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A sufficient condition for evasion in a nonlinear game. Part II

by

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We prove for a nonlinear differential game of evasion that under a certain condition discussed in [4] the evasion is possible for every initial state of the game. We construct a strategy of evasion and estimate the distance of the trajectory of the game from the terminal subspace.

1. Statement of the problem

In [4] we formulated a condition of evasion and a theorem of evasion for a nonlinear game. As shown there the condition is a generalization of the condition of evasion for a linear game given in [2]. The proof presented here even when applied to the linear case much differs from that in [2] especially in the part where certain integral equation is solved. We are able to construct there a strategy of evasion while in [2] only the existence of a relaxed strategy is shown, when the evader chooses at each moment a collection $(\mu_1, ..., \mu_r, v_1, ..., v_r)$, $\sum_{i=1}^{r} \mu_i=1$, $\mu_i \geqslant 0$, $v_i \in V$, $i=1, ..., r$, instead of one point v from his control set.

The game is given by the equation

$$\dot{z}=P_0(z)+f(z, u, v); \quad z \in R^n, \quad u \in U \subset R^p, \quad v \in V \subset R^q, \tag{1.1}$$

two compact control sets: U for the pursuer and V for the evader, and a linear subspace M of R^n such that docim $M \geqslant 2$. The right-hand side $P(z, u, v)=P_0(z)+f(z, u, v)$ is continuous in $R^n \times U \times V$, Lipschitzian in z in every compact subset of R^n uniformly with respect to u, v and form some constants A, B satisfies the growth condition: $|z \cdot P(z, u, v)| \leqslant A\, |z|^2+B$ for all $z \in R^n$, $u \in U$, $v \in V$. We assume moreover that $P_0(z)$ is continuously differentiable as many time as it is differentiated in the condition of evasion. Both players use measurable $u(t) \in U$ and $v(t) \in V$, respectively, as their control functions. The aim of the evader is to avoid the subspace M, that is to ensure that the trajectory of the game satisfies $z(t) \notin M$ for

$t \in [0, +\infty)$ whenever the initial state $z(t)=z_0$ does not belong to M, while the aim of the pursuer is opposite. We seek for a strategy for the evader $v^u(z_0; t)$ defined for all z_0 M such that any corresponding trajectory does not intersect M, where a mapping $v^u(z_0; t)$ is called strategy if for a fixed initial state z_0 it assigns to each pursuer's control function $u(t)$ an evader's control function $v(t)=v^u(z_0; t)$ in such a way that for any $T \in [0, +\infty)$ and any control functions $u^1(t)$, $u^2(t)$ the condition $u^1(t)=u^2(t)$ a.e. in $[0, T]$ implies that $v^{u_1}(z_0, t)=v^{u_2}(z_0, t)$ a.e. in $[0, T]$. Let us recall the condition of evasion.

Denote $C_0(z)=I$, $C_1(z)=DP_0(z)$ where $DP_0(z)$ is the derivative of the mapping $P_0(z)$ at point z

$$C_k(z)=D\left(C_{k-1}(z)\, P_0(z)\right) \quad \text{for } k=2, \ldots, p-1$$

and

$$F_{p-1}(t, z, u, v)=\sum_{i=0}^{p-1} C_i(z) f_i(z, u, v)\, t^i.$$

Let $z_* \in M$. Take a two-dimensional subspace L ortogonal to M and a linear mapping π_L of the form $\pi_L=AP_L$ where P_L s the orthogonal projection onto L, A is an isometric mapping of R^n which maps L onto $R^2=\{x \times R^n | x_i=0,\ i=3, \ldots, n\}$. Consider for z in a neighbourhood U_{z_*} of z_* and t in some interval $[0, T_{z_*}]$ the following representations of the mappings $\pi_L F_{p-1}(t, z, u, v)$:

$$\pi_L F_{p-1}(t, z, u, v)=H(t)\sum_{i=0}^{p-1} \psi_i(z, u, v)\, t^i+ \\ +\sum_{i=0}^{p-1} \alpha_i(z, u, v)\, t^i+\sum_{i=0}^{p-1} \beta_i\, t^i+R(t^p) \tag{1.2}$$

$$\text{for } t \in [0, T_{z_*}],\ u \in U,\ v \in V,\ z \in U_{z_*}$$

where $H(t)$ in an analytical in a neighbourhood of zero 2×2-matrix-function non--singular for $t \in (0, T_{z_*}]$. The latter implies (see [1] and [2]) that $H(t)$ may be written in the following form:

$$H(t)=A(t)\begin{vmatrix} t^{l_1}, & 0 \\ 0, & t^{l_2} \end{vmatrix} B(t)$$

where l_1, l_2 are integers $0 \leqslant l_1 \leqslant l_2$, which depend only on the function $H(t)$ and are called indices of the function $H(t)$, the matrix-functions $A(t)$, $B(t)$ ate analytical in a neighbourhood of zero and such that der $A(0) \neq 0$, det $B(0) \neq 0$. We consider representations of the form (1.2) which satisfy the following conditions:

