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# On some properties of two-layer parabolic free boundary value problems

by

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Existence, uniqueness and continuous dependence on given data of the solutions of two-layer parabolic free boundary problems describing dynamics of the underground gas reservoir are considered. It is shown that for considered boundary value problems maximum principle holds.

## 1. Introduction

Mathematical modelling of many physical processes related to filtration in porous media leads to so called one- or multi-layer free boundary problems. In particular such models describe dynamics of some underground gas reservoirs.

Mathematical models of filtration processes in the gas reservoir and properties of these models presented in the paper as well as numerical methods which will be presented in [13] may be useful in optimal control problems for pipeline networks containing underground gas reservoirs. The physical setting of problems related to filtration of gas and water in a porous medium and two equivalent mathematical models (describing pressure and filtration velocity distributions) are presented in [7, 12].

In this paper the correctness in the Hadamard sense and the maximum principle for parabolic free boundary problems are investigated.

Existence, uniqueness and stability (continuous dependence with respect to initial data, boundary data and coefficients) of classical solutions of multi-phase free boundary problems, known also as Stefan problem have been considered in many works [2, 3, 5, 15]. The problems we consider belong to the class of multi--layer free boundary problems known in Russian works as Verygin problems [8, 9, 15]. They differ in many aspects from Stefan problems [8].

Multi-layer free boundary problems have been investigated only in few works [6, 8, 9, 14, 16]. In [16] the analytical solutions for a particular case of constant initial and boundary data were obtained. In [8] existence (but not uniqueness) of the solution of the two-layer free boundary problem for linear parabolic equations in general form and with variable coefficients was proved by the use of Schauder's fixed point theorem. In [14, 15] existence and uniqueness for the Verygin problem with constant coefficients was obtained by employing Picard's method to the equivalent integral representation of the problem.

In [6] existence and uniqueness for the Cauchy free boundary problem was considered. We extend the results given in [6] to the case of free boundary value problems. Moreover we prove stability property of the solutions of considered problems.

In Section 3 we recall the formulation of Problems $(A_k)$, $(B_k)$, $k=1, 2$, introduced in [7, 12]. Problems $(A_k)$ are formulated in terms of the pressure distribution and $(B_k)$ — in terms of the velocity distribution in the underground gas reservoir.

In Section 4 we introduce the equivalent integral representation of the Problems $(B_k)$.

In Section 5 results relating to existence, uniqueness and stability of the solutions are presented together with shortening of the proof. This proof proceeds in much the same manner as in [6].

In Section 6 the maximum principle for Problems $(A_k)$, $(B_k)$ is demonstrated. In particular, it is shown that the extremal values of the solution cannot be attained on the free boundary.

## 2. Notations and conventions

$$D=\{(x,t)\mid x\in(0,l),\ t\in(0,T)\},$$
$$D_1=\{(x,t)\mid x\in\big(0,y(t)\big),\ t\in(0,T)\},$$
$$D_2=\{(x,t)\mid x\in\big(y(t),l\big),\ t\in(0,T)\},$$
$$D_i^0=\bar{D}_i\setminus\{(x,t)\in\bar{D}_i\mid t=0\},$$
$$\Gamma=\{(x,t)\mid x=y(t),\ t\in(0,T)\},$$
$$\Gamma^0=\bar{\Gamma}\setminus\{(y_0,0)\},$$
$$Z_1=\{(x,t)\mid x\in(0,y_0),\ t=0\},$$
$$Z_2=\{(x,t)\mid x\in(y_0,l),\ t=0\},$$
$$S_i=\{(l_i,t)\mid t\in(0,T)\},\ l_1=0,\ l_2=l,$$
$$S_i^0=\bar{S}_i\setminus\{(l_i,0)\},$$
$$Z=Z_1\cup Z_2,\ S=S_1\cup S_2$$

where $T>0$, $l>0$, $y(t)\in(0,l)$ for $t\in[0,T]$, $y(0)=y_0$; $y(t)$ denotes the location of the gas-water interface, $p_i(x,t)$ denotes the pressure and $u_i(x,t)$ — filtration velocity of fluid at the point $(x,t)\in D_i$.

Lower index $i=1$ corresponds everywhere to the gas domain and $i=2$ to the water domain.

By $\bar{Q}$ we denote the closure of the set $Q$.

According to the terminology of [15], the domains $D_i$ we call layers and $\Gamma$ — the free boundary between layers.

$C^m(Q)$ is the class of functions $m$—times continuously differentiable in the set $Q$;

$$L_\alpha = \frac{\partial}{\partial t} - \alpha \frac{\partial^2}{\partial x^2}, \qquad \alpha > 0;$$

$$E(x, t) = \begin{cases} (4\pi t)^{-\frac{1}{2}} \exp\left(-\frac{x^2}{4t}\right), & t > 0, \quad x \in R \\ 0 & , \quad t \leqslant 0, \quad x \in R \end{cases}$$

— fundamental solution of the heat equation $L_1 u = 0$;

$$G_i(x, \xi, t) = E(x - \xi, \alpha_1 t) + (-1)^i E(x + \xi, \alpha_1 t),$$

$$H_i(x, \xi, t) = E(x - \xi, \alpha_2 t) + (-1)^i E(x + \xi - 2l, \alpha_2 t),$$

$G_i$, $i = 1,2$—respectively Green and Neumann functions for the heat equation $L_{\alpha_1} u = 0$ in the domain $\{(x, t) \mid x > 0, \ t > 0\}$, $H_i$, $i = 1,2$—Green and Neumann functions for the equation $L_{\alpha_2} u = 0$ in the domain $\{(x, t) \mid x < l, \ t > 0\}$.

## 3. Formulation of two-layer free boundary value problems

We will consider the following problems:

*Problem* $(A_k)$, $k = 1,2$.

Find functions $p_1, p_2, y$ satisfying:

— system of parabolic equations

$$L_{\alpha_i} p_i = 0 \text{ in } D_i, \qquad i = 1,2; \tag{3.1}$$

— initial conditions

$$\begin{aligned} y(0) &= y_0, \quad \text{where } y_0 \in (0, l), \\ p_i(x, 0) &= p_{i0}(x) \quad \text{in } Z_i; \end{aligned} \tag{3.2}$$

— boundary conditions of the Dirichlet type in the case of Problem $(A_1)$

$$p_i(l_i, t) = f_i(t), \qquad t \in (0, T] \tag{3.3}$$

and conditions of the Neumann type in the case of Problem $(A_2)$

$$-a_i \frac{\partial p_i}{\partial x}(l_i, t) = F_i(t), \qquad t \in (0, T]; \tag{3.3'}$$

