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Nash feedback strategies for a two-person partial differential game with a Neumann-type parabolic system

by

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Nash equilibrium solution for a two-person nonzero-sum linear quadratic closed-loop partial differential game with a parabolic system having Neumann boundary conditions is developed using the optimal control approach. It is assumed that the first player exercises his control in the spatial domain and the other acts with his at the boundary. Feedback strategies are characterized by a system of integro-differential Riccati equations.

1. Introduction

Differential games with infinite dimensional systems have been recently widely studied, especially from the theoretical viewpoint. We recall the papers by Bensoussan, Grove and Papadakis, Chan, Ichikawa, and Underwood (references [1]–[5]). In Bensoussan's paper [1], the problem of Nash points for N-person games with linear systems in Hilbert spaces (including those of parabolic type) has been considered. Open-loop and closed-loop strategies have been derived and characterized by solutions of operator Riccati equations, similar to those of Starr and Ho [6].

Since infinite-dimensional models of real-life phenomena hardly exist yet in the fields where the game theory is applied, i.e. in economics and decision-making, so do partial differential games. It is worthwile to point out, however, that Beckmann [7] has developed some parabolic equations as models of economic diffusion processes for cases where spread of information about economic variables as well as movement of commodities are important. He has also mentioned wave equations in this context. Distributed models of investory replenishment policies involving deteriorating items (for instance food products) have been presented in [8]. In this case "spatial" variable denotes a deterioration state. Models like those developed in [7] and [8] make applications of partial differential games more realistic.

Here Nash strategies are derived for a two-person nonzero-sum linear quadratic feedback differential game with a parabolic system having Neumann-type boundary conditions. It is assumed that the first player uses his control in the spatial domain and the other is allowed to act on the boundary. The game problem is formulated in Section 2. Next, the first player's optimal feedback strategy is found under the assumption that the second player applies an arbitrary linear feedback strategy. In Section 4 we do the same for the second player and present a final system of Riccati integro-differential equations characterizing kernels of the Nash feedback operators. By Bensoussan's result [1], our Nash point is unique in the class of all feedback strategies.

2. Problem formulation

Let Ω be an open set in R^n with boundary Γ which is a C^∞-manifold of dimension $(n-1)$. Locally, Ω is on one side of Γ. Let t denote time, $t\in[0,T]$, $T<\infty$. We define

$$Q=\Omega\times(0,T), \quad \Sigma=\Gamma\times(0,T)$$

Let $a_{ij}(x,t)$, $i,j=1,2,\ldots,n$, and $a_0(x,t)$ be given functions in $L^\infty(Q)$ satisfying the conditions

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\,\xi_i\,\xi_j\geqslant\alpha\sum_{i=1}^{n}\xi_i^2$$

$$a_0(x,t)\geqslant\beta<0$$

for $(x,t)\in Q$, where $\alpha>0$, $\xi_i\in R$. Define on Σ another function $q(s,t)\in L^\infty(\Sigma)$ such that

$$q(s,t)\geqslant 0 \quad \text{for} \quad (s,t)\in\Sigma.$$

We consider the following scalar parabolic system of second order

$$\begin{cases} \dfrac{\partial y(x,t)}{\partial t}+(A(t)\,y(\cdot,t))(x)=u_1(x,t) & \text{on } Q,\\ \dfrac{\partial y(s,t)}{\partial\eta_A}+q(s,t)\,y(s,t)=u_2(s,t) & \text{on } \Sigma,\\ y(x,0)=y_0(x) & \text{on } \Omega, \end{cases} \tag{1}$$

where

$$(A(t)\,y(\cdot,t))(x)=-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial y(x,t)}{\partial x_j}\right)+a_0(x,t)\,y(x,t) \tag{2}$$

and

$$\frac{\partial y(s,t)}{\partial\eta_A}=\sum_{i,j=1}^{n}a_{ij}(s,t)\frac{\partial y(s,t)}{\partial x_j}\cos(\eta,x_i) \tag{3}$$

η — vector unit outward normal to Γ.

So the state y is given by the solution of a mixed problem (in the sense of Hadamard) with a Neumann boundary condition. External forces u_1 and u_2, called further policies or controls, are exercised in the spatial domain Ω and through its boundary Γ, respectively (Fig. 1). It is assumed that u_1 and u_2 are generated by two players in a two-person differential game.

