Control
and Cybernetics

VOL. 8 (1979) No. 1

Convex Control and Dual Approximations.*) Part I

by

WILLIAM W. HAGER

Department of Mathematics
Carnegie-Mellon University
Pittsburgh, Pennsylvania

Using the Lagrange dual formulation for convex control problems [2] we obtain error estimates for finite element approximations. The earlier analysis [4] of the Ritz-Trefftz method for quadratic cost problems with affine inequality state and control constraints is extended to treat the general convex case. Also two new variations of the Ritz-Trefftz metod are introduced and analyzed.

Introduction

Earlier [4] we considered Ritz-Trefftz approximations to quadratic cost control problems with affine inequality constraints on the state and the control. Two new variations of the Ritz-Trefftz method are now introduced, and the previous analysis is extended to treat the general convex problem. For other recent work in infinite element theory for contrained control problems, see papers by Malanowski [11–14], Lasiecka and Malanowski [7, 8], Holnicki [17] and Dontchev [18]. An outline of the paper follows:

Section 1 — Introduces the convex control problem along with its Lagrange dual, and discusses the regularity of solutions.

Section 2 — Develops minimum principles that are needed to evaluate the dual functional.

Section 3 — Formulates the finite element approximations.

Section 4 — Develops a general theory for estimating the error in dual approximations.

Section 5 — Bounds the error in approximating functions with restricted range.

Section 6 — Applies the results of the previous sections to estimate the error in dual approximations to control problems.

---

*) This research was supported in part by National Science Foundation Grant MCS75-09457 and Office of Naval Research Grant N00014-76-C-0369 and was presented at the International Conference on Methods of Mathematical Programming, Zakopane, Poland, September 12–16, 1977.

Acknowledgments

The author wishes to thank Professors George Fix and Victor Mizel for their helpful comments. In particular, the author would like to acknowledge stimulating discussions with Irena Lasiecka and Kazimierz Malanowski that helped initiate this paper during a visit to Poland in the fall, 1977. Finally, the author is grateful for financial support received from the Polish Academy of Sciences, the National Science Foundation and Carnegie-Mellon University.

1. Duality and Reqularity

The Ritz-Trefftz method involves two ideas:

(i) *Duality* — the original problem is replaced by its dual.

(ii) *Finite elements* — the dual multipliers are approximated by finite element subspaces.

In this section, we summarize both Lagrange duality principles and regularity properties for convex control problems. A knowledge of solution regularity is needed to estimate the error in piecewise polynomial approximation.

The following notation is used for spaces of functions f: $[0, 1] \to R^n$:

$C^p (R^n)$ Functions with continuous derivatives through order p.

$A (R^n)$ Absolutely continuous functions.

$BV (R^n)$ Functions of bounded variation that are left continuous on $[0, 1)$.

$L^\infty (R^n)$ Essentially bounded functions.

$L^p (R^n)$ Functions with $\int_0^1 |f(t)|^p \, dt < \infty$

where $|\cdot|$ denotes the Euclidean norm. The argument R^n above is omitted when the range is clear from context. Finally, we let $\|\cdot\|$ and $\lceil\cdot\rceil$ denote the L^2 and L^∞ norms on $[0, 1]$, respectively.

Consider the following control problem:

$$\text{(C)} \qquad \text{minimize } \{C(x, u) = \int_0^1 f(x(t), u(t), t) \, dt\}$$

$$\text{subject to } \dot{x}(t) = A(t) x(t) + B(t) u(t) \text{ for almost every } t \in [0, 1]$$

$$\left.\begin{aligned} K_c(u(t), t) \leqslant 0 \\ K_s(x(t), t) \leqslant 0 \end{aligned}\right\} \text{ for all } t \in [0, 1] \tag{1.1}$$