— conditions on the free boundary

$$\begin{aligned} & p_1(y(t), t) = p_2(y(t), t), \\ & a_1 \frac{\partial p_1}{\partial x}(y(t), t) = a_2 \frac{\partial p_2}{\partial x}(y(t), t), \\ & \frac{dy}{dt}(t) = -\beta a_1 \frac{\partial p_1}{\partial x}(y(t), t), \ t \in (0, T]. \end{aligned} \tag{3.4}$$

Here $\alpha_i>0$, $a_i>0$, $i=1,2$, $\beta>0$, $y_0$ are given constants and $p_{i0}$, $f_i$, $F_i$ are given functions. It was shown in [12] that under some additional regularity conditions for the data Problems $(A_k)$ can be formulated in terms of velocities $u_i$ which are related with pressures $p_i$ by the linear Darcy law [1]

$$u_i=-a_i\frac{\partial p_i}{\partial x}. \tag{3.5}$$

The formulation in terms of velocities has the following form:

*Problem* $(B_k)$, $k=1,2$

Find functions $u_1$, $u_2$, $y$ satisfying:
— system of equations $L_{\alpha_i}u_i=0$ in $D_i$, $i=1,2$;
— initial conditions

$$y(0)=y_0,$$

$$u_i(x,0)=u_{i0}(x) \text{ in } Z_i;$$

— boundary conditions of the Dirichlet type in the case of Problem $(B_1)$

$$u_i(l_i,t)=F_i(t), \quad t\in(0,T]$$

and conditions of the Neumann type in the case of Problem $(B_2)$

$$\frac{\partial u_i}{\partial x}(l_i,t)=\varphi_i(t), \quad t\in(0,T];$$

— conditions on the free boundary

$$u_1(y(t),t)=u_2(y(t),t),$$

$$\left(\frac{1}{a_1}-\frac{1}{a_2}\right)u_1(y(t),t)\frac{dy}{dt}(t)+\frac{\alpha_1}{a_1}\frac{\partial u_1}{\partial x}(y(t),t)=\frac{\alpha_2}{a_2}\frac{\partial u_2}{\partial x}(y(t),t),$$

$$\frac{dy}{dt}(t)=\beta u_1(y(t),t), \quad t\in(0,T].$$

Here

$$u_{i0}\stackrel{\Delta}{=}-a_i\frac{dp_{i0}}{dx}, \qquad \varphi_i\stackrel{\Delta}{=}-\frac{a_i}{\alpha_i}\frac{df_i}{dt}. \tag{3.6}$$

Solutions of all introduced problems we understand in the classical sense [12].

We make use of the following conditions:

(C1) $p_{10}\in C^1[0,y_0]$, $\quad p_{20}\in C^1[y_0,l]$;

(C2) $f_i\in C^1[0,T]$;

(C3) $\dfrac{\partial p_i}{\partial x}\in C^0(\bar{D}_i)$, $\quad \dfrac{\partial^2 p_i}{\partial x^2}\in C^0(D_i^0)$;

(C4) functions $p_i(y(t), t)$ are differentiable with respect to $t$ in the interval $(0, T]$;

(C5) functions $p_i$ satisfy equations (3.1) on $\Gamma^0$;

(C6) functions $p_i$ satisfy equations (3.1) on $S_i^0$.

Using the above conditions equivalence of Problems $(A_k)$ and $(B_k)$ can be formulated in the form of following lemmas:

LEMMA 3.1 [12]. If

(1) $u_i$ are related with $p_i$ by (3.5),

(2) functions $u_{i0}$, $\varphi_i$ are defined by (3.6),

(3) conditions (C1), (C2) are satisfied

then

(I) if $\{p_1, p_2, y\}$ is the solution of Problem $(A_2)$ satisfying conditions (C3)–(C5) then $\{u_1, u_2, y\}$ is the solution of Problem $(B_1)$;

(II) if $\{p_1, p_2, y\}$ is the solution of Problem $(A_1)$ such that conditions (C3)–(C6) are fulfilled then $\{u_1, u_2, y\}$ is the solution of Problem $(B_2)$.

LEMMA 3.2 [12]. If the assumptions (2), (3) of Lemma 3.1 are fulfilled then

(I) the solution $\{u_1, u_2, y\}$ of Problem $(B_1)$ uniquely determines the solution $\{p_1, p_2, y\}$ of Problem $(A_2)$ satisfying conditions (C3)–(C6) and the assumption (1) of Lemma 3.1;

(II) the solution $\{u_1, u_2, y\}$ of Problem $(B_2)$ uniquely determines the solution $\{p_1, p_2, y\}$ of Problem $(A_1)$ satisfying conditions (C3)–(C6) and the assumption (1) of Lemma 3.1.

## 4. Integral representations of Problems $(\mathbf{B}_k)$

Assume that boundary and initial data of Problems $(B_k)$ satisfy following regularity and compatibility conditions:

(H1) $F_i \in C^2[0, T]$,

(H2) $F_i(0) = u_{i0}(l_i)$,

(H3) $u_{10} \in C^2[0, y_0]$, $u_{20} \in C^2[y_0, l]$,

(H4) $u_{10}(y_0) = u_{20}(y_0)$,

$$\beta\left(\frac{1}{a_1} - \frac{1}{a_2}\right) u_{10}^2(y_0) + \frac{\alpha_1}{a_1} \frac{du_{10}}{dx}(y_0) = \frac{\alpha_2}{a_2} \frac{du_{20}}{dx}(y_0),$$

(H5) $\varphi_i \in C^1[0, T]$,

(H6) $\dfrac{du_{i0}}{dx}(l_i) = \varphi_i(0)$.

We assume also that

(H7) $y(t) \in (0, l)$ in the considered time interval $[0, T]$.

Let $M$ be a positive constant such that

$$\left|\frac{d^k F_i}{dt^k}\right|, \left|\frac{d^j \varphi_i}{dt^j}\right|, \left|\frac{d^k u_{i0}}{dx^k}\right| \leqslant M, \quad i=1,2; \quad j=0,1; \quad k=0,1,2 \tag{4.1}$$

in the appropriate domains.

Denote

$$v(t) \triangleq u_1(y(t), t) = u_2(y(t), t), \quad t \in [0, T]. \tag{4.2}$$

If the conditions (H1)–(H7) are satisfied and $v \in C^1[0, T]$ then Problems $(B_k)$ can be transformed into equivalent integral forms [12].

We are going to present the integral equations for function $v$ and to show that these equations are respectively equivalent to Problems $(B_k)$.

If $\{u_1, u_2, y\}$ is the solution of Problem $(B_k)$ then function $v$ defined by (4.2) satisfies some Volterra integral equation of second kind and on the contrary—having the function $v$ which satisfies such an integral equation it is possible to determine uniquely the solution $\{u_1, u_2, y\}$ of Problem $(B_k)$.