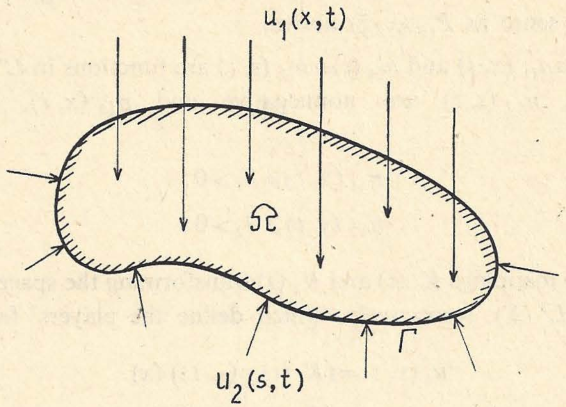

Fig. 1. Location of the player's controls: $u_1(x, t)$ in the domain Ω, $u_2(s, t)$—on the boundary Γ.

It is known that if the data u_1, u_2, y_0 are of appropriate classes L^2 then there exists a unique solution y to (1) with the properties

$$\begin{aligned} &y(x, t) \in L^2(0, T; H^1(\Omega)), \qquad (x, t) \in Q \\ &y(s, t) \in L^2(0, T; H^{1/2}(\Gamma)), \qquad (s, t) \in \Sigma, \end{aligned} \tag{4}$$

where $H^1(\Omega)$ and $H^{1/2}(\Gamma)$ are Sobolev spaces.

As in nonzero-sum games [6] we associate with each player a quadratic payoff functional given by $(i=1, 2)$

$$\begin{aligned} J_i(u_1, u_2) = \langle P_{iT}\, y(T), y(T)\rangle_{L^2(\Omega)} + \int_0^T \langle Q_i(t)\, y(t), y(t)\rangle_{L^2(\Omega)}\, dt + \\ + \int_0^T [\langle n_{i1}(t)\, u_1(t), u_1(t)\rangle_{L^2(\Omega)} + \\ + \langle n_{i2}(t)\, u_2(t), u_2(t)\rangle_{L^2(\Gamma)}]\, dt, \end{aligned} \tag{5}$$

where P_{iT} and $Q_i(t)$ are bounded linear mappings transforming $L^2(\Omega)$ into itself, such that

$$\begin{aligned} &(1) \quad P_{iT}\, y(T) = (P_{iT}\, y(\cdot, T))(x) = \int_\Omega P_{iT}(x, \xi)\, y(\xi, T)\, d\xi, \\ &P_{iT}(x, \xi) - \text{kernel of } P_{iT} \\ &P_{iT}(x, \xi) \in L^\infty(\Omega \times \Omega), \qquad P_{iT}(x, \xi) = P_{iT}(\xi, x) \\ &\int_\Omega P_{iT}(x, \xi)\, \varphi(\xi)^2\, d\xi \geq 0 \qquad \text{for all } \varphi \in L^2(\Omega). \end{aligned} \tag{6}$$

$$(2)\quad Q_i(t)\,y(t)=\big(Q_i(t)\,y(\cdot,t)\big)(x)=\int_\Omega Q_i(x,\xi,t)\,y(\xi,t)\,d\xi,$$

$$Q_i(x,\xi,t) \text{ – kernel of } Q_i(t) \tag{7}$$

$$Q_i(x,\xi,t)\in L^\infty(\Omega\times Q)$$

$Q_i(x,\xi,t)$ — symmetric with respect to x and ξ, and nonnegative definite for all $t\in(0,T)$ (in the sense as $P_{iT}(x,\xi)$ above).

In (5), $n_{i1}(t)\equiv n_{i1}(x,t)$ and $n_{i2}(t)\equiv n_{i2}(s,t)$ are functions in $L^\infty(Q)$ and $L^\infty(\Sigma)$, where $n_{21}(x,t)$, $n_{12}(s,t)$ are nonnegative and $n_{11}(x,t)$, $n_{22}(s,t)$ strictly positive, i.e.

$$\begin{aligned} n_{11}(x,t)&\geqslant\vartheta_1>0,\\ n_{22}(s,t)&\geqslant\vartheta_2>0. \end{aligned} \tag{8}$$

Introduce two mappings $K_1(t)$ and $K_2(t)$ transforming the space $L^2\big(0,T;L^2(\Omega)\big)$ into $L^2(Q)$ and $L^2(\Sigma)$, respectively, which define the players' feedback strategies

$$\begin{aligned} u_1(x,t)&=\big(K_1(t)\,y(\cdot,t)\big)(x)\\ u_2(s,t)&=\big(K_2(t)\,y(\cdot,t)\big)(s). \end{aligned} \tag{9}$$