(r) The indices of $H(t)$ are at most $(p-1)$, the functions $\psi_i(z, u, v) \in R^2$, $i=0, \ldots, p-1$, are continuous, $\beta_i \in R^2$, $i=0, \ldots, p-1$, are constant vectors; $R(t^p)=R(t, z, u, v)$ is such that $|R(t^p)/t^p|$ is bounded uniformly with respect to all variables, the functions $\alpha_i(z, u, v)$, $\alpha_i(z, u, v) \in R^2$, $i=0, \ldots, p-1$, sat sfy for some constant D the following estimation:

$$|\alpha_i(z, u, v)| \leqslant D\rho^{p-1}(z, M) \quad \text{for } z \in U_{z'_*}, \quad u \in U, \quad v \in V. \tag{1.3}$$

We say that conditon of evasion (F) is satisfied iff:

(F) For every point $z_* \in M$ there exist a compact neighbourhood U_{z_*} of z_*, a two-dimensional subspace $L=L(z_*)$ of R^n orthogonal to M, an integer $p=p(z_*)$ and $T=T(z_*)$, $T>0$, such that the mapping $\pi_L F_{p-1}(t, z, u, v)$ has a representation of the form (1.2) which satisfies (r) and such that:

(i) the set $\bigcap_{u \in U} \mathrm{co}\, \psi_0(z_*, u, v)$ contains an interior point with respect to R^2.

We prove the following theorem:

THEOREM 1.1. If for the game (1.1) the condition (F) is satisfied then there exists closed sets W, W_1, a strategy of evasion $v^u(z_0; t)$ defined for all $z_0 \notin M$, $t \in [0, +\infty)$ and positive functions $T(\xi)$, $\xi \in (0, +\infty)$, $T(\xi)<1$ and $\gamma(\xi_1, \xi_2)$, $\xi_1, \xi_2 \in (0, +\infty)$ such that $M \subset \operatorname{int} W_1 \subset \operatorname{int} W$ and any trajectory $z(t)$ corresponding to the strategy $v^u(z_0; t)$ sat'sfies:

if $z_0 \in W$ then $\rho(z(t), M) \geqslant \gamma(\rho(z_0, M), |z_0|$ for

$$t \in [0, T(|z_0|)] \text{ and } z(T(|z_0|)) \notin W,$$

if for some t_1 $z(t_1) \in W$ then $z(t) \notin W_1$ for all $t \geqslant t_1$,

if $z(t_1) \in W$ then for some $t_2 \in [t_1, t_1+T(|t_1)|)]$, $z(t_2) \notin W$.

We proceed to prove the theorem. For a detailed discussion of condition (F) see [4].

2. Proof of the evasion theorem

We split the proof into two parts. In part A we shall construct for each $z_* \in M$ a local strategy of evasion $v^u_{z_*}(z_0; t)$ defined for z_0 from some neighbourhood V_{z_*} of z_* and t from some interval $[0, T_{z_*}]$. Then in part B we shall describe a global strategy of evasion $v^u(z_0; t)$ and construct the sets W, W_1.

A. Take the trajectory corresponding to control functions $u(t)$, $v(t)$ and an initial condition $z(0)=z_0$, that is

$$z(t)=z_0+\int_0^t \big(P_0 z(\tau)\big)+f\big(z(\tau), u(\tau), v(\tau)\big)\, d\tau .$$

Integrating this p-times by parts we obtain the following formula:

$$z(t)=s_p(t; z_0)+\int_0^t \sum_{i=0}^{p-1} C_i\big(z(\tau)\big) f\big(z(\tau), u(\tau) v(\tau)\big) \frac{(t-\tau)^i}{i!}\, d\tau + R(t^{p+1}) \tag{2.1}$$

where $s_p(t, z_0)=z_0+P_0(z_0)\, t+\ldots+C_{p-1}(z_0)\, P_0(z_0)\frac{t^p}{p!}$ and the rest is of the form

$$R(t^{p+1})=\int_0^t C_p\big(z(\tau)\big)\big(P_0\big(z(\tau)\big)\big)+f\big(z(\tau), u(\tau), v(\tau)\big)\frac{(t-\tau)^p}{p!}\, d\tau .$$

The assumptions about the right-hand side of the game equation imply that on every compact interval of time $[0, T]$ all trajectories which start from a ball $K(0, r)$ of radius r around the origin remain in a certain ball $K(0, h_{r, T})$ of radius $h_{r, T}$. Thus there exists a constant $N_{r, T}$ such that for any $z_0 \in K(0, r)$ and any control functions $u(t), v(t)$ the following estimation holds:

$$|R(t^{p+1})| \leqslant N_{r, T}\, t^{p+1}, \quad t \in [0, T]. \tag{2.2}$$

Further we denote by $R(t^m)$ such terms that $|R(t^m)/t^m|$ is bounded uniformly with respect to all variables.

Let $z_* \in M$. Take a neighbourhood $\tilde{V}_{z_*}$ of z_* and $\tilde{T}_{z_*}$ such that each trajectory of (1.1) with an initial condition z_0 from $\tilde{V}_{z_*}$ remains in U_{z_*} for $t \in [0, T_{z_*}]$. Let $z_0 \in \in \tilde{V}_{z_*}$ and consider the image $\pi_L z(t)$ of a trajectory of (1.1). Since (2.1) we have for $p = p(z_*)$:

$$\pi_L z(t) = w_p(z_0; t) + \int_0^t \sum_{i=0}^{p-1} \pi_L C_i(z(\tau)) f(z(\tau), u(v), v(\tau)) \frac{(t-\tau)^i}{i!} + R(t^{p+1}) \tag{2.3}$$

where $w_p(z_0; t)$ is a curve in R^2 whose components are polynomials of degrees at most p and $R(t^{p+1})$ satisfies $|R(t^{p+1})| \leqslant N_{h, T_{z_*}} t^{p+1}$ (see (2.2)) where h is such that $V_{z_*} \subset K(0, h)$. Our aim is to construct a strategy $v^u_{z_*}(z_0; t)$ that ensures certain estimation from below of $\rho(z(t), M)$. Since $\rho(z(t), M) \geqslant |\pi_L z(t)|$, it sufficies to estimate the norm $|\pi z(t)|$. We shall use the following fundamental lemma:

LEMMA 2.1. Fix a cube Ω in R^2 and a number p. Then there exists a constant Θ such that for each curve $w_p(t)$ in R^2 whose components are polynomials of degrees not greater than p there exists a point $w \in \Omega$ such that the following holds:

$$|w_p(t) + wt^p| \geqslant \Theta t^p \quad \text{for } t \in [0, +\infty). \tag{2.4}$$

The Lemma is proved in [1]. Here we only descr·be briefly the idea of the proof. Assume that Ω is a squera whose sides are parallel to the axes. Divide Ω by a net of horizontal and vertical lines into r small squares whose interiors are mutually disjoint and consider the curve $w_p(t)/t^p$. Since components of $w_p(t)$ are polynomials of degrees not greater than p each of the lines is intersected by the curve at most p times and hence by a simple argument if the division is fine enough, namely if $r > > (2p+1)^2$ then there exists at least one among the small squares whose interior is disjoint with the curve. The center of this square is taken as w, then (2.4) holds with Θ equal to the half of the length of its side.

Put

$$\pi C_i(z) f(z, u, v) = f_i(z, u, v) \quad \text{for } i = 0, \ldots, p-1$$

$$I_{z, u, v}(t) = \int_0^t \sum_{i=0}^{p-1} f_i(z(\tau), u(\tau), v(\tau)) \frac{(t-\tau)^i}{i!} d\tau.$$

We shall show that there exists a ball $K(0, r)$ around the origin and a fixed curve $\chi_p(t)$ whose components are polynomials of degrees not greater than p such that for each z_0 from some neighbourhood v_{z_*} and each $w \in K(0, r)$ there can be constructed a strategy defined on some interval $[0, T_{z_*}]$ ensuring that the difference $|(\chi_p(t)+I_{z,u,v}(t))-wt^p|$ is sufficiently small. We shall take such a steraegy for $w=w(z_0)$ which corresponds to the curve $(w_p(z_0; t)-\chi_p(t))$ as in Lemma 2.1 and make use of the estimation (2.4).

We shall need the following.

LEMMA 2.2. Let $g_i(\tau)$, $\varphi_i(\tau)$, $i=0, ..., p-1$, be measurable bounded functions defined for $\tau \in [0, T]$ taking values in R^k, $H(t)$ an analytical $k\times k$ matrix-function $H(t)=\sum_{i=0}^{\infty} H_i t^i$, $t\in[0, T]$. Assume that for every $t\in[0, T]$ the following holds:

$$H(t)\sum_{i=0}^{p-1}\varphi_i(\tau)t^i+R(t^p)=\sum_{i=0}^{p-1}g_i(\tau)t^i \tag{2.5}$$

then for each $t\in[0, T]$

$$\left|\int_0^t\sum_{i=0}^{p-1}g_i(\tau)\frac{(t-\tau)^i}{i!}\,d\tau\right|\leqslant N_1\sup_{t\in[0,T]}\left|\int_0^t\sum_{i=0}^{p-1}\varphi_i(\tau)\frac{(t-\tau)^i}{i!}\,d\tau\right|+N_2t^{p-1} \tag{2.6}$$

for $N_1=\sum_{i=0}^{p-1}\|H_i\|T^i$, $N_2=F\sum_{j=0}^{p-1}\|H_i\|T^j$, where F is such a constant that $|\varphi_i(\tau)|<F$ for $\tau\in[0, T]$, $i=0, ..., p-1$.

Proof. Since the assumption (2.5), we have for $\tau\in[0, T]$

$$g_i(\tau)=\sum_{j=0}^{i}H_j\varphi_{i-j}(\tau),\qquad i=0, ..., p-1$$

therefore for all $t, \tau\in[0, T]$

$$\sum_{i=0}^{p-1}g_i(\tau)\frac{t^i}{i!}=\sum_{i=0}^{p-1}\frac{t^i}{i!}\sum_{j=0}^{i}H_j\varphi_{i-j}(\tau)=\sum_{i,j=0}^{p-1}H_j\varphi_i(\tau)\frac{t^{i+j}}{(i+j)!}+R(t^p) \tag{2.7}$$

where $|R(t^p)|\leqslant N_2t^p$ for $t\in[0, T]$. Recall the following formula that holds for each measurable, bounded function $\varphi(\tau)$:

$$\int_0^t\varphi(\tau)\frac{(t-\tau)^k}{k!}=\int_0^t\int_0^{\tau_1}\dots\int_0^{\tau_k}\varphi(\tau)\,d\tau\,d\tau_k\dots d\tau_1. \tag{2.8}$$

Using (2.7) and (2.8) we obtain:

$$\left|\int_0^t\sum_{i=0}^{p-1}g_i(\tau)\frac{(t-\tau)^i}{i!}d\tau\right|\leqslant\left|\int_0^t\sum_{i,j=0}^{p-1}H_i\varphi_j(\tau)\frac{(t-\tau)^{i+j}}{(i+j)}\right|d\tau+N_2t^{p+1}\leqslant$$

$$\leqslant\left|\sum_{i=0}^{p-1}H_i\int_0^t\sum_{j=0}^{p-1}\varphi_j(\tau)\frac{(t-\tau)^{i+j}}{(i+j)!}d\tau\right|+N_2t^{p+1}\leqslant$$

$$\leqslant N_1\sup_{[0,T]}\left|\int_0^t\sum_{i=0}^{p-1}\varphi_i(\tau)\frac{(t-\tau)^i}{i!}d\tau\right|+N_2t^{p+1},$$

what completes the proof of Lemma 2.2.