To do that first we calculate function $y$ according to the expression

$$y(t) = y_0 + \beta \int_0^t v(\tau)\, d\tau, \quad t \in [0, T] \tag{4.3}$$

and then we solve appropriate boundary value problems in the specified domains $D_i$, $i=1,2$ (Problems $(F_i^k)$, $i=1,2$ in the case of $(B_k)$).

*Problem* $(F_i^1)$, $i=1,2$

Find function $u_i$ defined in the domain $D_i$ satisfying

(1) equation $L_{\alpha_i} u_i = 0$ in $D_i$

and the following conditions:

(2) $u_i(x, 0) = u_{i0}(x)$ in $Z_i$,

(3) $u_i(y(t), t) = v(t)$, $t \in (0, T]$,

(4) $u_i(l_i, t) = F_i(t)$, $t \in (0, T]$.

*Problem* $(F_i^2)$, $i=1,2$

Find function $u_i$ defined in the domain $D_i$ satisfying conditions (1)–(3) of Problem $(F_i^1)$ and

$$\frac{\partial u_i}{\partial x}(l_i, t) = \varphi_i(t), \quad t \in (0, T].$$

Solutions of these problems obtained by the use of thermal potentials are given in [12].

Denote by $\Phi_i$ the following mappings:

$$\Phi_i: C^1[0,T] \dashrightarrow C^0(0,T], \quad i=1,2.,$$

$$[\Phi_i(v)](t) \triangleq \lim_{x\to y(t)+(-1)^i\cdot 0} \frac{\partial u_i}{\partial x}(x,t),$$

where functions $u_i$ are solutions of Problems $(F_i^1)$. If $u_i$ are solutions of Problems $(F_i^2)$ then similarly defined mappings we denote by $\Psi_i$.

The above definitions are used in process of derivation of the integral representations for Problems $(B_k)$. According to the method of Fulks and Guenther [6], in [12] it has been shown that in the case of Problem $(B_k)$ function $v$ satisfies the following nonlinear integral Volterra equation of the second kind

$$v(t)=v(0)+\frac{2\sqrt{\alpha_1\alpha_2}}{\gamma_1\sqrt{\pi\alpha_2}+\gamma_2\sqrt{\pi\alpha_1}}\int_0^t J_\kappa(t,\sigma,y(\sigma),v(\sigma))\,d\sigma$$

$$k=1,2, \qquad t\in[0,T] \tag{4.4}$$

where $y$ is defined by (4.3) and $\gamma_i=\alpha_i/a_i$, $i=1,2$.

The above Volterra operators are defined in Appendix.

Denote $C_a^j[0,T]=\{f\in C^j[0,T]\mid f(0)=a\}$, $j=0,1$.

The right-hand sides of the equations (4.4) define transformations

$$Q_\kappa: C_a^1[0,T]\dashrightarrow C_a^0[0,T], \quad k=1,2, \quad \text{where } a=v(0),$$

$$[Q_\kappa(v)](t)=a+\frac{2\sqrt{\alpha_1\alpha_2}}{\gamma_1\sqrt{\pi\alpha_2}+\gamma_2\sqrt{\pi\alpha_1}}\int_0^t J_\kappa(t,\sigma,y(\sigma),v(\sigma))\,d\sigma. \tag{4.5}$$

It can be shown that the following lemma is satisfied:

LEMMA 4.1 [12]. If $v\in C_a^1[0,T]$ then $Q_\kappa(v)\in C_a^1[0,T]$.

It has been proved in [12] that the integral equations (4.4) are the integral representations of Problems $(B_k)$. Here we recall without proof the result obtained in [12].

THEOREM 4.1.

(I)

(1) If $\{u_1,u_2,y\}$ is the solution of Problem $(B_1)$ and conditions (H1)–(H4), (H7) are satisfied then function $v\in C_a^1[0,T]$ defined by (4.2) is the solution of integral equation

$$v=Q_1(v). \tag{4.6}$$

(2) If $v$ is the solution of equation (4.6), function $y$ is defined by (4.3) and functions $u_i$ are the appropriate solutions of Problems $(F_i^1)$ then $\{u_1,u_2,y\}$ is solution of Problem $(B_1)$.

(II)

(1) If $\{u_1, u_2, y\}$ is the solution of Problem $(B_2)$ and conditions (H3)–(H7) are satisfied then function $v \in C_a^1 [0, T]$ defined by (4.2) satisfies integral equation

$$v = Q_2(v). \tag{4.7}$$

(2) If $v$ satisfies equation (4.7), function $y$ is defined by (4.3) and functions $u_i$ are the appropriate solutions of Problems $(F_i^2)$ then $\{u_1, u_2, y\}$ is solution of Problem $(B_2)$.

## 5. Existence, uniqueness and stability of solutions

In this section we are going to show that Problems $(A_k)$ and $(B_k)$ are properly posed in the sense of Hadamard i.e. for some class of initial and boundary conditions these problems have unique solution continuously depending upon the given data.

### 5.1. Existence and uniqueness

THEOREM 5.1. If

(1) for Problem $(B_1)$ conditions (H1)–(H4), (H7) are satisfied,

(2) for Problem $(B_2)$ conditions (H3)–(H7) are satisfied

then there exists unique solution $\{u_1, u_2, y\}$ of Problem $(B_k)$, $k=1,2$ in the interval $[0, T]$, where $y \in C^2 [0, T]$.

To prove this theorem we use the techniques presented in [6]. Because in our case the proof is almost identical to that given in [6], we are going to recall here only the main steps of it. In the discussed problems on the contrary to those considered in [6] the integral equations have components including boundary conditions and furthermore the operators include Green or Neumann functions instead of the fundamental solution $E$.

We introduce in the space $C^1 [0, T]$ the following family of seminorms

$$\|v\|_\sigma = \sup_{0 \leqslant t \leqslant \sigma} |v(t)| + \sup_{0 \leqslant t \leqslant \sigma} |v'(t)|, \quad \sigma \in [0, T].$$

As in [6] it can be shown that there exist finite positive constants $N = N(M, \alpha_1, \alpha_2, a_1, a_2, \beta)$ and $\sigma_0 = \sigma_0(N, M, T, \alpha_1, \alpha_2, a_1, a_2, \beta)$ such that $\|v\|_\sigma \leqslant N$ implies $\|Q_\kappa(v)\|_\sigma \leqslant N$, $k=1, 2$ for $\sigma \in [0, \sigma_0]$. Furthermore it can be shown that if $v_1, v_2 \in C_a^1 [0, T]$ and $\|v_1\|_T, \|v_2\|_T \leqslant N$ then for $\sigma \in [0, \sigma_0]$

$$\|Q_\kappa(v_1) - Q_\kappa(v_2)\|_\sigma \leqslant B\sigma \|v_1 - v_2\|_\sigma,$$

where $B = B(M, N, T, \alpha_1, \alpha_2, a_1, a_2, \beta) < \infty$.