Let $K_1(t)$ and $K_2(t)$ be such that the following equation

$$\begin{cases} \dfrac{\partial y(x,t)}{\partial t}+\big(A(t)\,y(\cdot,t)\big)(x)=\big(K_1(t)\,y(\cdot,t)\big)(x) & \text{on } Q\\ \dfrac{\partial y(s,t)}{\partial\eta_A}+q(s,t)\,y(s,t)=\big(K_2(t)\,y(\cdot,t)\,(s)\big) & \text{on } \Sigma\\ y(x,0)=y_0(x) & \text{on } \Omega \end{cases} \tag{10}$$

has a unique solution, so the game makes sense.

Using (9), one can formally write $J_i(u_1,u_2)$ as

$$\tilde J_i(K_1,K_2)=J_i(K_1\,y,K_2\,y). \tag{11}$$

The players to play best should perform minimization of $\tilde J_1$ and $\tilde J_2$ with respect to $K_1(t)$ and $K_2(t)$, respectively.

The game problem: Find optimal feedback strategies $\hat K_1(t)$ and $\hat K_2(t)$ defined by the following inequalities

$$\begin{aligned} \tilde J_1(\hat K_1,\hat K_2)&\leqslant\tilde J_2(K_1,\hat K_2)\\ \tilde J_2(\hat K_1,\hat K_2)&\leqslant\tilde J_2(\hat K_1,K_2) \end{aligned} \tag{12}$$

for all $K_1(t)$ and $K_2(t)$ such that the system (10) is well posed.

The pair $\big(\hat K_i(t),\hat K_2(t)\big)$ is called a Nash equilibrium solution to the partial differential game.

3. First player's feedback strategy

For the time being we assume that $K_2(t)$ is linear. Since $u_2=K_2 y$, so for the first player the system (1) formally looks like

$$\begin{cases} \dfrac{\partial y(x,t)}{\partial t}+\big(A(t)\,y(\cdot,t)\big)(x)=u_1(x,t) & \text{on } Q \\ \dfrac{\partial y(s,t)}{\partial \eta_A}+q(s,t)\,y(s,t)=\big(K_2(t)\,y(\cdot,t)\big)(s) & \text{on } \Sigma \\ y(x,0)=y_0(x) & \text{on } \Omega \end{cases} \tag{13}$$

and the payoff functional $J_1(u_1, u_2)$ becomes

$$J_{1f}(u_1)=J_1(u_1, K_2 y)=\langle P_{1T}\,y(T), y(T)\rangle_{L^2(\Omega)}+ \int_0^T \big\langle \big(Q_1(t)+L_1(t)\big)\,y(t), y(t)\big\rangle_{L^2(\Omega)}\,dt+ \int_0^T \langle n_{11}(t)\,u_1(t), u_1(t)\rangle_{L^2(\Omega)}\,dt\,, \tag{14}$$

where

$$L_1(t)\,y(t)=\big(K_2^*(t)\,(n_{12}(\cdot,t)\,\big(K_2(t)\,y(\cdot\cdot,t)\big)(\cdot)\big)(x)\,. \tag{15}$$

$K_2^*(t)$ is defined by

$$\langle \varphi, K_2(t)\,\psi\rangle_{L^2(\Gamma)}=\langle K_2^*(t)\,\varphi, \psi\rangle_{L^2(\Omega)} \tag{16}$$

for all $\varphi\in L^2(\Gamma)$ and $\psi\in L^2(\Omega)$. Assumed linearity of $K_2(t)$ justifies (16).

Now we have two standard problems to solve

(1) Find optimal open-loop policy $\hat{u}_1\in L^2(Q)$ defined by

$$J_{1f}(\hat{u}_1)\leqslant J_{1f}(u_1)$$

for all $u_1\in L^2(Q)$. The existence of such $\hat{u}_1$ may be proved by methods given in [9] and is assumed here.

(2) Express $\hat{u}_1$ in a feedback form

$$\hat{u}_1(x,t)=\big(\hat{K}_1(t)\,\hat{y}(\cdot,t)\big)(x), \qquad \hat{y}=y(\hat{u}_1) \tag{17}$$

determining the optimal feedback strategy $\hat{K}_1(t)$ (in dependence on $K_2(t)$).

