is similar (although not identical) to the concept of mixed strategies, which plays a crucial role in the classical theory of games [4, 7]. The use of the rather sophisticated topological space of relaxed controls is also justified by the fact, that the existence of the equilibrium solution in the class of these controls implies the existence of the ε-equilibrium in the class of ordinary controls, what makes the theory applicable to practical situations.

The principal result of this Section is the theorem stating the sufficient conditions for the existence of the equilibrium solution for the general differential game Γ^d, in the class of relaxed controls. To formulate this theorem we shall need the following facts concerned with the concept of relaxed controls (see [14] Chap. IV). Consider an optimal control problem, or a differential game, where control functions are constrained to take values in a compact subset S of R^m, i.e. $u(t) \in S$ a.e. on T, where T is a given, finite time interval. Let rpm (S) be the set of nonnegative unit measures (probability measures) on R^m, wholly concentrated on S. By Riesz's theorem rpm (S) can be identified with a subset of the space $C(S)^*$. Thus, we can consider rpm (S) as the topological space with the relative weak star topology of $C(S)^*$.

DEFINITION. A relaxed control is any function $\sigma(\cdot): T \to \text{rpm}(S)$, which is measurable with respect to the Lebesgue measure on T.

Let $\mathscr{P}(S)$ be the set of all relaxed controls corresponding to a given T. By Danford-Pettis' theorem $\mathscr{P}(S)$ can be identified with a subset of the space $L_1(T, C(S))^*$ ($\varphi \in L_1(T, C, (S))$ iff $\varphi: T \times S \to R$, $\varphi(\cdot, s)$ integrable for $s \in S$, and $\varphi(t, \cdot)$ continuous for $t \in T$). The set $\mathscr{P}(S)$ turns out to be convex and compact in the weak star topology of $L_1(T, C(S))^*$. Suppose $\sigma \in \mathscr{P}(S)$, then

$$\varphi(t, \sigma(t)) \stackrel{\text{def}}{=} \int_{R^m} \varphi(t, s)\, d\sigma(t) \tag{27}$$

for any $t \in T$, and $\varphi(t, \sigma(t))$ is integrable on T for all $\varphi \in L_1(T_3\, C(S))$. Let σ^k, $\sigma \in \mathscr{P}(S)$, $k=1, 2, \ldots$, then $\lim_{k \to \infty} \sigma^k = \sigma$ in the weak star topology of $L_1(T, C(S))^*$ iff

$$\lim \int_{t_0}^{t_f} \varphi(t, \sigma^k(t))\, dt = \int_{t_0}^{t_f} \varphi(t, \sigma(t))\, dt \tag{28}$$

for all $\varphi \in L_1(T, C(S))$.

Consider the differential game (2), (3) and assume, that the i-th player chooses its control u_i in such way, that $u_i(t) \in S_i$ a.e. on T, where S_i is a compact subset of R^m. Let $\sigma_i \in \mathscr{P}(S_i)$ denotes a relaxed control of the i-th player, and let $\sigma = (\sigma_1, \dots, \sigma_N)$ and $S = S_1 \times \dots \times S_N \subset R^m$. The game (2), (3) considered in the class of relaxed controls takes the form

$$J_i(\sigma) = \int_{t_0}^{t_f} L_i(x(t), \sigma_1(t), \dots, \sigma_N(t), t)\, dt + K_i(x(t_f)) \tag{29}$$

$$\dot{x}(t) = f(x(t), \sigma_1(t), \dots, \sigma_N(t), t), \; x(t_0) = x_0 \tag{30}$$

or equivalently

$$J_i(\sigma) = \int_{t_0}^{t_f} dt \int_{R^{m_1}} \dots \int_{R^{m_N}} L_i(x(t), s_1, \dots, s_N, t)\, d\sigma_1(t) \dots d\sigma_N(t) +$$

$$+ K_i(x(t_f)) = \int_{t_0}^{t_f} dt \int_{R^m} L_i(x(t), s, t)\, d\sigma(t) + K_i(x(t_f)) \tag{29'}$$

$$\dot{x}(t) = \int_{R^{m_1}} \dots \int_{R^{m_N}} f(x(t), s_1, \dots, s_N, t)\, d\sigma_1(t) \dots d\sigma_N(t) =$$

$$= \int_{R^m} f(x(t), s, t)\, d\sigma(t), \; x(t_0) = x_0 . \tag{30'}$$

THEOREM 3. Assume the following conditions are satisfied:

C1. The exists a $c > 0$ such that

$$|f(x, u, t)| \leqslant c(|x| + 1) \text{ for all } (x, u, t) \in R^n \times S \times T.$$

C2. The functions $f\colon R^n \times R^m \times T \to R^n$ and $L_i\colon R^n \times R^m \times T \to R$ and their partial derivatives $\partial f/\partial x$, $\partial L_i/\partial x$ exist and are continuous on $R^n \times S \times T$.

Then, there exists a relaxed control $\sigma^* = (\sigma_1^*, \dots, \sigma_N^*) \in \mathscr{P}(S)$ such that

$$J_i(\sigma^*) \leqslant J_i(\sigma_1^*, \dots, \sigma_{i-1}^*, \sigma_i, \sigma_{i+1}^*, \dots, \sigma_N^*) \tag{31}$$

for all $\sigma_i \in \mathscr{P}(S_i)$ and $i \in I$.

Proof. We have to show, that the game (29), (30) satisfies the assumptions (A1)—(A4) of Lemma 1. For $i \in I$ we know the set $\mathscr{P}(S_i)$ to be convex, compact subset of locally convex space $L_1(T, C(S_i))^*$ with weak star topology. We now show, using the method of Warga ([15] Chap. V), that J_i is continuous in this topology, in all its arguments $\sigma_1, \dots, \sigma_N$. First, observe, that under conditions (C1), (C2), for any relaxed control σ, there exists an absolutly continuous function $x(\cdot, \sigma)\colon T \to R^n$, that is the unique solution to (30). Furthermore, there exists a $c_1 > 0$ such that

$$\|x(\cdot, \sigma)\|_{\sup} \leqslant c_1 \quad \text{for} \quad \sigma \in \mathscr{P}(S). \tag{32}$$

Next, by the continuity of $\partial f/\partial x$, there exists a c_2 such that

$$\|(\partial/\partial x) f(x(\cdot, \sigma), \sigma, \cdot)\|_{\sup} \leqslant c_2 \quad \text{for all} \quad \sigma \in \mathscr{P}(S) \tag{33}$$

(33) implies, that f is Lipschitz continuous in x. Now suppose, that $\sigma^k \to \sigma^o$ in the weak star topology of $L_1(T, C(S))^*$, and $\sigma^k, \sigma^o \in \mathscr{P}(S)$. Let $x^k(\cdot) = x(\cdot, \sigma^k)$. Then

$$|x^k(t) - x^o(t)| \leqslant \left| \int_{t_0}^{t} [f(x^k(\tau), \sigma^k(\tau), \tau) - f(x^o(\tau), \sigma^o(\tau), \tau)]\, d\tau \right| \leqslant$$

$$\leqslant \left| \int_{t_0}^{t} [f(x^o(\tau), \sigma^k(\tau), \tau) - f(x^o(\tau), \sigma^o(\tau), \tau)[\, d\tau \right| +$$

$$+ \int_{t_0}^{t} \left| f(x^k(\tau), \sigma^k(\tau), \tau) - f(x^o(\tau), \sigma^k(\tau), \tau) \right| d\tau \leqslant$$

$$\leqslant c_2 \int_{t_0}^{t} |x^k(\tau) - x^o(\tau)|\, d\tau + \left| \int_{t_0}^{t} f(x^o(\tau), \sigma^k(\tau) - \sigma^o(\tau), \tau)\, d\tau \right|. \tag{34}$$

The last inequality follows from (33) and from the fact, that for any σ, σ', x and τ we have

$$f(x, \sigma, \tau) - f(x, \sigma', \tau) = \int_{R^m} f(x, s, \tau)\, d\sigma(\tau) - \int_{R^m} f(x, s, \tau)\, d\sigma'(\tau) =$$

$$= \int_{R^m} f(x, s, \tau)\, d(\sigma(\tau) - \sigma'(\tau)) = f(x, \sigma(\tau) - \sigma'(\tau), \tau). \tag{35}$$