Therefore if there is chosen such $\sigma_1>0$ that $\sigma_1\leqslant\sigma_0$ and $B\sigma_1\leqslant r<1$ then in the interval $[0,\sigma_1]$ mappings $Q_\kappa$ have the contraction property. Hence there exists unique solution of Problem $(B_k)$ in the interval $[0,\sigma_1]$.

In order to demonstrate existence of the solution in the whole interval $[0,T]$ (when $\sigma_1<T$) we transform the origin to the point $(0,\sigma_1)$ and reset the problem. Using the same arguments as previously we can conclude existence and uniqueness of solution in some interval $[\sigma_1,\sigma_2]$. We must yet show that after finite number of steps we obtain existence of solution in the whole interval $[0,T]$. To show this we make use of the Theorem 2.D [6] which can be extended to problems considered in our paper. From this theorem follows existence of such finite positive constant $M'=M'(M,N,T,\sigma_0,\alpha_1,\alpha_2,a_1,a_2,\beta)$ that

$$|u_i|,\left|\frac{\partial u_i}{\partial x}\right|,\left|\frac{\partial^2 u_i}{\partial x^2}\right|\leqslant M' \quad \text{in} \quad \bar{D}_i\cap\{(x,t)\,|\,t\leqslant\sigma_0\},\qquad i=1,2. \tag{5.1}$$

In view of definitions of $N$ and $\sigma_0$ the constant $M'$ depends in fact only on $M$, $T$, $\alpha_1$, $\alpha_2$, $a_1$, $a_2$, $\beta$, so the inequalities (5.1) are a priori- estimates for functions $u_i$.

Hence for every reset problem we obtain estimates of appropriate initial conditions in the same form as in (4.1). Taking into account definition of $\sigma_1$ we conclude that for all the reset problems the interval $[0,\sigma_1]$ in which there exists unique solution is the same. This remark completes the proof of Theorem 5.1.

In view of the equivalence of Problems $(A_1)$ and $(B_2)$, $(A_2)$ and $(B_1)$ (see Lemmas 3.1, 3.2) conditions (H1)-(H6) correspond to following ones expressed in terms of functions $p_i$:

(H8) $F_i\in C^2[0,T]$,

(H9) $F_i(0)=-a_i\dfrac{dp_{i0}}{dx}(l_i)$,

(H10) $p_{10}\in C^3[0,y_0]$, $p_{20}\in C^3[y_0,l]$,

(H11) $a_1\dfrac{dp_{10}}{dx}(y_0)=a_2\dfrac{dp_{20}}{dx}(y_0)$, $p_{10}(y_0)=p_{20}(y_0)$,

(H12) $f_i\in C^2[0,T]$,

(H13) $p_{i0}(l_i)=f_i(0)$.

COROLLARY 5.1. If

(1) for Problem $(A_2)$ conditions (H7)-(H11) are satisfied,

(2) for Problem $(A_1)$ conditions (H7), (H10)-(H13) are satisfied then Problem $(A_k)$, $k=1,2$ has unique solution in the interval $[0,T]$ and furthemore $y\in C^2[0,T]$.

**5.2. Stability.**

Using the integral representation of Problems $(B_k)$ and results of L. I. Kamynin [10, 11] we are ready to prove stability property of the solutions of these problems.

Assume that $\{u_1, u_2, y\}$ is the solution of Problem $(B_1)$ corresponding to the given data $F_i$, $u_{i0}$, $y_0$, $\alpha_1$, $a_i$, $\beta$, $i=1,2$ while $\{u_1^*, u_2^*, y^*\}$ is the solution corresponding to $F_i^*, \ldots, \beta^*$. Similarly for Problem $(B_2)$ let $\{u_1, u_2, y\}$ be the solution corresponding to the data $\varphi_i$, $u_{i0}$, $y_0$, $\alpha_i$, $a_i$, $\beta$, $i=1,2$ and $\{u_1^*, u_2^*, y^*\}$ — corresponding to $\varphi_i^*, \ldots, \beta^*$.

Assume that the functions $F_i$, $\varphi_i$, $u_{i0}$ and $F_i^*$, $\varphi_i^*$, $u_{i0}^*$ satisfy conditions (H1)–(H6) and for these functions estimates (4.1) hold. Denote by $A$ a positive constant such that

$$\alpha_i, \alpha_i^*, a_i, a_i^*, \beta, \beta^* \leqslant A .$$

We may demonstrate the following

THEOREM 5.2. Assume that

(1) for Problem $(B_1)$

$$\left| \frac{d^k F_i}{dt^k}(t) - \frac{d^k F_i^*}{dt^k}(t) \right| < \delta, \quad 0 \leqslant t \leqslant T, \quad i=1,2, \quad k=0, 1, 2, \tag{5.2}$$

$$\left| \frac{d^j u_{10}}{dx^j}(x) - \frac{d^j u_{10}^*}{dx^j}(x) \right| < \delta, \quad 0 \leqslant x \leqslant \min\{y_0, y_0^*\},$$

$$\left| \frac{d^j u_{20}}{dx^j}(x) - \frac{d^j u_{20}^*}{dx^j}(x) \right| < \delta, \quad \max\{y_0, y_0^*\} \leqslant x \leqslant l, \quad j=0, 1, 2, \tag{5.3}$$

$$|y_0 - y_0^*| < \delta, \quad |\alpha_i - \alpha_i^*| < \delta, \quad |a_i - a_i^*| < \delta, \quad |\beta - \beta^*| < \delta; \tag{5.4}$$

(2) for Problem $(B_2)$ (5.3), (5.4) and the following estimates hold

$$\left| \frac{d^k \varphi_i}{dt^k}(t) - \frac{d^k \varphi_i^*}{dt^k}(t) \right| < \delta, \quad 0 \leqslant t \leqslant T, \quad i=1, 2, \quad k=0, 1 . \tag{5.5}$$

Then Problem $(B_k)$ has property of the local stability i.e. for each $\varepsilon > 0$ there exists such $\delta > 0$ that in some interval $[0, T_1]$, where $T_1$ is independent of $\varepsilon$, the estimates (5.2)–(5.5) imply the following inequalities

$$\begin{aligned} |u_i(x, t) - u_i^*(x, t)| < \varepsilon \quad & \text{for } (x, t) \in S^i(y, y^*; T_1), \\ |y(t) - y^*(t)| < \varepsilon \quad & \text{for } t \in [0, T_1], \end{aligned} \tag{5.6}$$

where

$$S^1(y, y^*; T_1) \triangleq \{(x, t) |\, 0 \leqslant x \leqslant \min\{y(t), y^*(t)\},\ 0 \leqslant t \leqslant T_1\} .$$

$$S^2(y, y^*; T_1) \triangleq \{(x, t) \mid \max\{y(t), y^*(t)\} \leqslant x \leqslant l,\ 0 \leqslant t \leqslant T_1\} .$$

Proof. We are going to demonstrate this theorem for Problem $(B_1)$. For Problem $(B_2)$ the proof is analogous.