Take a vector $w \in R^n$ and consider

$$I_{z,u,v}(t) - \left(\frac{1}{p!} w\right) t^p = \int_0^t \sum_{i=0}^{p-1} f_i(z(\tau), u(\tau), v(\tau)) \frac{(t-\tau)^i}{t!} - w \frac{(t-\tau)^{p-1}}{(p-1)!} d\tau. \tag{2.9}$$

From the condition (F) for all $t \in [0, T(z_*)]$, $u \in U$, $v \in V$, $z \in U_{z_*}$ we have:

$$\sum_{i=0}^{p-1} f_i(z, u, v) t^i - wt^{p-1} = H(t) \sum_{i=0}^{p-1} \psi_i(z, u, v) t^i +$$

$$+ \sum_{i=0}^{p-1} \alpha_i(z, u, v) t^i + \sum_{i=0}^{p-1} \beta_i t^i + wt^{p-1} + R(t^p). \tag{2.10}$$

Take $w_0, \tilde{r}$ such that $K(w_0, \tilde{r}) \subset \bigcap_{u \in U} \operatorname{co} \psi_0(z_*, u, v)$ where $K(w_0, r)$ denotes the ball of radius $\tilde{r}$ around w_0. Because of (2.10)

$$\sum_{i=0}^{p-1} f_i(z, u, v) t^i - wt^{p-1} = H(t) (\psi_0(z, u, v) - w_0 +$$

$$+ \sum_{i=1}^{p-1} \psi_i(z, u, v) t^i - wt^{p-1} + \sum_{i=0}^{p-1} \chi_i t^i + \sum_{i=0}^{p-1} \alpha_i(z, u, v) t^i + R(t^p) \tag{2.11}$$

where $\chi_i = \beta_i + H_i w_0$, the part $\sum_{i=p}^{\infty} H_i w_0$ has been included into $R(t^p)$. Since the indices of the function $H(t)$ are at most $(p-1)$ the function $t^{p-1} H^{-1}(t)$ is analytical around zero; that is, $t^{p-1} H^{-1}(t) = \sum_{i=0}^{\infty} F_i t^i$. Assume that $w \in K(0, \tilde{r})$ then $\sum_{i=p}^{\infty} (F_i(w) t^{i-p}$ is bounded uniformly with respect to w and we may include $H(t) \sum_{i=p}^{\infty} (F_i w) t^i$ into $R(t^p)$. Because of (2.11) we have

$$\sum_{i=0}^{p-1} f_i(z, u, v) t^i - wt^{p-1} - \sum_{i=0}^{p-1} \chi_i t^i - \sum_{i=0}^{p-1} \alpha_i(z, u, v) t^i =$$

$$= H(t) ((\psi_0(z, u, v) - w_0) + \sum_{i=1}^{p-1} \psi_i(z, u, v) t^i + \sum_{i=0}^{p-1} (F_i w) t^i) + R(t^p). \tag{2.12}$$

Lemma 2.2 gives then that there exists a constant N_{z_*} such that for any interval $[0, T_{z_*}] \subset [0, T_{z_*}]$ and any trajectory of (1.1) with an initial condition in V_{z_*} the following inequality holds for all $t \in [0, T_{z_*}]$

$$\left| \int_0^t - \sum_{i=0}^{p-1} f_i(z(\tau), u(\tau), v(\tau)) \frac{(t-\tau)^i}{i!} + w \frac{(t-\tau)^{p-1}}{(p-1)!} d\tau + \right.$$

$$\left. + \int_0^t - \sum_{i=0}^{p-1} \chi_i \frac{(t-\tau)^i}{i!} d\tau + \int_0^t \sum_{i=0}^{p-1} \alpha_i(z(\tau), u(\tau), v(\tau)) \frac{(t-\tau)^i}{i!} d\tau \right| \leqslant$$

$$\leqslant N_{z_*} \sup_{t \in [0, T_{z_*}]} \left| \int_0^t \big(\psi_0 (z(\tau), u(\tau), v(\tau)) - w_0\big) + \right.$$

$$\left. + \sum_{i=1}^{p-1} \psi_i \big((z(\tau), u(\tau), v(\tau)\big) \frac{(t-\tau)^i}{i!} - \sum_{i=0}^{p-1} (F_i w) \frac{(t-\tau)^i}{i!} d\tau \right| + N_{z_*} t^{p+1}. \tag{2.13}$$