Let

$$h_k(t) \overset{\text{def}}{=} \int_{t_0}^{t} f(x^o(\tau), \sigma^k(\tau) - \sigma^o(\tau), \tau)\, d\tau =$$

$$= \int_{t_0}^{t_f} \chi_{[t_0, t]} f(x^o(\tau), \sigma^k(\tau) - \sigma^o(\tau), \tau) d\tau \tag{36}$$

and

$$\varphi_t(\tau, \sigma^k(\tau) - \sigma^o(\tau)) = \chi_{[t_0, t]} f(x^o(\tau), \sigma^k(\tau) - \sigma^o(\tau), \tau). \tag{37}$$

Then $\varphi_t \in L_1(T, C(S))$.

Combining (28) with (37) gives

$$\sigma^k \to \sigma^o \Rightarrow h_k(t) \to 0 \quad \text{for any} \quad t \in T. \tag{38}$$

Thus $h^k(\cdot)$ converges to 0 a.e. on T. Next, we see, that there exists a c_3 such that

$$|f(x^o(\tau), \sigma^k(\tau) - \sigma^o(\tau), \tau)| \leqslant c_3 \quad \text{for all} \quad \tau \in T \quad \text{and} \quad k = 1, 2, \dots. \tag{39}$$

It means that the functions $\{h^k(\cdot)\}$ are equicontinuous and uniformly bounded. Hence, by [15] theorem 1.5.3 $h^k(\cdot) \to 0$ uniformly on T. Combining (34) with the Gronwell inequality gives

$$|x^k(t) - x^o(t)| \leqslant h^k(t) + c \int_{t_0}^{t} h^k(\tau)\, d\tau, \tag{40}$$

where $c = c_2 \exp(c_2(t_f - t_0))$.

(38) implies

$$\|x^k(\cdot) - x^o(\cdot)\|_{\sup} \leqslant \|h^k(\cdot)\|_{\sup} (1 + c(t_f - t_0)) \xrightarrow[k \to \infty]{} 0. \tag{41}$$

We have shown, that if $\sigma^k \to \sigma^o$ in the assumed topology, then $x(\cdot, \sigma^k) \to x(\cdot, \sigma^o)$ in the strong topology of $C(T, R^n)$, i.e., that the function $\sigma \to x(\cdot, \sigma): \mathscr{P}(S) \to \to C(T, R^n)$ is continuous. It is now easy to show, that $\sigma^k \to \sigma^o$ implies $|J_i(\sigma^k) - -J_i(\sigma^o)| \to 0$ for $i \in I$, what means the continuity of the functionals (28).

To complete the proof we have to show the convexity of $\psi(\sigma)$, where ψ is defined by (7), with σ substituted for u and $\mathscr{P}(S_i)$ for U_i. Let $\sigma_i^1, \sigma_i^2 \in \psi_i(\sigma)$, $\lambda \in (0, 1)$ and $\bar{J}_i = J_i(\sigma_1, \ldots, \sigma_{i-1}, \sigma_i^k, \sigma_{i+1}, \ldots, \sigma_N)$, where $k=1$ or 2. Then

$$J_i(\sigma_1, \ldots, \lambda\sigma_i^1 + (1-\lambda)\sigma_i^2, \ldots, \sigma_N) = \\ = \lambda J_i(\sigma_i, \ldots, \sigma_i^1, \ldots, \sigma_N) + (1-\lambda) J_i(\sigma_1, \ldots, \sigma_i^2, \ldots, \sigma_N) = \bar{J}_i \tag{43}$$

and $\lambda\sigma_i^1 + (1-\lambda)\sigma_i^2 \in \psi_i(\sigma)$.

REMARK 1. Instead of (C1) one can assume more generally, that for any σ there exists a unique solution to (30) and (32) is satisfied for a $c_1 > 0$ and $\sigma \in \mathscr{P}(S)$.