$$x(0) = x_0, \; x \in A(R^n), \; u \in L^2(R^m)$$

where K_c and K_s are vector valued with range in R^{m_c} and R^{m_s}, respectively. (C) is assumed to satisfy conditions (1.2)–(1.4) below, but first some notation is needed: Given two symmetric matrices M_1 and M_2, the statement $M_1 > M_2$ means that $M_1 - M_2$ is positive definite. If g: $R^{m_1} \times R^{m_2} \times \ldots \times R^{m_l} \to R$, we let $\nabla_j g$ and $\nabla_j^2 g$ denote the gradient and the Hessian respectively of $g(y_1, \ldots, y_l)$ with respect to y_j where $y_k \in R^{m_k}$ for $k=1, \ldots, l$. We assume the following:

A and B are Lipschitz continuous while f, K_c, K_s, and $\nabla_1 K_s(\cdot, \cdot)$ are C^2. (1.2)

Both $f(\cdot, \cdot, t)$ and the components of $K_s(\cdot, t)$ and $K_c(\cdot, t)$ are convex for all $t \in [0, 1]$. Moreover, there exists $\alpha > 0$ such that

$$\nabla_1^2 f(z, t) > \alpha I \tag{1.3}$$

for all $z \in R^{n+m}$ and $t \in [0, 1]$.

There exists a continuous control $\bar{u}$, a corresponding trajectory $\bar{x}$, and a constant $\beta < 0$ such that

$$K_c(\bar{u}(t), t)_j < \beta > K_s(\bar{x}(t), t)_i \tag{1.4}$$

for all $t \in [0, 1]$, $j=1, \ldots, m_c$, and $i=1, \ldots, m_s$.

Using classical techniques in convex analysis, (1.2)–(1.4) imply for (C) the existence of an optimal control $u^* \in L^2$ and a corresponding trajectory $x^* \in A$, and all optimal controls are equal almost everywhere. Furthermore, by Appendix 2, $u^* \in L^\infty$.

The dual of (C) is now introduced. Given a measurable set $I \subset [0, 1]$, let $\langle \cdot, \cdot \rangle_I$ denote the L^2 inner product on I and set $\langle \cdot, \cdot \rangle \equiv \langle \cdot, \cdot \rangle_{[0,1]}$. If $I \subset [0, 1]$ is the union of intervals and $f \in BV(R^n)$, define the functional $[f, \cdot]_I$ as follows:

$$[f, g]_I = \int_I g(t)^T \, df(t)$$

for all continuous functions g, and set $[\cdot, \cdot] \equiv [\cdot, \cdot]_{[0,1]}$.

The *Lagrange dual function* associated with (C) is given by:

$$\mathscr{L}(p, \lambda, v) = \inf\{\mathscr{J}(x, u) \equiv C(x, u) + \langle p, \dot{x} - Ax - Bu \rangle + \langle \lambda, K_c(u) \rangle + [v, K_s(x)]:$$

$$x(0) = x_0,\ x \in A(R^n),\ u \in L^\infty(R^m)\}; \tag{1.5}$$

and the Lagrange dual to (C) becomes:

(D) $\quad \sup\,[\mathscr{L}(p, \lambda, v): (p, v) \in BV,\ \lambda \in L^1,\ \lambda \geqslant 0,\ v(1) = 0,\ v$ nondecreasing$\}$.

Now recall our strong duality result [2]:

THEOREM 1.1.. *If (1.2)–(1.4) hold, then there exist optimal solutions (x^*, u^*) to (C) and (p^*, λ^*, v^*) to (D) with $C(x^*, u^*) = \mathscr{L}(p^*, \lambda^*, v^*)$. Moreover, (x^*, u^*) achieve the minimum in (1.5) for $(p, \lambda, v) = (p^*, \lambda^*, v^*)$ and the following complementary slackness conditions hold:*

$$\langle \lambda^*, K_c(u^*) \rangle = 0 = [v^*, K_s(x^*)].$$

To guarantee Lipschitz continuity properties for solutions of (C) and (D), we assume that the binding constraints satisfy an independence condition:

There exists $\gamma>0$ such that for all $t\in[0, 1]$ and all z, we have:

$$|[G_c(t)^T, B(t)^T G_s(t)^T]\, z| \geqslant \gamma\, |z| \tag{1.6}$$

where $G_c(t)$ is the matrix whose rows are the gradients evaluated at $u^*(t)$ of components of $K_c(\cdot, t)$ corresponding to binding constraints for $u^*(t)$. The matrix $G_s(t)$ is defined similarly.