We have the following result

PROPOSITION 1. The first player's optimal policy $\hat{u}_i$ and the resulting solution $\hat{y}$ minimize the payoff functional (14) if and only if

$$\hat{u}_1(x,t)=-n_{11}(x,t)^{-1}\,p_1(x,t) \tag{18}$$

for almost all $(x,t)\in Q$, where $p_1(x,t)$ is the solution of the adjoint system*

$$\begin{cases} -\dfrac{\partial p_1(x,t)}{\partial t}+\big((A^*(t)-K_2^*(t))\,p_1(\cdot,t)\big)(x)= \\ \qquad\qquad =\big((Q_1(t)+L_1(t))\,\hat{y}(\cdot,t)\big)(x) & \text{on } Q \\ \dfrac{\partial p_1(s,t)}{\partial\eta_{A^*}}+q(s,t)\,p_1(s,t)=0 & \text{on } \Sigma \\ p_1(x,T)=(P_{1T}\,\hat{y}(\cdot,T))(x)\,. & \text{on } \Omega \end{cases} \tag{19}$$

The minimal value of (15) is given by

$$J_{1f}(\hat{u}_1)=\int_\Omega p_1(x,0)\,y_0(x)\,dx\,. \tag{20}$$

Proof. See Appendix A.

Since for a linear $K_2(t)$ the solution p_1 to (19) has the same properties as $\hat{y}$ in (4) we can apply the Schwartz kernel theorem [9] and seek a representation

$$p_1(x,t)=\int_\Omega P_1(x,\xi,t)\,\hat{y}(\xi,t)\,d\xi \tag{21}$$

such that the function kernel P_1 is symmetric with respect to x and ξ

$$p_1(x,\xi,t)=P_1(\xi,x,t)\,. \tag{22}$$

We claim that

PROPOSITION 2. The kernel P_1 of (21) satisfies the following equation

$$\begin{aligned}\int_\Omega\Big[&-\frac{\partial P_1(x,\xi,t)}{\partial t}+\big(A_x^*(t)\,P_1(\cdot,\xi,t)\big)(x)+A_\xi^*(t)\,P_1(x,\cdot,t)\big)(\xi)+\\ &+\int_\Omega P_1(x,\zeta,t)\,n_{11}(\zeta,t)^{-1}\,P_1(\zeta,\xi,t)\,d\zeta-\\ &-\big(K_2^*(t)\,P_1(\cdot,\xi,t)\big)(x)-Q_1(x,\xi,t)\Big]\hat{y}(\xi,t)\,d\xi=\\ &=\int_\Gamma P_1(x,s,t)\,\big(K_2(t)\,y(\cdot,t)\big)(s)\,ds+\\ &+\big(K_2^*(t)\,n_{12}(\cdot,t)\,\big(K_2(t)\,y(\cdot\cdot,t)\big)(\cdot)\big)(x) \qquad \text{on } Q\end{aligned} \tag{23}$$

with the terminal and boundary conditions

$$\begin{cases} P_1(x,\xi,T)=P_{1T}(x,\xi) & \text{on } \Omega\times\Omega \\ \dfrac{\partial P_1(x,s,t)}{\partial\eta_{A^*}}+q(s,t)\,P_1(x,s,t)=0 & \text{on } \Omega\times\Sigma \\ \dfrac{\partial P_1(s,\xi,t)}{\partial\eta_{A^*}}+q(s,t)\,P_1(s,\xi,t)=0 & \text{on } \Gamma\times Q\,. \end{cases} \tag{24}$$

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* Adjoint $A^*(t)$ to $A(t)$ is defined by the Green's formula

$$\langle A(t)\,\varphi,\psi\rangle_{L^2(\Omega)}+\left\langle\frac{\partial\varphi}{\partial\eta_A},\psi\right\rangle_{L^2(\Gamma)}=\langle\varphi,A^*(t)\,\psi\rangle_{L^2(\Omega)}+\left\langle\varphi,\frac{\partial\psi}{\partial\eta_{A^*}}\right\rangle_{L^2(\Gamma)}.$$

Proof. The proof is given in Appendix B.

Using (21), the optimal control $\hat{u}_1$ can be written in the following form

$$\hat{u}_1(x,t)=-n_{11}(x,t)^{-1}\int_\Omega P_1(x,\xi,t)\,\hat{y}(\xi,t)\,d\xi=\big(\hat{K}_1(t)\,\hat{y}(\cdot,t)\big)(x) \tag{25}$$

which defines the first player's optimal feedback strategy $\hat{K}_1(t)$ in dependence on some linear $K_2(t)$.