Denote

$$\chi_p(t) = \int_0^t - \sum_{i=1}^{p-1} \chi_i \frac{(t-\tau)^i}{i!} d\tau.$$

We have then from (2.13), (2.9) and (1.3):

$$\left| I_{z,u,v}(t) + \chi_p(t) - \left(\frac{1}{p!} w\right) t^p \right| \leqslant$$

$$\leqslant N_{z_*} \sup_{t \in [0, T_{z_*}]} \left| \int_0^t \big(\psi_0 (z(\tau), u(\tau), v(\tau)) - w_0\big) + \right.$$

$$\left. + \sum_{i=1}^{p-1} \psi_i \big(z(\tau), u(\tau), v(\tau)\big) \frac{(t-\tau)^i}{i!} - \sum_{i=0}^{p-1} (F_i w) \frac{(t-\tau)^i}{i!} d\tau \right| +$$

$$+ N_{z_*} t^{p+1} + D \int_0^t \sum_{i=1}^{p-1} \rho^{p-1} (z(\tau), M) \frac{(t-\tau)^i}{i!} d\tau. \tag{2.14}$$

Fix now T_{z_*}, r, V_{z_*} such that $r \leqslant \tilde{r}$, $T_{z_*} \leqslant \tilde{T}_{z_*}$, $V_{z_*} \subset \tilde{V}_{z_*}$

$$\sum_{i=0}^{p-1} \|F_i\| \, T^i_{z_*} r + \sum_{i=1}^{p-1} |\psi_i(z, u, v)| \, T^i_{z_*} \leqslant \frac{\tilde{r}}{4} \quad \text{for } u \in U, \quad v \in V, \quad z \in U_{z_*} \tag{2.15}$$

and any trajectory of (1.1) with an initial condition $z_0 \in V_{z_*}$ remains for $t \in [0, T]_{z_*}$ in a neighbourhood $\tilde{\tilde{v}}_{z_*}$ such that

$$|\psi_0(u, u, v) - \psi_0(z_*, u, v)| \leqslant \frac{\tilde{r}}{4} \quad \text{for } z \in v_{z_*}, \quad u \in U, \quad v \in V. \tag{2.16}$$

We shall show that for every $z_0 \in V_{z_*}$, every $w \in K(0, r)$ and every $\varepsilon > 0$ a strategy $v^u(t) = v_z(z_0, w, \varepsilon; t)$ can be constructed such that for each control function $u(t)$ the following inequality holds:

$$\sup_{t \in [0, T_{z_*}]} \left| \int_0^t \big(\psi_0 (z(\tau), u(\tau), v^u(\tau)) - w_0\big) + \right.$$

$$\left. + \sum_{i=1}^{p-1} \psi_i \big(z(\tau), u(\tau), v^u(\tau)\big) \frac{(t-\tau)^i}{i!} + \sum_{i=1}^{p-1} + (F_i w) \frac{(t-\tau)^i}{i!} d\tau \right| < \varepsilon \tag{2.17}$$

that is

$$\sup_{t\in[0,T_{z_*}]}\left|\int_0^t \tilde{\psi}_0(z(\tau),u(\tau),v^u(\tau))+k_{z,u,v,w}(\tau)\,d\tau\right|<\varepsilon \tag{2.18}$$

where

$$k_{z,u,v,w}(\tau)=-F_0w+\sum_{i=1}^{p-1}\int_0^{\tau}\int_0^{\tau_1}\dots\int_0^{\tau_{i-1}}\big(\psi_i(z(s),u(s),v^u(s))-$$

$$-F_i w\,ds\,d\tau_{i-1}\dots d\tau,\ \tilde{\psi}_0(z,u,v)=\psi_0(z,u,v)-w_0\,.$$

By (2.15) along any trajectory $z(s)$ such that $z_0\in V_{z_*}$, for each $\tau\in[0,T_{z_*}]$ and each $w\in K(0,r)$ we have:

$$|k_{z,u,v,w}(\tau)|\leqslant\frac{\tilde{r}}{4}\,. \tag{2.19}$$

Since the definitions of $\tilde{r}$ and $\tilde{\psi}_0$, $K(0,r)\subset\bigcap_{u\in U}\operatorname{co}\tilde{\psi}_0(z_*,u,V)$. This implies that for any vector $s\in R^2$ and each u there exists v^u such that

$$\langle\tilde{\psi}_0(z_*,u,v^u),s\rangle\geqslant|s|\,r\,.$$

Take a constant R such that

$$|\tilde{\psi}_0(z,u,v)|+\frac{\tilde{r}}{4}<R\quad\text{for}\quad z\in U_{z_*},\quad u\in U,\quad v\in V. \tag{2.20}$$

Fix $z_0\in v_{z_*}$, $\varepsilon>0$, $w\in K(0,r)$. Divide the interval $[0,T_{z_*}]$ into n intervals $I_1,\dots,I_n$ of the length $\delta=T_{z_*}/n$, $I_1\cup\dots\cup I_n=[0,T_{z_*}]$ such that

$$\frac{T_{z_*}}{\sqrt{n}}R<\varepsilon\,. \tag{2.21}$$

We shall construct a strategy $v^u(t)$ step-by-step on each of the intervals $I_1,\dots,I_n$. Fix an element $\bar{v}\in V$ and put

$$v^u(t)\equiv\bar{v}\quad\text{for}\quad t\in I_1\,.$$

In order to define $v^u(t)$ on I_j take:

$$s_{j-1}=\int_0^{(j-1)\delta}\big(\tilde{\psi}_0(z(\tau),u(\tau),v^u(\tau))+k_{z,u,v,w}(\tau)\big)\,d\tau$$

and choose a measurable $v^u(t)$ such that

$$\langle\tilde{\psi}_0(z_*,u(t),v^u(t)),-s_{j-1}\rangle\geqslant|s_{j-1}|\,\tilde{r}\,. \tag{2.22}$$