By [3] we have:

THEOREM 1.2. *If* (1.2)–(1.4) *and* (1.6) *hold, there exist optimal solutions* (x^*, u^*) *to* (C) *and* (p^*, λ^*, ν^*) *to* (D) *such that* $(u^*, p^*, \lambda^*, \nu^*)$ *ate Lipschitz continuous on* $[0, 1)$, *and* $(x^*(\cdot), q^*(\cdot)\equiv \nabla_1 K_s(\cdot^*(\cdot), \cdot)^T \nu^*(\cdot)-p^*(\cdot))$ *have Lipschitz continuous derivative on* $[0, 1)$.

As noted in [3], the interval of Lipschitz continuity is $[0, 1)$ since ν^* may be discontinuous at $t=1$. Although the regularity given by Theorem 1.2 is sharp, it was noted in [4] that the primal and dual solutions are generally very smooth except at the points where constraints change from binding to nonbinding. At these "contact points", the first derivatives of $(u^*, p^*, \lambda^*, \nu^*)$ and the second derivatives of (x^*, q^*) are usually discontinuous. Throughout this paper, we let (x^*, u^*) and (p^*, λ^*, ν^*) denote optimal solutions to (C) and (D) respectively, which possess at least the regularity given by Theorem 1.2.

REMARK 1.3: *Although many of the theorems in this paper are valid under much weaker assumptions than* (1.2)–(1.4), *we often begin theorems with the phrase "suppose that* (1.2)–(1.4) *are satisfied" in order to shorten the exposition.*

2. Minimum Principles

In this section, we study the evaluation of the dual functional. Since the value of $\mathscr{L}$ is given through a minimization problem, the (x, u) pair achieving the minimum in (1.5) shall be characterized. Also recalling that the error in finite element approximation depends on the regularity of the variables being approximated, we change from the unknown p in the dual functional, to smoother variables.

If p and $\dot{\nu}$ are absolutely continuous on $[0, 1]$ and $[0, 1)$, respectively, we can integrate by parts $\mathscr{I}(x, u)$ in (1.5) to get:

$$\mathscr{I}(x, u)=\mathscr{I}_0(x, u)+p(1)^T x(1)-p(0)^T x_0-K_s(x(1), 1)^T \nu(1^-) \tag{2.1}$$

where

$$\mathscr{I}_0(x, u)=C(x, u)-\langle \dot{p}, x\rangle-\langle p, Ax+Bu\rangle+\langle \lambda, K_c(u)\rangle+\langle \dot{\nu}, K_s(x)\rangle. \tag{2.2}$$

Moreover, $\mathscr{J}_0$ can be expressed in the form

$$\mathscr{J}_0(z)=\int_0^1 h\big(z(t),\eta(t),t\big)\,dt \tag{2.3}$$

where

$$\begin{cases} z(t)^T=\big(x(t)^T,u(t)^T\big) \\ \eta(t)^T=\big(\dot p(t)^T,p(t)^T,\lambda(t)^T,\dot v(t)^T\big)\end{cases} \tag{2.4}$$

$$\begin{cases} h\big(z(t),\eta(t),t\big)=f\big(z(t),t\big)+\eta(t)^T g\big(z(t),t\big) \\ g\big(z(t),t\big)=\begin{bmatrix} -x(t) \\ -A(t)\,x(t)-B(t)\,u(t) \\ K_c\big(u(t),t\big) \\ K_s\big(x(t),t\big)\end{bmatrix}. \end{cases} \tag{2.5}$$