Let $v$ be the solution of the integral equation (4.4), $k=1$. This equation is equivalent to Problem $(B_1)$. According to considerations given in the proof of Theorem 5.1 there exists such interval $[0, \sigma_1]$ that mapping $Q_1 : C_a^1 [0, \sigma_1] \to C_a^1 [0, \sigma_1]$ has the contraction property.

Denote $v^*(t)=u_1^*(y^*(t),t)=u_2^*(y^*(t),t)$, $t\in[0,T]$. If functions $F_i^*, u_{i0}^*$, $i=1,2$ satisfy conditions (H1)–(H4) then there exists interval $[0,\sigma_1^*]$ such that integral equation $v^*=Q_1^*(v^*)$ has unique solution of the class $C_{a^*}^1[0,\sigma_1^*]$, where $a^*=$ $=v^*(0)$. The mapping $Q_1^*$ is defined by the expression (4.5) in which $F_i, \ldots, \beta$ are replaced by $F_i^*, \ldots, \beta^*$ and mappings $\Phi_i$ — by $\Phi_i^*$.

Let

$$T_1=\min\{\sigma_1,\sigma_1^*\}. \tag{5.7}$$

Observe that the following estimates hold

$$\|v-v^*\|_{T_1}=\|Q_1(v)-Q_1^*(v^*)\|_{T_1}\leqslant\|Q_1(v)-Q_1(v^*)\|_{T_1}+\|Q_1(v^*)-Q_1^*(v^*)\|_{T_1}\leqslant$$
$$\leqslant r\|v-v^*\|_{T_1}+\|Q_1(v^*)-Q_1^*(v^*)\|_{T_1},\quad 0<r<1.$$

Hence

$$\|v-v^*\|_{T_1}\leqslant\frac{1}{1-r}\|Q_1(v^*)-Q_1^*(v^*)\|_{T_1}. \tag{5.8}$$

Assume that $\|v^*\|_T\leqslant N$. In order to estimate the integral operator (4.5), $k=1$ we make use of the estimates obtained in [15]. It can be shown that there exists a positive constant $C_1=C_1(M,N,T_1,A)$ such that inequalities (5.2)–(5.4) imply

$$\|Q_1(v^*)-Q_1^*(v^*)\|_{T_1}\leqslant C_1\,\delta \tag{5.9}$$

and $C_1\to 0+$ when $T_1\to 0+$. From (5.8) and (5.9) it follows that

$$\|v-v^*\|_{T_1}\leqslant C_2\,\delta,\qquad C_2=\frac{C_1}{1-r}. \tag{5.10}$$

Now we will demonstrate (5.6). Assume that functions $v$ and $v^*$ are known in the interval $[0,T_1]$. Then according to (4.3) we can define in $[0,T_1]$ functions $y$ and $y^*$.

Denote

$$D_1(T_1;y)\triangleq\{(x,t)\mid x\in(0,y(t)),\quad t\in(0,T_1)\},$$
$$D_2(T_1;y)\triangleq\{(x,t)\mid x\in(y(t),l),\quad t\in(0,T_1)\}.$$

Let $u_i$, $i=1,2$ be the solution of Problem $(F_i^1)$ in $D_i(T_1;y)$-domain corresponding to boundary functions $v, F_i, u_{i0}$ and parameter $\alpha_i$. Let $u_i^*$ be the solution of Problem $(F_i^1)$ in $D_i(T_1;y^*)$-domain corresponding to boundary functions $v^*, F_i^*, u_{i0}^*$ and parameter $\alpha_i^*$. By $\bar{u}_i$ we denote solution of Problem $(F_i^1)$ in domain $D_i(T_1;y)$, corresponding to boundary functions $v^*, F_i^*, u_{i0}^*$ and parameter $\alpha_i^*$.

Observe that

$$|u_i(x,t)-u_i^*(x,t)|\leqslant|u_i(x,t)-\bar{u}_i(x,t)|+|\bar{u}_i(x,t)-u_i^*(x,t)|$$
$$\text{for } (x,t)\in S^i(y,y^*;T_1). \tag{5.11}$$

Solutions of boundary value problems for linear parabolic equation are continuously dependent on boundary data and parameters [11]. Therefore from inequalities (5.2)-(5.4) and (5.10) follows

$$|u_i(x,t)-\bar{u}_i(x,t)|\leqslant C_3\,\delta\quad\text{for } (x,t)\in D_i(T_1;y), \tag{5.12}$$

where $C_3>0$ does not depend on $\delta$.

In order to estimate the second term on the right-hand side of (5.11) we make use of results of L. I. Kamynin given in [10]. The continuous dependence of the solution of linear parabolic equation on the boundary of domain has been investigated in that work.

On the basis of the main theorem proved in [10] we can conclude that

$$|\bar{u}_i(x,t)-u_i^*(x,t)|\leqslant C_4\,\|y-y^*\|_{T_1} \quad \text{for } (x,t)\in S^i(y,y^*;T_1), \tag{5.13}$$

where $C_4>0$ is independent of $y$ and $y^*$.

Taking into account definition of functions $y$, $y^*$ and inequality (5.10) we obtain

$$\left|\frac{dy}{dt}(t)-\frac{dy^*}{dt}(t)\right|\leqslant C_2\,\delta$$

$$|y(t)-y^*(t)|\leqslant C_2\,\delta t+\delta, \qquad t\in[0,T_1].$$

Hence

$$\|y-y^*\|_{T_1}\leqslant(1+C_2+C_2\,T_1)\,\delta. \tag{5.14}$$

From (5.11)-(5.14) it follows that

$$|u_i(x,t)-u_i^*(x,t)|\leqslant[C_3+C_4(1+C_2+C_2\,T_1)]\,\delta.$$

This completes the proof of the Theorem 5.2.

We have proved that solutions of Problems $(B_k)$ have the stability property only in an interval $[0, T_1]$ which is in general less than the given interval $[0, T]$. It can be easily shown that Theorem 5.2 is also valid in the whole interval $[0, T]$. To this end observe that $T_1$ defined by (5.7) is independent of $\varepsilon$ and $\delta$. The considerations included in the proof of Theorem 5.1 imply that $T_1$ is also independent of the moment $t_0$ $(0\leqslant t_0<T)$ which is assumed to be initial. Thus for every $\varepsilon>0$ there exists such $\delta>0$ that in all intervals $[t_0, t_0+T_1]$ inequalities (5.2)-(5.5) imply (5.6). Taking in turn $t_0=kT_1$, $k=1, \ldots, n$, after a finite number of steps we may demonstrate validity of Theorem 5.2 in the interval $[0, T]$.