Namely, for each t we choose from all $v^u(t)$ satisfying (2.22) the lexicographical maximum with respect to a certain fixed basis in R^q; that is, from all vectors $v^u(t)$ satisfying (2.22) we choose the ones whose first coordinate is maximal, from those the ones whose second coordinate is maximal and so forth. In this way for each measurable function $u(t)$ the function $v^u(t)$ is uniquely defined and measurable. Since (2.16), (2.19) and (2.22) we have that

$$\langle \tilde{\psi}_0(z(\tau), u(\tau), v^u(\tau)) + k(\tau), -s_{j-1}\rangle \geq \frac{1}{2}|s_{j-1}|\tilde{r} \geq 0, \quad \tau \in I_j, \quad j=1, \dots, n. \tag{2.23}$$

(2.19), (2.20) and (2.23) imply that for $t \in I_j, j=1, \dots, n$, the following holds:

$$\left|\int_0^t \left(\tilde{\psi}_0(z(\tau), u(\tau), v^u(\tau)) + k(\tau)\right) d\tau\right| = \left| s_{j-1} + \right.$$

$$\left. + \int_{(j-1)\delta}^t \left(\tilde{\psi}_0(z(\tau), u(\tau), v^u(\tau)) + k(\tau)\right) d\tau \right| \leq \sqrt{|s_{j-1}|^2 + R^2\delta^2}.$$

Thus an introduction argument gives the following inequality:

$$\left|\int_0^t \left(\tilde{\psi}_0(z(\tau), u(\tau), v^u(\tau)) + k(\tau)\right) d\tau\right| \leq R\delta\sqrt{j} \quad \text{for } t \in I_j, \quad j=1, \dots, n. \tag{2.24}$$

(2.24) together with (2.21) imply (2.18) and (2.17).

Thus we have proved that for every $z_0 \subset V_{z_*}$, every $\varepsilon > 0$ and every $w \in K(0, r^*)$, $r^* = \frac{1}{p!} r$, there exists a strategy $v^u(t) = v^u(z_0, \varepsilon, w; t)$ defined for $t \in [0, T_{z_*}]$ that ensures the following inequality for $t \in [0, T_{z_*}]$:

$$|\chi_p(t) + I_{z,u,v}(t) - wt^p| \leq \varepsilon + N_{z_*} t^{p+1} + D\int_0^t \sum_{i=0}^{p-1} \rho^{p-1}(z(\tau), M)\frac{(t-\tau)^i}{i!} d\tau. \tag{2.25}$$

Take a constant C_{z_*} such that

$$|P(z, u, v)| < \frac{C_{z_*}}{2} \quad \text{for } z \in U_{z_*}, \quad u \in U, \quad v \in V. \tag{2.26}$$

Assume moreover that $\rho(z_0, M) \leq C_{z_*}$ for all $z_0 \in V_{z_*}$ what will be convenient later. Each trajectory of (1.2) with $z_0 \in V_{z_*}$ satisfies

$$\rho(z(t), M) \geq \rho(z_0, M) - \frac{C_{z_*}}{2} t \quad \text{for } t \in [0, T_{z_*}]$$

therefore

$$\rho(z(t), M) \geq \frac{\rho(z_0, M)}{2} \quad \text{for } t \in \left[0, \frac{\rho(z_0, M)}{C_{z_*}}\right]. \tag{2.27}$$

For $t \in \left[\frac{\rho(z_0, M)}{C_{z_*}}, T_{z_*}\right]$ the last term in (2.25) can be estimated in the following way:

$$\int_0^t \rho^{p-1}(z(\tau), M)\frac{(t-\tau)^i}{i!} d\tau \leq t^i \int_0^t \left(\rho(z_0, M) + \tau\frac{C_{z_*}}{2}\right)^{p-1} d\tau \leq$$

$$\leq t^{i+1}\left(\rho(z_0, M) + t\frac{C_{z_*}}{2}\right)^{p-i} \leq t^{p+1}\left(C_{z_*} + \frac{C_{z_*}}{2}\right)^{p-i}, \quad i=0, \dots, p-1. \tag{2.28}$$

Apply Lemma 2.1 now that is, take a cube $\Omega_{z_*} \subset K(0, r_*)$ and for $z_0 \in V_{z_*}$ choose $w=w(z_0)$ corresponding as in Lemma 2.1 to the curve $w_p(t)=w_p(t; z_0)-\chi_p(t)$. Then since (2.3), (2.5), (2.25) and (2.28) the strategy $v^u(z_0, w(z_0), \varepsilon; t)$ ensures

$$|\pi z(t)| \geqslant \Theta_{z_*} t^p - F_{z_*} t^{p+1} - \varepsilon \quad \text{for } t \in \left[\frac{\rho(z_0, M)}{C_{z_*}}, T_{z_*}\right] \tag{2.29}$$

where $F_{z_*}=N_{z*}+N_{h, \tilde{\gamma} z_*}+Dp\left(\frac{3}{2} C_{z_*}\right)^p$, Θ_{z_*} is a constant corresponding as in Lemma 2.1 to the cube Ω_{z_*}. We can assume making eventually T_{z_*} smaller that $T_{z_*} \leqslant \leqslant \frac{\Theta_{z_*}}{2F_{z_*}}$ and then

$$\Theta_{z_*} t^p - F_{z_*} t^{p+1} \geqslant \frac{\Theta_{z_*}}{2} t^p \quad \text{for all } t \in [0, T_{z_*}]. \tag{2.30}$$