From the identity (2.1), we see that the minimization (1.5) can be uncoupled as follows:

$$\mathscr{L}(p,\lambda,v)=\mathscr{L}_0(p,\lambda,\dot v)+\mathscr{L}_1\big(p(1),v(1^-)\big)-p(0)^T x_0 \tag{2.6}$$

where

$$\mathscr{L}_0(p,\lambda,\dot v)=\inf\Big\{\int_0^1 h\big(z(t),\eta(t),t\big)\,dt: z\in AxL^\infty\Big\} \tag{2.7}$$

$$\mathscr{L}_1(\gamma,\delta)=\inf\{\gamma^T x-K_s(x,1)^T\delta: x\in R^n\}. \tag{2.8}$$

Defining the functional

$$\mathscr{L}_1(\gamma)=\inf\{\gamma^T x: x\in R^n,\ K_s(x,1)\leqslant 0\}, \tag{2.9}$$

recall the classical duality result:

$$\mathscr{L}_1(\gamma)=\sup\{\mathscr{L}_1(\gamma,\delta):\delta\leqslant 0\} \tag{2.10}$$

and if $\mathscr{L}_1(\gamma)>-\infty$, there exists $\delta(\gamma)$ such that $\mathscr{L}_1(\gamma)=\mathscr{L}_1\big(\gamma,\delta(\gamma)\big)$. Combining (1.5), (2.6), and (2.10), we have:

LEMMA 2.1. *If* (1.2)–(1.4) *hold, then program* (D) *with the additional constraint that p and v are absolutely continuous on* [0, 1] *and* [0, 1) *respectively, is equivalent to the program:*

$$\sup\{\mathscr{L}_0(p,\lambda,w)+\mathscr{L}_1\big(p(1)\big)-p(0)^T x_0:(\lambda,w)\geqslant 0,\ p\in A,\ (\lambda,w)\in L^1\}. \tag{2.11}$$

That is, the value of (2.11) *and* (D) *are equal. If* (p^*,λ^*,v^*) *solve* (D), *then* $(p^*,\lambda^*,w^*=\dot v^*)$ *solve* (2.11). *Conversely, if* (p^*,λ^*,w^*) *solve* (2.11), *then* (p^*,λ^*,v^*) *solve* (D) *where*

$$v^*(t)=\delta\big(p^*(1)\big)+\int_1^t w^*(s)\,ds.$$

■

The evaluation of $\mathscr{L}_0$ is now considered. Using Lemmas 2.2 and 2.3 below, we shall establish the following:

(i) For all $t \in [0, 1]$, there exists $\tilde{z}(t)$ which minimizes $h(\cdot, \eta(t), t)$.

(ii) $\tilde{z}(\cdot)$ is measurable and absolutely integrable.

(iii) $\mathscr{L}_0(p, \lambda, \dot{v}) = \int_0^1 h(\tilde{z}(t), \eta(t), t)\, dt$.

Our earlier paper [2], in contrast to (i)–(iii), assumed the existence of $\tilde{z} \in A \times L^\infty$ achieving the minimum in (2.7) and in [2, Theorem 5] we stated that $\tilde{z}$ satisfied a minimum principle. Now we show that the dual function can be evaluated by integrating a pointwise minimum. Moreover, $\tilde{z}(\cdot)$ need only be L^1.