In view of the equivalence of Problems $(A_k)$ and $(B_k)$ we can conclude that also solutions of Problems $(A_k)$ have the stability property in the interval $[0, T]$.

## 6. Maximum principle

In this section we demonstrate the maximum principle for Problems $(A_k)$ and $(B_k)$. The results are based on the maximum principles in weak and strong formulations for parabolic equations [5] and Vyborny-Friedman Theorem [4, 17]. We recall here this theorem. We shall use the following notations:

$\Omega^t\subset\{(x,\tau)\in R^{n+1}\,|\,0\leqslant\tau\leqslant t\}$—some domain with sufficiently regular boundary,

$\Omega_s=\{(x,\tau)\in\Omega^t\,|\,\tau=s\}$,

$\Sigma_s$—boundary of the domain $\Omega_s$ in $R^n$,

$$\Sigma^t=\bigcup_{0<s\leqslant t}\Sigma_s, \qquad \Xi^t=\overline{\Sigma^t\cup\Omega_0}.$$

THEOREM 6.1 (Vyborny-Friedman [4, 17]). Assume that:

(1) $u$ is a function continuous in the closure of the domain $\Omega^t$ and

$$\frac{\partial u}{\partial t}-a^2\Delta u\geqslant 0 \quad (\leqslant 0) \qquad \text{in int } \Omega^t ;$$

(2) at some point $P_0=(x_0, \tau_0)\in \Xi^t$ function $u$ attains its minimal (respectively maximal) value $M$ in $\bar{\Omega}^t$;

(3) there exists a ball $K\subset R^{n+1}$ (with center $P_K$) such that:

(a) $\bar{K}^t=\overline{\{(x, \tau)\in K \mid \tau\leqslant t\}}\subset \bar{\Omega}^t$ and the point $P_0\in \partial K$,

(b) vector $\overrightarrow{P_K P_0}$ is not parallel to $0t$-axis,

(c) $u(x, \tau)>M\ (<M)$ for all $(x, \tau)\in K^t$;

(4) there exists the derivative $\dfrac{\partial u}{\partial v}(x_0, \tau_0)$ where $\vec{v}$ denotes vector with the beginning at $P_0$, internal with respect to $K$.

Under such assumptions

$$\frac{\partial u}{\partial v}(x_0, \tau_0)>0 \qquad (\text{respectively} \frac{\partial u}{\partial v}(x_0, \tau_0)<0) .$$

REMARK. This theorem holds for general linear parabolic equation [4].

Now we are going to prove the following properties of solutions of Problems $(B_k)$.

THEOREM 6.2. Let $\{u_1, u_2, y\}$ be the solution of Problem $(B_k)$.

(1) If

$$G(t)\triangleq\beta\left(\frac{1}{a_1}-\frac{1}{a_2}\right)u_1^2(y(t), t)\geqslant 0, \qquad t\in(0, T] \tag{6.1}$$

then

$$\max_{(x, t)\in\bar{D}} u(x, t)=\max_{(x, t)\in\overline{Z\cup S}} u(x, t) \tag{6.2}$$

where $u(x, t)\triangleq u_i(x, t)$ for $(x, t)\in\bar{D}_i$;

(2) if $G(t)\leqslant 0$ for $t\in(0, T]$ then

$$\min_{(x, t)\in\bar{D}} u(x, t) = \min_{(x, t)\in\overline{Z\cup S}} u(x, t) ; \tag{6.3}$$

(3) if

$$\beta u_i^2(y(t), t)<\alpha_i\frac{\partial u_i}{\partial x}(y(t), t), \qquad i=1, 2, \qquad t\in(0, T] \tag{6.4}$$

then both (6.2) and (6.3) are satisfied.

Proof. Denote by $M$ and $m$ respectively the maximal and the minimal value of the solution $u$ on the boundary $\overline{Z \cup S}$ i.e.

$$M = \max_{(x,t)\in\overline{Z \cup S}} u(x,t), \quad m = \min_{(x,t)\in\overline{Z \cup Z}} u(x,t). \tag{6.5}$$

If we apply the weak maximum principle for parabolic equations [5] in $D_i$-domains, $i=1,2$, we obtain

$$\max_{(x,t)\in\bar{D}_i} u(x,t) = \max_{(x,t)\in\overline{S_i \cup Z_i \cup \Gamma}} u(x,t)$$

and

$$\min_{(x,t)\in\bar{D}_i} u(x,t) = \min_{(x,t)\in\overline{S_i \cup Z_i \cup \Gamma}} u(x,t).$$

If we show that

$$\max_{(x,t)\in\bar{\Gamma}} u(x,t) \leqslant M \tag{6.6}$$

then we obtain equalities

$$\max_{(x,t)\in\bar{D}_i} u(x,t) = \max_{(x,t)\in\overline{S_i \cup Z_i}} u(x,t), \quad i=1,2.$$

Hence in view of the following obvious equality

$$\max_{(x,t)\in\bar{D}} u(x,t) = \max\{\max_{(x,t)\in\bar{D}_i} u(x,t); \quad i=1,2\}$$

we obtain (6.2). Similarly if

$$\min_{(x,t)\in\bar{\Gamma}} u(x,t) \geqslant m \tag{6.7}$$

then we conclude that the condition (6.3) is satisfied.

(1) Assume that $G(t)\geqslant 0$ for $t\in(0,T]$. We are going to show that in this case the estimate (6.6) is satisfied. Let function $v$ defined by (4.2) attain at $t_1\in(0,T)$ its maximal value $M'>M$ in the interval $(0,T)$ i.e.

$$\max_{t\in(0,T)} v(t) = v(t_1) = M' > M.$$

Hence according to the weak maximum principle

$$\max_{(x,t)\in\bar{D}_i} u_i(x,t) = v(t_1).$$

Now we are going to show that the assumptions of the Vyborny-Friedman Theorem are fulfilled in both domains $D_i$. Actually, functions $u_i$ satisfy condition (1) of the theorem and at the point $P_0=(y(t_1), t_1)\in\Gamma$ functions $u_i$ attain their maximal values in $\bar{D}_i$. To show that conditions (3) are also satisfied observe first that since $y\in C^2[0,T]$ then there exist balls $\bar{K}_i\subset\bar{D}_i$ (with centers $P_K^i$) such that the point $P_0\in\partial K_i$, $i=1,2$, and the vectors $\overrightarrow{P_K^i P_0}$, i=1,2, are not parallel to $0t$-axis. In order to show that condition (3) (c) holds we make use of the strong maximum principle [5]. Assume the opposite, namely let function $u_i$ attain the value $M'$ at some point $P'=(x', t') \in \operatorname{int} K_i$.