4. Second player's feedback trategy

This time, instead of writing u_1 with the help of $K_1 y$, we employ the expression (25) explicitly. Thus the second player wishes to minimize the cost

$$\begin{aligned} J_{2f}(u_2)=\langle P_{2T}\,y(T),y(T)\rangle_{L^2(\Omega)}&+\\ &+\int_0^T \big\langle (Q_2(t)+L_2(t))\,y(t),y(t)\big\rangle_{L^2(\Omega)}\,dt+\\ &+\int_0^T \langle n_{22}(t)\,u_2(t),u_2(t)\rangle_{L^2(\Gamma)}\,dt \end{aligned} \tag{26}$$

with respect to $u_2\in L^2(\Sigma)$, where

$$\begin{aligned} L_2(t)\,y(t)=\int_\Omega\Big[\int_\Omega P_1(x,\zeta,t)\,n_{11}(\zeta,t)^{-1}\,n_{21}(\zeta,t)\,n_{11}(\zeta,t)^{-1}\cdot\\ \cdot P_1(\zeta,\xi,t)\,d\zeta\Big]\,y(\xi,t)\,d\xi, \end{aligned}$$

subject to the restriction

$$\begin{cases} \dfrac{\partial y(x,t)}{\partial t}+\big(A(t)\,y(\cdot,t)\big)(x)=-n_{11}(x,t)^{-1}\displaystyle\int_\Omega P_1(x,\xi,t)\,y(\xi,t)\,d\xi\\ \dfrac{\partial y(s,t)}{\partial\eta_A}+q(s,t)\,y(s,t)=u_2(s,t)\\ y(x,0)=y_0(x). \end{cases} \tag{27}$$

Deriving similarly as in Appendix A one obtains

(i) optimal control

$$\hat{u}_2(s,t)=-n_{22}(s,t)^{-1}\,p_2(s,t) \tag{28}$$

for almost all $(s,t)\in\Sigma$;

(ii) adjoint system

$$\begin{cases} -\dfrac{\partial p_2(x,t)}{\partial t}+\big(A^*(t)\,p_2(\cdot,t)(x)+\displaystyle\int_\Omega P_1(x,\xi,t)\,n_{11}(\xi,t)^{-1}\,p_2(\xi,t)\,d\xi=\\ \qquad\qquad=\big((Q_2(t)+L_2(t))\,\hat{y}(\cdot,t)\big)(x)\\ \dfrac{\partial p_2(s,t)}{\partial\eta_{A^*}}+q(s,t)\,p_2(s,t)=0\\ p_2(x,T)=\big(P_{2T}\,\hat{y}(\cdot,T)\big)(x) \end{cases} \tag{29}$$

$$\hat{y}(x,t)=y(x,t;\hat{u}_2);$$

(iii) minimal value of (26)

$$J_{2f}(\hat{u}_2)=\int_\Omega p_2(x,0)\,y_0(x)\,dx\,. \tag{30}$$

As before we take

$$p_2(x,t)=\int_\Omega P_2(x,\xi,t)\,\hat{y}(\xi,t)\,d\xi \tag{31a}$$

$$P_2(x,\xi,t)=P_2(\xi,x,t)\,. \tag{31b}$$

Using (31a, b) in (29) yields (compare Appendix B)

(iv) integro-differential system of Riccati type

$$\begin{cases} -\dfrac{\partial P_2(x,\xi,t)}{\partial t}+\big(A_x^*(t)\,P_2(\cdot,\xi,t)\big)(x)+\big(A_\xi^*(t)\,P_2(x,\cdot,t)\big)(\xi)- \\ \quad -\int_\Omega P_1(x,\xi,t)\,n_{11}(\xi,t)^{-1}\,n_{21}(\zeta,t)\,n_{11}(\zeta,t)^{-1}\,P_1(\zeta,\xi,t)\,d\zeta+ \\ \quad\quad +\int_\Omega P_1(x,\zeta,t)\,n_{11}(\zeta,t)^{-1}\,P_2(\zeta,\xi,t)\,d\zeta+ \\ \quad\quad +\int_\Omega P_2(x,\zeta,t)\,n_{11}(\zeta,t)^{-1}\,P_1(\zeta,\xi,t)\,d\zeta+ \\ \quad\quad +\int_\Gamma P_2(x,s,t)\,n_{22}(s,t)^{-1}\,P_2(s,\xi,t)\,ds=Q_2(x,\xi,t) \\ P_2(x,\xi,T)=P_{2T}(x,\xi) \\ \dfrac{\partial P_2(x,s,t)}{\partial\eta_{A^*}}+q(s,t)\,P_2(x,s,t)=0 \\ \dfrac{\partial P_2(s,\xi,t)}{\partial\eta_{A^*}}+q(s,t)\,P_2(s,\xi,t)=0\,. \end{cases} \tag{32}$$