LEMMA 2.2. *Let $K \subset R^n$ be nonempty, closed, and convex, $E \subset R^m$, $h: R^n \times R^m \to R$ be differentiable in its first n arguments on $K \times R^m$, and assume that there exists $\alpha > 0$ such that*

$$h(y, \xi) \geqslant h(z, \xi) + \nabla_1 h(z, \xi)(y - z) + \alpha |y - z|^2 \tag{2.12}$$

for all $y, z \in K$ and $\xi \in E$. Then for all $\xi \in E$, there exists a unique $z(\xi) \in K$ satisfying

$$h(z(\xi), \xi) = \text{minimum}\, \{h(y, \xi): y \in K\}; \tag{2.13}$$

and given $\bar{z} \in K$, we have

$$|z(\xi) - \bar{z}| \leqslant |\nabla_1 h(\bar{z}, \xi)|/\alpha. \tag{2.14}$$

Also, if $\nabla_1 h(z, \cdot)$ is continuous for all $z \in K$, then $z(\cdot)$ is continuous on E, and if, moreover, $\nabla_2 \nabla_1 h(\cdot, \cdot)$ is continuous, then $z(\cdot)$ is Lipschitz continuous on bounded subsets of E.

Proof. For related results, see [3] and [7, Theorem 1]. The existence of $z(\xi)$ satisfying (2.13) follows classically from (2.12) and the fact the K is closed. To obtain (2.14), insert $z = \bar{z}$ and $y = z(\xi)$ into (2.12) and observe that $h(z(\xi), \xi) - h(\bar{z}, \xi) \leqslant 0$. Now consider the continuity results.

Given $\xi_1, \xi_2 \in E$ define $z_1 = z(\xi_1)$ and $z_2 = z(\xi_2)$. Since h satisfies (2.12), we have the classical [11, p. 11] monotonicity of derivative:

$$2\alpha |z_1 - z_2|^2 \leqslant [\nabla_1 h(z_1, \xi_1) - \nabla_1 h(z_2, \xi_1)](z_1 - z_2). \tag{2.15}$$

By the convexity of K, the following inequalities also hold [11, p. 10]:

$$\begin{cases} 0 \leqslant \nabla_1 h(z_1, \xi_1)(z_2 - z_1) \\ 0 \leqslant \nabla_1 h(z_2, \xi_2)(z_1 - z_2). \end{cases} \tag{2.16}$$

Adding the relations (2.15)–(2.16), we get:

$$2\alpha |z_1 - z_2|^2 \leqslant [\nabla_1 h(z_2, \xi_2) - \nabla_1 h(z_2, \xi_1)](z_1 - z_2);$$

and by the Schwarz inequality, we conclude that:

$$|z_1 - z_2| \leqslant |\nabla_1 h(z_2, \xi_2) - \nabla_1 h(z_2, \xi_1)|/2\alpha. \tag{2.17}$$

If $\nabla_1 h(z, \cdot)$ is continuous for all $z \in K$, then (2.17) implies that $z(\cdot)$ is continuous on E. Moreover, if $\nabla_2 \nabla_1 h(\cdot, \cdot)$ is continuous, (2.17) gives us:

$$|z_1 - z_2| \leqslant \frac{|\xi_1 - \xi_2|}{2\alpha} \max_{0 \leqslant s \leqslant 1} |\nabla_2 \nabla_1 h(z_2, \xi_1 + s(\xi_2 - \xi_1))| . \tag{2.18}$$

By (2.14) and the fact that $\nabla_1 h(\bar{z}, \cdot)$ is continuous, we see that $z(\cdot)$ is uniformly bounded on any bounded subset $E_0 \subset E$. Hence the second derivative in (2.18) is bounded uniformly for all $\xi_1, \xi_2 \in E_0$, and the proof is complete. ■

LEMMA 2.3: *Suppose that* $\varphi: R^n \times [0, 1] \to R$, $\tilde{z}: [0, 1] \to R^n$, *and* $K: R^n \times [0, 1] \to R^m$ *have the following properties*:

(1) $\tilde{z} \in L^1$, $K(\cdot, \cdot)$ *and* $\varphi(\cdot, t)$ *are continuous for all* $t \in [0, 1]$, *and* $\varphi(y, \cdot)$ *is measurable for all* $y \in R^n$.