Then it follows from the strong maximum principle that $u_i = M'$ in $\bar{D}_i \cap \{(x, t) \mid t \leqslant t'\}$ what contradicts (6.5). Condition (4) is obviously satisfied for vector $\vec{v}$ parallel to the $0x$-axis.

Thus we have verified that all the assumptions of the Theorem 6.1 hold. From this theorem we get

$$\frac{\partial u_1}{\partial x}(y(t_1), t_1) > 0, \qquad \frac{\partial u_2}{\partial x}(y(t_1), t_1) < 0.$$

Taking into account that on the free boundary the following condition is satisfied

$$G(t) + \frac{\alpha_1}{a_1}\frac{\partial u_1}{\partial x}(y(t), t) = \frac{\alpha_2}{a_2}\frac{\partial u_2}{\partial x}(y(t), t) \quad \text{for } t \in (0, T] \tag{6.8}$$

we arrive at contradiction. Therefore

$$v(t) \leqslant M \text{ for } t \in [0, T]. \tag{6.9}$$

If the function $v$ attains its maximal value at the point $t_1 = T$ then we can take the final value of time interval at some point $T+\varepsilon$ $(\varepsilon > 0)$. Using the same arguments we conclude that in this case (6.9) is also satisfied. So we have proved that the estimate (6.6) actually holds.

(2) If we assume $G(t) \leqslant 0$ for $t \in (0, T]$ then by similar considerations to those of the previous case we conclude that the estimate (6.7) holds.

(3) Assume that inequalities (6.4) are fulfilled. We are going to show that $v(t) \in [m, M]$ for $t \in [0, T]$, so that both conditions (6.6) and (6.7) are satisfied. Let us rewrite the equality (6.8) in the following form

$$\frac{\alpha_1}{a_1}\frac{\partial u_1}{\partial x}(y(t), t) + \frac{\beta}{a_1}u_1^2(y(t), t) = \frac{\alpha_2}{a_2}\frac{\partial u_2}{\partial x}(y(t), t) + \frac{\beta}{a_2}u_2^2(y(t), t), \quad t \in (0, T].$$

It is easy to observe that if we assume existence of time moments $t \in (0, T]$ at which $v(t) > M$ or $v(t) < m$, in view of the Theorem 6.1 and the assumption (6.4) we arrive at contradiction. This completes the proof of Theorem 6.2.

From Theorem 6.2 immediately follows

COROLLARY 6.2. Let $\{u_1, u_2, y\}$ be solution of Problem $(B_1)$.

(1) If $a_2 \geqslant a_1$ then

$$\begin{gathered} u(x, t) \leqslant B, \quad (x, t) \in \bar{D}, \\ \frac{dy}{dt}(t) \leqslant \beta B, \quad y(t) \leqslant y_0 + \beta B t, \quad t \in [0, T], \end{gathered} \tag{6.10}$$

where

$$B = \max\{\max u_{i0}(x), \max F_i(t);\ i = 1, 2\};$$

(2) if $a_2 \leqslant a_1$ then

$$u(x,t) \geqslant b, \quad (x,t) \in \bar{D},$$
$$\frac{dy}{dt}(t) \geqslant \beta b, \quad y(t) \geqslant y_0 + \beta b t, \quad t \in [0,T] \tag{6.11}$$

where $b = \min\{\min u_{i0}(x), \min F_i(t);\ i=1,2\}$;

(3) if estimates (6.4) are satisfied then conditions (6.10) and (6.11) hold.

Using inequalities (6.10), (6.11) we can a priori estimate on the basis of given initial and boundary data the fluids velocity distribution and location of the gas--liquid interface.

To apply the result of Theorem 6.2 to Problems $(A_k)$ observe that in this case $G(t) \equiv 0$. Therefore we have immediately the following.

COROLLARY 4.3. If $\{p_1, p_2, y\}$ is the solution of Problem $(A_k)$ then conditions (6.2) and (6.3) are satisfied for function $p(x,t) = p_i(x,t)$, $(x,t) \in \bar{D}_i$.

## 7. Remarks

The results obtained in this paper will be used in [13] where finite-difference approximations of multi-layer free boundary problems and proof of their convergence will be presented as well as results of numerical experiments will be discussed.

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**Appendix**

The integral operators in (4.4) are defined in the following way:

$$
\begin{aligned}
J_1\big(t,\sigma,y(\sigma),v(\sigma)\big) = & -\beta\left(\frac{1}{a_1}-\frac{1}{a_2}\right)(t-\sigma)^{1/2}\frac{d}{d\sigma}[v^2(\sigma)]+\\
& +2\beta\gamma_1 u_{10}'(y_0)(t-\sigma)^{1/2} v(\sigma) G_1\big(y(\sigma),y_0,\sigma\big)-\\
& -2\beta\gamma_1(t-\sigma)^{1/2} v(\sigma)\int_0^{y_0} G_1\big(y(\sigma),\xi,\sigma\big)u_{10}''(\xi)\,d\xi+\\
& +2\alpha_1\gamma_1 u_{10}'(y_0)(t-\sigma)^{1/2}\frac{\partial G_1}{\partial x}\big(y(\sigma),y_0,\sigma\big)-\\
& -2\alpha_1\gamma_1(t-\sigma)^{1/2}\int_0^{y_0}\frac{\partial G_1}{\partial x}\big(y(\sigma),\xi,\sigma\big)u_{10}''(\xi)\,d\xi+\\
& +2\beta\gamma_2 u_{20}'(y_0)(t-\sigma)^{1/2} v(\sigma) H_1\big(y(\sigma),y_0,\sigma\big)-\\
& -2\beta\gamma_2(t-\sigma)^{1/2} v(\sigma)\int_{y_0}^{l} H_1\big(y(\sigma),\xi,\sigma\big)u_{20}''(\xi)\,d\xi+\\
& +2\alpha_2\gamma_2 u_{20}'(y_0)(t-\sigma)^{1/2}\frac{\partial H_1}{\partial x}\big(y(\sigma),y_0,\sigma\big)-\\
& -2\alpha_2\gamma_2(t-\sigma)^{1/2}\int_{y_0}^{l}\frac{\partial H_1}{\partial x}\big(y(\sigma),\xi,\sigma\big)u_{20}''(\xi)\,d\xi+\\
& +2\beta\gamma_1(t-\sigma)^{1/2} v(\sigma)\int_0^{\sigma}\frac{\partial G_2}{\partial x}\big(y(\sigma),0,\sigma-\tau\big)F_2'(\tau)\,d\tau+\\
& +2\gamma_1 F_1'(0)(t-\sigma)^{1/2} G_2\big(y(\sigma),0,\sigma\big)+\\
& +2\gamma_1(t-\sigma)^{1/2}\int_0^{\sigma} G_2\big(y(\sigma),0,\sigma-\tau\big)F_1''(\tau)\,d\tau+
\end{aligned}
$$