By (28) and (31a) one obtains

(iiv) second player's feedback strategy

$$\hat{u}_2(s,t)=-n_{22}(s,t)^{-1}\int_\Omega P_2(s,\xi,t)\,\hat{y}(\xi,t)\,d\xi=\big(\hat{K}_2(t)\,\hat{y}(\cdot,t)\big)(s) \tag{34}$$

determining the optimal operator $\hat{K}_2(t)$.

Its adjoint $\hat{K}_2^*(t)$ can be written as (see (16))

$$\big(\hat{K}_2^*(t)\,\varphi(\cdot)\big)(x)=-\int_\Gamma P_2(x,s,t)\,n_{22}(s,t)^{-1}\,\varphi(s)\,ds$$

for $\varphi\in L^2(\Gamma)$. Using $\hat{K}_2(t)$ and $\hat{K}_2^*(t)$ in (23) yields

$$-\frac{\partial P_1(x,\xi,t)}{\partial t}+\big(A_x^*(t)\,P_1(\cdot,\xi,t)\big)(x)+\big(A_\xi^*(t)\,P_1(x,\cdot,t)\big)(\xi)+$$
$$+\int_\Omega P_1(x,\zeta,t)\,n_{11}(\zeta,t)^{-1}\,P_1(\zeta,\xi,t)\,d\xi+$$
$$+\int_\Gamma P_1(x,s,t)\,n_{22}(s,t)^{-1}\,P_2(s,\xi,t)\,ds-$$

$$+\int_\Gamma P_2(x,s,t)\,n_{22}(s,t)^{-1}P_1(s,\xi,t)\,ds-$$
$$-\int_\Gamma P_2(x,s,t)\,n_{22}(s,t)^{-1}\,n_{12}(s,t)\,n_{22}(s,t)^{-1}\,P_2(s,\xi,t)\,ds=$$
$$=Q_1(x,\xi,t) \qquad (35)$$

with terminal and boundary conditions given by (24).

Thus we have found that the Nash feedback strategies $\hat{K}_1(t)$ and $\hat{K}_2(t)$ are linear integral operators with the kernels

$$-n_{11}(x,t)^{-1}P_1(x,\cdot,t) \quad \text{and} \quad -n_{22}(s,t)^{-1}P_2(s,\cdot,t),$$

respectively, where P_1 and P_2 satisfy the system of Riccati equations (32), (35). Minimal payoffs are given by

$$J_1(\hat{u}_1,\hat{u}_2)=\int_\Omega\int_\Omega y_0(x)\,P_1(x,\xi,0)\,y_0(\xi)\,dx\,d\xi$$
$$J_2(\hat{u}_1,\hat{u}_2)=\int_\Omega\int_\Omega y_0(x)\,P_2(x,\zeta,0)\,y_0(\xi)\,dx\,d\xi. \qquad (36)$$

Our theory has been developed under the initial restriction that the second player applies linear strategies only. Hence in the pair $(\hat{K}_1(t),\hat{K}_2(t))$, $\hat{K}_2(t)$ should be interpreted as the best strategy of linear strategies and $\hat{K}_1(t)$ as the best of all strategies (including nonlinear ones) for which the game makes sense. The other way of derivation is also possible, i.e. assuming a linear $K_1(t)$ we can find $\hat{K}_2(t)$ first and then $\hat{K}_1(t)$. Now $K_2(t)$ would be the best of all strategies. The two approaches suggest that the initial restriction concerning linearity might be withdrawn. This hypothesis has been proved in [1] (Theorem 3.1). Thus our pair $(\hat{K}_1(t),\hat{K}_2(t))$ constitutes a Nash point for all feedback strategies $(K_1(t),K_2(t))$.

Bensoussan has also shown [1] that Riccati operator equations associated with nonzero-sum differential games in Hilbert spaces posses unique solutions in $L^\infty(0,T;\mathcal{L}(H,H))$ (here $H=L^2(\Omega)$). So our Nash point is unique and the function kernels $P_1(x,\xi,t), P_2(x,\xi,t)$ are members of $L^\infty(0,T;D'(\Omega\times\Omega))$, where $D'(\Omega\times\Omega)$ denotes the space of distributions on $\Omega\times\Omega$ [9].