Take a positive constant K_{z_*} such that $K_{z_*} < \frac{\Theta_{z_*}}{4}$, $K_{z_*} < \frac{C^p_{z_*}}{2}$. Choose now for each $z_0 \in V_{z_*}$, $\varepsilon(z_0)=\frac{\Theta_{z_*}}{4}\left(\frac{\rho(z_0, M)}{C_{z_*}}\right)^p$. Then since (2.29), (2.27), (2.30) the strategy $v^u_{z_*}(z_0; t)=v^u(z_0, w(z_0), \varepsilon(z_0); t)$ defined for all $z_0 \in V_{z_*}$ and $t \in [0, T_{z_*}]$ ensures:

$$\rho(z)(t), M \geqslant K_{z_*} t^{p(z^*)}, \rho(z(t), M) \geqslant K_{z_*}\left(\frac{\rho(z_0, M)}{C_{z_*}}\right)^{p(z_*)} \quad \text{for } t \in [0, T_{z_*}]. \tag{2.31}$$

B. We proceed to construct a strategy of evasion $v^u(z_0; t)$. Take a sequence $0=r_0<r_1<\ldots<r_i<r_{i+1}<\ldots$, $i=1,2, \ldots$, such that for every $i=1,2, \ldots$ if $z_0 \in K(0, r_i)$ then for any tarjectory $z(t)$ of (1.1) $z(t) \in \operatorname{int} K(0, r_{i+1})$ for $t \in [-1,1]$ where as befor $K(0, r_i)$ denotes the closed ball around the origin of radius r_i.

Denote $M_i=M \cap K(0, r_i)$ $(V_{z_*})_{z_* \in M_i}$ is an open covering of the compact set M_i. Choose then a finite covering $V_{z^1_*, i}, \ldots, V_{z^{m_i}_*, i}$ and define

$$K_1=\min\{K_{z^1_*, 1}, \ldots, K_{z^{m_i}_*, 1}\}, \qquad K_i=\min\{K_{i-1}, K_{z^1_*, i}, \ldots, K_{z^{m_i}_*, i}\} \quad \text{for } i=2, 3, \ldots,.$$

Put $T_0=1$, $p_0=1$, $C_0=1$ and define for $i=1, 2, \ldots$

$$T_i=\min\{T_{i-1}, T_{z^1_*, i}, \ldots, T_{z^{m_i}_*, i}\},$$

$$p_i=\max\{p_{i-1}, p_{z^1_*, i}, \ldots, p_{z^{m_i}_*, i}\}.$$

$$C_i=\max\{C_{i-1}, C_{z^1_*, i}, \ldots, C_{z^{m_i}_*, i}\}.$$

Therefore each of strategies $v^u_{z^j_*, i}(z_1; t)$, $j=1, \ldots, m_i$, ensures

$$\rho(z(t), M) \geqslant K_i t^{p_i}, \quad \rho(z(t), M) \geqslant K_i\left(\frac{\rho(z_0, M)^{p_i}}{C_i}\right) \quad \text{for } t \in [0, T_i]. \tag{2.32}$$

Take for every $i=1, 2, \ldots$ a cylinder $W(\rho_i, M)=\{z \in R^n \mid \rho(z, M) \leqslant \rho_i\}$ such that:

$$W(\rho_i, M) \cap K(0, r_i) \subset V_{z^1_*, i} \cup \ldots \cup V_{z^{m_i}_*, i} \quad i=1, 2, \ldots \tag{2.33}$$

and a sequence of positive numbers $\sigma_0, \sigma_1, ..., \sigma_i, ...$ such that

$$\sigma_0=\sigma_1, \quad \sigma_i \leqslant \rho_i, \quad \sigma_i < K_{i+1} T_{i+1}^{p_{i+1}}; \quad \sigma_i \leqslant \sigma_{i-1} \quad \text{for } i=1, 2, 3, \ldots . \tag{2.34}$$

Define

$$\eta_i = \frac{K_{i+1}\, \sigma_{i+1}^{p_{i+1}}}{2C_{i+1}^{p+1}}, \qquad i=1, 2, \ldots$$

and

$$W=\bigcup_{i=1}^{\infty} W(\sigma_i, M) \cap K(0, r_i), \qquad W_1=\bigcup_{i=1}^{\infty} W(\eta_i, M) \cap K(0, r_i).$$

We can describe now a strategy of evasion $v^u(z_0; t)$. Fix an element $\bar{v} \in V$. Let at first $z_0 \notin W$, then no matter what the pursuer's control function is put $v^u(z_0; t) \equiv \bar{v}$ as long as the corresponding trajectory satisfies $z(t) \notin W$. Let t_1 be the first moment such that $z(t_1) \in W$. Denote $z(t_1)=z_1$ and let $|z_1| \in (r_{i-1}, r_i]$. Then $z_1 \in K(0, r_i) \cap$ $\cap\, W(\sigma_i, M)$. Denote by j_0 the smallest of all integers j for which $z_1 \in V_{z_*^j, i}$ and define $v^u(z_0; t)=v_{z_*^{j_0}, i}^{\tilde{u}}(z_1; t-t_1)$ for $t \in [t_1, t_1+T_i]$, where $\tilde{u}(t-t_1)=u(t)$.