(2) $K(\tilde{z}(t), t) \leqslant 0$ *for all* $t \in [0, 1]$ *and* $\varphi(\tilde{z}(t), t) = \inf\{\varphi(z, t): z \in R^n, K(z, t) \leqslant 0\}$ *for almost every* $t \in [0, 1]$.

(3) *There exists* $g: R \times [0, 1] \to [0, \infty)$ *such that* $g(s, \cdot) \in L^1$ *for all* $s \geqslant 0$, $g(\cdot, t)$ *is monotone nondecreasing, and for all* $z \in C^\infty$, *we have*:

$$|\varphi(z(t), t)| \leqslant g(|z|, t)$$

for almost every $t \in [0, 1]$.

(4) *The components of* $K(\cdot, t)$ *are convex for all* $t \in [0, 1]$, *and there exists* $\gamma < 0$ *and* $\bar{z} \in C^\infty$ *such that*

$$K(\bar{z}(t), t)_j < \gamma \textit{ for all } t \in [0, 1] \textit{ and } j = 1, 2, \ldots, m.$$

Then we have

$$\int_0^1 \varphi(\tilde{z}(t), t)\, dt = \inf\Big\{\int_0^1 \varphi(z(t), t)\, dt : z \in C^\infty,$$
$$K(z(t), t) \leqslant 0 \textit{ for all } t \in [0, 1]\Big\}. \tag{2.19}$$

In our present applications, K is vacuous; however, the inclusion of the constraint $K(z(t), t) \leqslant 0$ only increases the length of the proof slightly and in a subsequent paper [5], we consider cases where K is nonvacuous.

Proof of Lemma 2.3. By assumptions (1)–(3), $\varphi(\tilde{z}(\cdot), \cdot)$ is measurable and lies below the L^1 function $g(|\tilde{z}|, \cdot)$ almost everywhere on $[0, 1]$; hence, the integral of $\varphi(\tilde{z}(\cdot), \cdot)$ exists. We first prove the following assertion:

Given $\varepsilon > 0$, there exists a closed set $E \subset [0, 1]$ and $y \in C^\infty$ such that

(i) meas $(E) \geqslant 1 - \varepsilon$,

(ii) $K(y(t), t) < 0$ for all $t \in E$,

(iii) $|\varphi(\tilde{z}(t), t) - \varphi(y(t), t)| \leqslant \varepsilon$ for almost every $t \in E$, (2.20)

where meas $(\cdot)$ denotes Lebesgue measure.

Since $\tilde{z}$ is measurable, for all $\varepsilon>0$ there exists a continuous function z and a closed set E_1 such that meas $(E_1)\geq 1-\varepsilon/3$ and $\tilde{z}(t)=z(t)$ for all $t\in E_1$ (Lusin's Theorem). Define the function $z_\lambda=\lambda\bar{z}+(1-\lambda)z$ where $\bar{z}$ was given in assumption (4). By the convexity of $K(\cdot, t)$, we see that

$$K(z_\lambda(t), t)_j\leq\lambda\gamma \text{ for all } t\in E_1, \lambda\in[0, 1], \tag{2.21}$$

and $j=1, 2, \ldots, m$. Since $\varphi(\cdot, t)$ is continuous and $|z-z_\lambda|\leq\lambda|z-\bar{z}|$, Egoroff's Theorem implies the existence of a closed set $E_2\subset E_1$ and $\lambda>0$ such that meas $(E_1-E_2)\leq\varepsilon/3$ and

$$|\varphi(z(t), t)-\varphi(z_\lambda(t), t)|\leq\varepsilon/2$$

for all $t\in E_2$.

Now use a mollifier to approximate z_λ with a C^∞ function. That is, select $\theta\in C^\infty$ with support $(\theta)\subset[-1, 1]$ and

$$\int_{-1}^{+1}\theta(s)\,ds=1.$$

For $t\in[0, 1]$, define:

$$z^k(t)=k\int_{t-1/k}^{t+1/k} z_\lambda(s)\,\theta(k(s-t))\,ds$$

where we set $z_\lambda(s)=z_\lambda(0)$ for $s\leq 0$ and $z_\lambda(s)=z_\lambda(1)$ for $s\geq 1$.