We summarize our results in the following statement.

Conclusion. The pair of optimal feedback strategies $(\hat{K}_1(t),\hat{K}_2(t))$, being a unique Nash point for the nonzero-sum differential game with the Neumann-type parabolic system (1) and the quadratic payoff functionals (5), is given by the formulae (25) and (34), where $P_1(x,\xi,t)$, $P_2(x,\xi,t)$ satisfy the system of Riccati equations (32), (35).

5. Final remarks

Feedback strategies for a two-person nonzero-sum partial differential game have been derived using the familiar optimal control approach. The case has been considered where the first player exercises his control in the spatial domain and the other acts with his on the boundary. The strategies have been characterized by a system of Riccati integro-differential equations.

In [10], Cruz and Chen have developed a perturbation technique for iterative computing of the ordinary Nash feedback strategies introduced by Starr and Ho [6]. Their algorithm may be applied in our case after expansion of the Riccati equations with respect to eigenfunctions of the operator $A(t)$.

APPENDIX A

It is well known that for positive quadratic forms like (14) the minimum condition

$$I_1(u_1-\hat{u}_1)=\left\langle \left.\frac{\partial J_{1f}(u_1)}{\partial u_1}\right|_{\hat{u}_1},\ u_1-\hat{u}_1 \right\rangle_{L^2(Q)}=0$$

for all $u_1 \in L^2(Q)$, is both necessary and sufficient.

Denote $v_1=u_1-\hat{u}_1$, so $I_1(v_1)$ becomes

$$I_1(v_1)=\langle P_{1T}\,\hat{y}(T), y(T)\rangle+\int_0^T \left\langle (Q_1(t)+L_1(t))\,\hat{y}(t), y(t)\right\rangle dt+ \\ +\int_0^T \langle n_{11}(t)\,\hat{u}_1(t), v_1(t)\rangle\, dt, \tag{A1}$$

where $y(t)=y(t;v_1)$ and the subscripts $L^2(\Omega)$ at $\langle\cdot,\cdot\rangle$ have been omitted for brevity. Observe that

$$y(x,0;v_1)=y(x,0;u_1)-y(x,0;\hat{u}_1)=0. \tag{A2}$$

We shall show that if (13), (18), and (19) are satisfied then

$$I_1(v_1)=0 \tag{A3}$$

for all $v_1 \in L^2(Q)$. Using the adjoint equation, the second term of $I_1(v_1)$ can be transformed as follows

$$\int_0^T \left\langle (Q_1(t)+L_1(t))\,\hat{y}(t), y(t)\right\rangle dt= \\ =\int_0^T \left\langle -\frac{\partial p_1(t)}{\partial t}+(A^*(t)-K_2^*(t))\,p_1(t), y(t)\right\rangle dt= \\ =\langle p_1(0), y(0)\rangle-\langle p_1(T), y(T)\rangle+\int_0^T \left\langle p_1(t), \frac{\partial y(t)}{\partial t}\right\rangle dt+ \\ +\int_0^T \left[\langle p_1(t), A(t)\,y(t)\rangle-\left\langle \frac{\partial p_1(t)}{\partial \eta_{A*}}, y(t)\right\rangle_{L^2(\Gamma)}+ \right. \\ \left. +\left\langle p_1(t), \frac{\partial y(t)}{\partial \eta_A}\right\rangle_{L^2(\Gamma)}\right] dt-\int_0^T \langle K_2^*(t)\,p_1(t), y(t)\rangle\, dt=$$

$$=-\langle p_1(T), y(T)\rangle + \int_0^T \Big[\langle p_1(t), v_1(t)\rangle -$$

$$-\Big\langle \frac{\partial p_1(t)}{\partial \eta_{A^*}} + q(t)\, p_1(t), y(t) \Big\rangle_{L^2(\Gamma)} +$$

$$+\langle p_1(t), K_2(t)\, y(t)\rangle_{L^2(\Gamma)} \Big] dt - \int_0^T \langle K_2^*(t)\, p_1(t), y(t)\rangle\, dt, \qquad \text{(A4)}$$

where integration by parts with respect to t, identity $y(0)=0$ given by (A2), Green's formula, state equation (13) and its boundary conditions have been applied in the sequence. Employing boundary conditions of (19) in (A4) and using the result in (A1) yields

$$I_1(v_1) = \langle P_{1T}\, \hat{y}(T) - p_1(T), y(T)\rangle + \int_0^T \langle p_1(t) + n_{11}(t)\, \hat{u}_1(t), v_1(t)\rangle\, dt.$$

Hence (A3) holds by (18) and the terminal condition of (19).