REMARK 2. Theorem 3 remains true for the game (29), (30) with additional constrains on control and state variables, of the type

$x(t_f) \in A_1$ for a compact, convex set $A_1 \subset R^n$ and/or

$\forall t \in T\, x(t) \in A_2$ for a convex, closed set $A_2 \subset R^n$.

REMARK 3. A result concerned with the existence of the equilibrium solution of the differential game, in class of the relaxed controls was also obtained in [9], but under considerably stronger assumptions about the form of the state equation and the cost functionals.

Theorem 3 implies the existence of ε-equilibria in class of ordinary controls, for a large class of differential games satisfying conditions (C1) and (C2). It follows from the fact, that the set of measurable functions u such that $u(t) \in S$ a.e. on T is dense in $\mathscr{P}(S)$ (u is identified with $\rho \in \mathscr{P}(S)$ such that for $t \in T\, \rho(t)$ is wholly concentrated in the point $u(t)$). We state this result in the form of the following corollary.

COROLLARY 2. If the differential game (2), (3) satisfies conditions (C1), (C2) for some compact set $S \subset R^m$, then, for any $\varepsilon > 0$ there exist sets V_i^ε, $i \in I$, of measurable functions $u_i: T \to S_i$ such that for any $u^\varepsilon \in V^\varepsilon = V_i^\varepsilon \times \ldots \times V_N^\varepsilon$ and all $i \in I$

$$J_i(u^\varepsilon) \leqslant \inf J_i(u_1^\varepsilon, \ldots, u_{i-1}^\varepsilon, u_i, u_{i+1}^\varepsilon, \ldots, u_N^\varepsilon) + \varepsilon. \tag{44}$$

Proof. For any $\varepsilon > 0$ define the set V^ε as the intersection of the set U of ordinary controls with the set

$$\tilde{V}^\varepsilon = \{\sigma \mid J_i(\sigma) \leqslant J_i(\sigma^*) + \varepsilon,\ i \in I\} \tag{45}$$

where σ^* is the equilibrium solution in the class of relaxed controls. As all J_i are continuous on $\mathscr{P}(S)$ and U is dense in $\mathscr{P}(S)$, the proof is completed.

5. Differential games with unbounded sets of controls

We shall now consider the differential game without constraints, i.e. the game, where the i-th player minimizes the functional (2) subject to (3) or (4), choosing its strategy u_i from the whole space $L_2^{m_i}(T)$. Clearly, the methodology used in the previous Sections does not apply to this problem and only much weaker results, concerned with the existence of the equilibrium solution, can be obtained. The theorem below, which is due to Varaiya [12], gives the sufficient conditions for the local existence of an equilibrium solution for the differential game (2), (4) with convex cost functionals.

THEOREM 4. Consider the differential game (2), (4) without constraints, and assume, that for $i \in I L_i$ is of the form

$$L_i(x, u_1, \ldots, u_N, t) = g_i(u_i, t) + h_i(x, t). \tag{46}$$

Assume furthermore, that the following conditions are satisfied

D1. The matrices $A, B_i, i \in I$, are bounded, measurable functions of t.

D2. $\forall i\ K_i(x), g_i(u_i, t), h_i(x, t)$ are continuous in all variables, bounded from below, and for each t, they are convex in remaining variables.

D3. $\forall i\ K_i'(x),\ K_i''(x),\ g_{iu_i}'(u_i, t),\ g_{iu_iu_i}''(u_i, t),\ h_{ix}'(x, t),\ h_{ixx}''(x, t)$ are continuous in all variables and, furthermore, there exist positive numbers $\varepsilon_1, \varepsilon_2$ such that for all t, x, u_i

$$g_{iu_iu_i}''(u_i, t) \geqslant \varepsilon_1 I,\ \varepsilon_2 I \geqslant h_{ixx}''(x, t),\ \varepsilon_2 I \geqslant K_i''(x), \tag{47}$$

where I denotes the identity matrix.

Then, there is a $T_0 > 0$ such that if the duration $T = t_f - t_0$ of the game satisfies $T < T_0$, then this game has an equilibrium solution.