$$+2\beta\gamma_2 (t-\sigma)^{1/2} v(\sigma) \int\limits_0^{\sigma} \frac{\partial H_2}{\partial x} \big(y(\sigma), l, \sigma-\tau\big) F_2'(\tau)\, d\tau +$$

$$+2\gamma_2 F_2'(0)(t-\sigma)^{1/2} H_2\big(y(\sigma), l, \sigma\big)+$$

$$+2\gamma_1 (t-\sigma)^{1/2} \int\limits_0^{\sigma} H_2\big(y(\sigma), l, \sigma-\tau\big) F_2''(\tau)\, d\tau -$$

$$-\alpha_1 \gamma_1 (t-\sigma)^{-1/2} \int\limits_0^{\sigma} \frac{\partial G_1}{\partial x} \big(y(\sigma), y(\tau), \sigma-\tau\big) [\Phi_1(v)](\tau)\, d\tau -$$

$$-\alpha_2 \gamma_2 (t-\sigma)^{-1/2} \int\limits_0^{\sigma} \frac{\partial H_1}{\partial x} \big(y(\sigma), y(\tau), \sigma-\tau\big) [\Phi_2(v)](\tau)\, d\tau +$$

$$+(t-\sigma)^{-1/2} \int\limits_0^{\sigma} (\sigma-\tau)^{-1/2} [\gamma_1 (4\pi\alpha_1)^{-1/2} + \gamma_2 (4\pi\alpha_2)^{-1/2} -$$

$$-\gamma_1 (\sigma-\tau)^{1/2} G_2\big(y(\sigma), y(\tau), \sigma-\tau\big) -$$

$$-\gamma_2 (\sigma-\tau)^{1/2} H_2\big(y(\sigma), y(\tau), \sigma-\tau)\big] v'(\tau)\, d\tau .$$

$$J_2\big(t, \sigma, y(\sigma), v(\sigma)\big) = -\beta\left(\frac{1}{a_1} - \frac{1}{a_2}\right)(t-\sigma)^{1/2} \frac{d}{d\sigma} [v^2(\sigma)] +$$

$$+2\beta\gamma_1 u_{10}'(y_0)(t-\sigma)^{-1/2} v(\sigma) G_2\big(y(\sigma), y_0, \sigma\big) -$$

$$-2\beta\gamma_1 (t-\sigma)^{1/2} v(\sigma) \int\limits_0^{y_0} G_2\big(y(\sigma), \xi, \sigma\big) u_{10}''(\xi)\, d\xi +$$

$$+2\alpha_1 \gamma_1 u_{10}'(y_0)(t-\sigma)^{1/2} \frac{\partial G_2}{\partial x} \big(y(\sigma), y_0, \sigma\big) -$$

$$-2\alpha_1 \gamma_1 (t-\sigma)^{1/2} \int\limits_0^{y_0} \frac{\partial G_2}{\partial x} \big(y(\sigma), \xi, \sigma\big) u_{10}''(\xi)\, d\xi +$$

$$+2\beta\gamma_2 u_{20}'(y_0)(t-\sigma)^{1/2} v(\sigma) H_2\big(y(\sigma), y_0, \sigma\big) -$$

$$-2\beta\gamma_2 (t-\sigma)^{1/2} v(\sigma) \int\limits_{y_0}^{l} H_2\big(y(\sigma), \xi, \sigma\big) u_{20}''(\xi)\, d\xi +$$

$$+2\alpha_2 \gamma_2 u_{20}'(y_0)(t-\sigma)^{1/2} \frac{\partial H_2}{\partial x} \big(y(\sigma), y_0, \sigma\big) -$$

$$-2\alpha_2 \gamma_2 (t-\sigma)^{1/2} \int\limits_{y_0}^{l} \frac{\partial H_2}{\partial x} \big(y(\sigma), \xi, \sigma\big) u_{20}''(\xi)\, d\xi +$$

$$+2\beta\gamma_1(t-\sigma)^{1/2}v(\sigma)\int_0^\sigma G_2\big(y(\sigma),0,\sigma-\tau\big)\varphi_1'(\tau)\,d\tau+$$

$$+2\alpha_1\gamma_1(t-\sigma)^{1/2}\int_0^\sigma\frac{\partial G_2}{\partial x}\big(y(\sigma),0,\sigma-\tau\big)\varphi_1'(\tau)\,d\tau+$$

$$+2\beta\gamma_2(t-\sigma)^{1/2}v(\sigma)\int_0^\sigma H_2\big(y(\sigma),l,\sigma-\tau\big)\varphi_2'(\tau)\,d\tau+$$

$$+2\alpha_2\gamma_2(t-\sigma)^{1/2}\int_0^\sigma\frac{\partial H_2}{\partial x}\big(y(\sigma),l,\sigma-\tau\big)\varphi_2'(\tau)\,d\tau-$$

$$-\alpha_1\gamma_1(t-\sigma)^{-1/2}\int_0^\sigma\frac{\partial G_2}{\partial x}\big(y(\sigma),y(\tau),\sigma-\tau\big)[\Psi_1(v)](\tau)\,d\tau-$$

$$-\alpha_2\gamma_2(t-\sigma)^{-1/2}\int_0^\sigma\frac{\partial H_2}{\partial x}\big(y(\sigma),y(\tau),\sigma-\tau\big)[\Psi_2(v)](\tau)\,d\tau+$$

$$+(t-\sigma)^{-1/2}\int_0^\sigma(\sigma-\tau)^{-1/2}\big[\gamma_1(4\pi\alpha_1)^{-1/2}+\gamma_2(4\pi\alpha_2)^{-1/2}-$$

$$-\gamma_1(\sigma-\tau)^{1/2}G_1\big(y(\sigma),y(\tau),\sigma-\tau\big)-$$

$$-\gamma_2(\sigma-\tau)^{1/2}H_1\big(y(\sigma),y(\tau),\sigma-\tau\big)\big]v'(\tau)\,d\tau.$$

## O pewnych własnościach dwuwarstwowych parabolicznych zagadnień brzegowych ze swobodną granicą

W artykule badano problemy istnienia, jednoznaczności i ciągłej zależności od danych wejściowych rozwiązań dwuwarstwowych parabolicznych zagadnień brzegowych ze swobodną granicą, opisujących dynamikę podziemnego zbiornika gazu. Udowodniono, że dla badanych zagadnień brzegowych obowiązuje zasada maksimum.

## О некоторых свойствах двухслойных параболических краевых задач со свободной границей

В статье рассмотрены вопросы существования, единственности и непрерывной зависимости от входных данных решений двухслойных параболических краевых задач со свободной границей описывающих динамику подземного газохранилища. Доказано, что для рассматриваемых краевых задач действует принцип максимума.