To prove (20) notice that the only difference between $J_{1f}(\hat{u}_1)$ and $I_1(\hat{u}_1)$ is nonzero initial state $y_0(x)$ in general, instead of the condition (A2). Therefore, transforming $J_{1f}(\hat{u}_1)$ in the same way as $I_1(v_1)$ above one could not omit the term $\langle p_1(0), y_0\rangle$ in (A4). Hence (20) holds.

APPENDIX B

We use the formula (21) in the adjoint equation. Its elements become

$$\text{(1)} \qquad -\frac{\partial p_1(x,t)}{\partial t} = -\int_\Omega \frac{\partial P_1(x,\xi,t)}{\partial t}\, \hat{y}(\xi,t)\, d\xi - $$

$$-\int_\Omega P_1(x,\xi,t) \frac{\partial \hat{y}(\xi,t)}{\partial t}\, d\xi,$$

where

$$-\int_\Omega P_1(x,\xi,t) \frac{\partial \hat{y}(\xi,t)}{\partial t}\, d\xi = \int_\Omega P_1(x,\xi,t)\, (A_\xi(t)\, \hat{y}(\cdot,t))(\xi)\, d\xi +$$

$$+\int_\Omega \int_\Omega P_1(x,\xi,t)\, n_{11}(\xi,t)^{-1}\, P_1(\xi,\zeta,t)\, \hat{y}(\zeta,t)\, d\xi\, d\zeta,$$

$$\int_\Omega P_1(x,\xi,t)\, (A_\xi(t)\, \hat{y}(\cdot,t))(\xi)\, d\xi = \int_\Omega (A_\xi^*(t)\, P_1(x,\cdot,t))(\xi)\, \hat{y}(\xi,t)\, d\xi +$$

$$+\int_\Gamma \frac{\partial P_1(x,s,t)}{\partial \eta_{A^*}}\, \hat{y}(s,t)\, ds - \int_\Gamma P_1(x,s,t) \frac{\partial \hat{y}(s,t)}{\partial \eta_A}\, ds,$$

$$-\int_\Gamma P_1(x,s,t) \frac{\partial \hat{y}(s,t)}{\partial \eta_A}\, ds = \int_\Gamma P_1(x,s,t)\, q(s,t)\, \hat{y}(s,t)\, ds -$$

$$-\int_\Gamma P_1(x,s,t)\, (K_2(t)\, y(\cdot,t))(s)\, ds;$$

(2) $$\left(A^*(t)\,p_1(\cdot, t)\right)(x) = \int_{\Omega} \left(A_x^*(t)\,P_1(\cdot, \xi, t)\right)(x)\,\hat{y}(\xi, t)\,d\xi;$$

(3) $$-\left(K_2^*(t)\,p_1(\cdot, t)\right)(x) = -\int_{\Omega} \left(K_2^*(t)\,P_1(\cdot, \xi, t)\right)(x)\,\hat{y}(\xi, t)\,d\xi.$$

Collecting transformed and remaining terms in (19) yields the equation (23) and the first boundary condition of (24). The other holds by spatial symmetry. The terminal condition in (24) follows that of (19).

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Pozycyjne strategie Nasha dla dwuosobowej cząstkowej gry różniczkowej z układem parabolicznym typu Neumanna

Stosując metodę optymalnego sterowania wyznaczono punkt równowagi Nasha pewnej dwuosobowej pozycyjnej liniowo-kwadratowej gry różniczkowej z niezerową sumą dla układu parabolicznego mającego warunki brzegowe typu Neumanna. Założono, że pierwszy gracz korzysta ze sterowania w domenie przestrzennej, natomiast drugi dysponuje swoim sterowaniem na granicy. Optymalne strategie pozycyjne scharakteryzowano za pomocą różniczkowo-całkowych równań Riccatiego.

Позиционные стратегии Нэша для распределенной дифференциальной игры двух лиц с параболической системой типа Неймана

Употребляя метод оптимального управления найдена точка равновесия Нэша некоторой позиционной линейно-квадратичной дифференциальной игры двух лиц с ненулевой суммой для параболической системы имеющей краевые условия типа Неймана. Предположено, что первий игрок использует своё управление во всей области, а второй употребляет своё на границы. Оптимальные позиционные стратегии характеризованных с помощью интегродифференциальных уравнений Риккати.