The proof of this theorem can be found in [12]. The theorem remains valid also for the game with nonlinear dynamics, provided f is twice continuously differentiable with respect to (x, u). (It follows from the fact, that for small $T f$ can be linearized in x and u). In [6], where games with linear dynamics and quadratic cost functionals are considered, it is shown, that, in general, the theorem is false for arbitrary duration T. Finally, it is worth to note, that sufficient conditions for the existence of the equilibrium for linear-quadratic games can be obtained from the analysis of the Riccati equations [1]. The form of such conditions is, however, rather complicated and their verification may not be an easy task.

6. Conclusion

The analysis of open-loop games is only one, and possibly less important aspect of the differential game theory. However, in the situation, where there exists no result concerned with the existence of closed-loop equilibria, it is interesting to note, that the existence of open-loop equilibria for linear games, and open-loop ε-equilibria for nonlinear ones, is not a rare phenomenon.

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O istnieniu punktów równowagi Nasha dla gier różniczkowych z liniowymi i nieliniowymi równaniami stanu

Przedstawiono warunki wystarczające istnienia rozwiązania równowagi Nasha bez sprzężenia zwrotnego dla gier różniczkowych nieantagonistycznych (z niezerową sumą). W pierwszej części podsumowano znane z literatury wyniki dotyczące gier z liniowymi równaniami stanu i wypukłymi funkcjonałami kosztów, w drugiej części natomiast przedstawiono nowy rezultat stwierdzający istnienie punktu równowagi w klasie uogólnionych sterowań dla dowolnych gier nieliniowych. Dowody twierdzeń o istnieniu punktów równowagi oparto na twierdzeniu o punkcie stałym Fana-Glicksberga.

О существоваию точек равновесия Нэша для дифференциальных игр с линейными и нелинейными уровнениями состоянйя

В статьи рассматривается проблема существования программных точек равновесия Нэша неантагонистических дифференциальных игр многих лиц. Формулируются достаточные условия существования точек равновесия в чистых стратегиях для игр с линейными уровнениями состояния и выпуклыми функционалами качества и достаточные условия существования точек равновесия в обобщенных стратегиях для общего класса игр с произвольными уровнениями состояния. Для доказательства приведеных теорем используется известную теорему Фана-Гликсберга о неподвижной точке.

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STANISŁAW KURCYUSZ

1947—1978

On March 12, 1978 died tragically in Cleveland, USA, Dr. STANISLAW KURCYUSZ, a talented Polish control engineer and mathematician.

Dr. Stanislaw Kurcyusz was born on December 12, 1947, in Warsaw, Poland. He received his M.Sc. degree in Electronics from Technical University of Warsaw in 1971, with distinction. At the same time he studied Mathematics at the University o Warsaw. He obtained his Ph.D. degree in Automatic Control in 1974 from the Technical University of Warsaw after defending the dissertation "Necessary optimality conditions for problems with function space constraints", with special distinction. From 1974 he was an Assistant Professor in the Institute of Automatic Control, Faculty of Electronic Engineering, Technical University of Warsaw. From August 1977 he was on a leave to the Case Western Reserve University, Department of System Engineering, Computer Engineering and Information Sciences where he gave lectures as a Visiting Professor. He was an excellent and dedicated teacher, beloved by his students.

Dr. Stanislaw Kurcyusz in few years become an author of many important contributions to the field o control and optimization theory and applications. He developed regularity and normality conditions or general optimization problems in Banach spaces with operator equality and inequality constraints. Applied abstract methods to systems with delays and obtained regularity conditions and necessary optimality conditions for optimal control problems with state constraints of function space type. He worked also on the problems of projection on cones, generalized lagrangians and penalty function methods in optimization; the results in this field are of great importance both for theory and for applications in numerical algorithms. His recent brilliant work on Φ-convexity concept is a generalization of well-known separation theorems to the case of nonlinear separating or cupporting functionals. His late research was related to various types of penalty algorithms for optimization. He received many awards for his work, including an award of the Polish Academy of Sciences. The scientific findings of S. Kurcyusz are kept alive and are being extended by his students in Warsaw and colleagues in Poland and other countries.

Dr Stanislaw Kurcyusz was a man of great culture and broad horizons, of a gentle and warm character. The charm of his personality will be long remembered by those who knew him well.

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