Since $\{z_k\}$ converges uniformly to z_λ and $\varphi(\cdot, t)$ is continuous, Egoroff's Theorem again implies the existence of a closed set $E_3\subset E_2$ and $\bar{k}>0$ such that meas $(E_2-E_3)\leq\varepsilon/3$ and

$$|\varphi(z_\lambda(t), t)-\varphi(z^k(t), t)|\leq\varepsilon/2$$

for all $t\in E_3$ and $k\geq\bar{k}$. Furthermore, since K is continuous and (2.21) holds, there exists $\hat{k}>0$ such that

$$K(z^k(t), t)<0 \text{ for all } t\in E_3 \text{ and } k\geq\hat{k}.$$

Selecting $E=E_3$ and $y=z^k$ for any $k\geq\max\{\bar{k}, \hat{k}\}$, we get (2.20).

Now given $y\in C^\infty$ and a closed set $E\subset[0, 1]$ such that $K(y(t), t)<0$ for all $t\in E$, we establish the following:

> For all $\delta>0$, there exists an open set D and $z\in C^\infty$ such that $E\cap D$ is empty, meas $(D)<\delta$, $z(t)=y(t)$ for all $t\in E$, $z(t)=\bar{z}(t)$ for all $t\notin E\cup D$, $|z|\leq|y|+|\bar{z}|$, and $K(z(t), t)\leq 0$ for all $t\in[0, 1]$. (2.22)

Since we have E closed, $K(y(\cdot), \cdot)$ uniformly continuous on $[0, 1]$, and $K(y(t), t)<0$ for all $t\in E$, there exists $\Delta>0$ such that $K(y(s), s)<0$ whenever $d(s, E)\equiv\inf\{|s-t|: t\in E\}\leq\Delta$. Given $\delta>0$, recall that for Lebesgue measure there exists an open set $\mathfrak{D}_1\supset E$ such that meas $(\mathfrak{D}_1-E)<$minimum $\{\delta, \Delta\}$ (see [16, Theorem 2.17]). Defining the open set $\mathfrak{D}_2=\{t: d(t, E)<\Delta\}$, we set $\mathfrak{D}=\mathfrak{D}_1\cap\mathfrak{D}_2$.

By [1, Theorem 3.14], there exists $w \in C^\infty$ such that $0 \leqslant w \leqslant 1$, $w(t)=1$ for all $t \in E$, and support $(w) \subset \mathfrak{D}$. Therefore, define $z = wy + (1-w)\bar{z}$ and $D = \mathfrak{D} - E$. Observe that

$$\begin{cases} z(t) = y(t) \text{ for all } t \in E, \\ z(t) = \bar{z}(t) \text{ for all } t \in ([0, 1] - \mathfrak{D}), \text{ and} \\ K(y(t), t) \leqslant 0 \text{ for all } t \in D \subset \mathfrak{D}_2. \end{cases}$$

Since $K(\bar{z}(t), t) \leqslant 0$ and $K(\cdot, t)$ is convex, we see that the convex combination $z = wy + (1-w)\bar{z}$ satisfies $K(z(t), t) \leqslant 0$ for all $t \in D$. Hence $K(z(t), t) \leqslant 0$ for all $t \in [0, 1]$.

To complete the proof of Lemma 2.3, we consider two cases:

Case 1. $\varphi(\bar{z}) \in L^1$.