We have $z_1 \in \partial W$ and hence $\rho(z_1, M) \geqslant \sigma_{i+1}$ thus (2.32) gives that

$$\rho(z(t), M) \geqslant K_i (t-t_1)^{p_i}, \qquad \rho(z(t), M) \geqslant K_i \frac{\sigma_{i+1}^{p_i}}{C_i^{p_i}} \qquad \text{for } t \in [t_1, t_1+T_i]. \tag{2.35}$$

Therefore $\rho(z(t_1+T_i), M) \geqslant K_i T_i^{p_i} > \sigma_{i-1} \geqslant \sigma_i \geqslant \sigma_{i+1}$ and by definition of the sequence $r_0, ..., r_i, ...$ $|z(t_1+T_i)| \in (r_{i-2}, r_{i+1})$ if $i \geqslant 2$ and $|z(t_1+T_i)| \in (0, r_2)$ if $i=1$ thus $z(t_1+T_i) \notin W$. Put again $v^u(z_0; t) \equiv \bar{v}$ till the next moment t_2 such that $z(t_2) \in W$, when as before one of the strategies $v_{z_*^i, j}^{u}(z_2, t-t_2)$ is swiched on. If $z_0 \in W$ and $|z_0| \in (r_{i-1}, r_i]$ we switch on at once the strategy $v_{z_*^{j_0}, j}^{u}(z_0; t)$, that is we put $v^u(z_0; t)=v_{z_*^{j_0}, i}^{u}(z_0; t)$ for $t \in [0, T_i]$ where j_0 is the smallest of all integers j for which $z_0 \in V_{z_*^j, i}$. Since (2.32), we have

$$\rho(z(t), M) \geqslant K_i\, t^{p_i}, \rho(z(t), M) \geqslant K_i \frac{\rho(z_0, M)^{p_i}}{C^{p_i}} \qquad \text{for } t \in [0, T_i]. \tag{2.36}$$

Thus $z(T_i) \notin W$ and we proceed as befor.

Let $T_i(j)$, $j=1, 2, ...,$ $i(j) \in \{1, 2, ...\}$ denotes the duration of the j-th local manoeuvre of evasion, s_j, $j=1, 2, ...,$ the time between the (j-1)-th and the j-th manoeuvre.

The game goes the way described above over the interval $[0, \sum_{j=1}^{\infty} s_j+T_{i(j)}]$. Suppose that $\sum_{j=1}^{\infty} s_j+T_{i(j)} < +\infty$. Then the trajectory remains over the interval in a certain ball, hence $i(j) \in \{1, 2, ..., n_0\}$ for some n_0. But this implies that $\sum_{i=1}^{\infty} T_{i(j)} =$ $=+\infty$. Therefore, $\sum_{j=1}^{\infty} T_{i(j)}+s_j=+\infty$, and the procedure defines the strategy $v^u(z_0, t)$ for all $t \in [0, +\infty)$ and $z_0 \notin M$.

Define

$$T(\xi)+T_i \quad \text{for } \xi\in(r_{i-1}, r_i], \quad i=1,2,...$$

$$\gamma(\xi_1,\xi_2)=\frac{K_i}{C_i^{p_i}}\xi_1^{p_i} \quad \text{for } \xi_1\in(0,+\infty), \quad \xi_2\in(r_{i-1},r_i], \quad i=1,2,\dots.$$

The functions $T(\xi)$, $\gamma(\xi_1,\xi_2)$ have the properties required in. Theorem 1.1. So does the set W_1 defined above. Indeed, assume that $z(t_1)\notin W$ and take $t>t_1$. Let $|z(t)|\in(r_{i-1}, r_i]$. If $\rho(z(t), M)\geqslant\sigma_{i+1}$ then $z(t)\notin W_1$ as $\sigma_{i+1}>\eta_i$ If $\rho(z(t), M)<\sigma_{i+1}$ then $z(t)\in \operatorname{int} W$ and the trajectory is on the course of action of a local manoeuvre of evading which began at some earlier moment at a point $z_2\in\partial W$, $|z_2|\in(r_{i-2}, r_{i+1})$. Then (2.35) gives that $\rho(z(t), M)\geqslant K_{i+1}\frac{\sigma_{i+1}^{p_i+1}}{C_{i+1}^{p_i+1}}>\eta_i$ thus $z(t)\notin W_1$. This completes the proof of the evasion theorem.

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Warunek wystarczający dla ucieczki w grze nieliniowej. Część II

Dla nieliniowej gry różniczkowej ucieczki dowiedziono, że przy pewnym warunku omówionym w pracy [4] ucieczka jest możliwa dla każdego stanu początkowego gry. Skonstruowano strategię ucieczki i oceniono odległość trajektorii gry od podprzestrzeni końcowej.

Достаточное условие уклонения в некоторой нелинейной игре. Часть II

Для нелинейной дифференциальной игры уклонения доказывается, что при некотором условии, представленном в [4] уклонения является возможным для любого начального состояния игры. Разрабатывается стратегия уклонения и оценивается расстояние траектории игры от конечного подпространства.