By assumption (3), $\varphi(\bar{z}(\cdot), \cdot) \in L^1$. Applying the continuity property of the Lebesgue integral [17, p. 32, Exercise 12], for all $\rho > 0$, there exists $\varepsilon < \rho$ such that

$$\left.\begin{aligned} \int_F |\varphi(\bar{z}(t), t)|\, dt < \rho \\ \int_F |\varphi(\bar{z}(t), t)|\, dt < \rho \end{aligned}\right\} \text{ whenever meas } (F) \leqslant \varepsilon. \tag{2.23}$$

Now choose ε such that (2.23) holds, let E and y satisfy (2.20), and set $M = |y| + |\bar{z}|$. Similar to (2.23), there exists $\delta > 0$ such that

$$\int_F g(M, t)\, dt \leqslant \rho \text{ whenever meas } (F) \leqslant \delta.$$

Letting D and z satisfy (2.22), observe that

$$\left| \int_0^1 [\varphi(\bar{z}(t), t) - \varphi(z(t), t)]\, dt \right| \leqslant \int_E |\varphi(\bar{z}(t), t) - \varphi(z(t), t)|\, dt +$$

$$+ \int_{E^C} |\varphi(\bar{z}(t), t)|\, dt + \int_D |\varphi(z(t), t)|\, dt + \int_{E^C} |\varphi(\bar{z}(t), t)|\, dt \leqslant$$

$$\leqslant \varepsilon \text{ meas } (E) + \rho + \int_D g(M, t)\, dt + \rho \leqslant 4\rho \tag{2.24}$$

since $\varepsilon \leqslant \rho$, meas $(E) \leqslant 1$, and meas $(D) \leqslant \delta$ (we use the notation $E^c = [0, 1] - E$). As $\rho \to 0$, we get (2.19).

Case 2. $\varphi(\bar{z}) \notin L^1$.

Let $\rho > 0$ be fixed; by the construction in Case 1, for all $\beta > 0$, there exists a closed set E and $z \in C^\infty$ such that meas $(E^C) \leqslant \beta$, $K(z(t), t) \leqslant 0$ for all $t \in [0, 1]$, and

$$\left| \int_E \varphi(\bar{z}(t), t)\, dt - \int_0^1 \varphi(z(t), t)\, dt \right| \leqslant 3\rho. \tag{2.25}$$

Thus the proof is complete since

$$\int_E \varphi(\bar{z}(t), t)\, dt \to -\infty \text{ as meas } (E^C) \to 0. \qquad \blacksquare$$

Let us apply Lemmas 2.2–2.3 to evaluate $\mathcal{L}_0$. Introducing the function

$$\bar{h}(t)=\inf\{h(z, \eta(t), t): z \in R^{n+m}\}, \tag{2.26}$$

we have:

COROLLARY 2.4. *If* (1.2) *and* (1.3) *hold,* $(\dot{p}, \dot{v}, \lambda) \in L^1$, *and* $(\lambda, \dot{v}) \geqslant 0$, *then there exists* $\bar{z} \in L^1$ *such that*

$$\bar{h}(t)=h(\bar{z}(t), \eta(t), t) \text{ for almost every } t \in [0, 1] \text{ and}$$

$$\mathcal{L}_0(p, \lambda, \dot{v})=\int_0^1 h(\bar{z}(t), \eta(t), t)\, dt. \tag{2.27}$$

Proof. Defining $\xi=(\eta(t), t)$, Lemma 2.2 implies the existence of $\bar{z}(t)$ satisfying (2.27) for all $t \in [0, 1]$, and $\bar{z}(t)$ depends continuously on ξ. Since $\eta(\cdot)$ is measurable, we conclude that $\bar{z}(\cdot)$ is measurable. Applying (2.14) with $\bar{z}=0$ and integrating over $[0, 1]$, we obtain:

$$\|\bar{z}\|_{L^1} \leqslant |\nabla_1 f(0, \cdot)|+|\nabla_1 g(0, \cdot)|\, \|\eta\|_{L^1};$$

therefore, $\bar{z} \in L^1$.

Finally, we utilize Lemma 2.3 with $\varphi(z, t)=h(z, \eta(t), t)$ and K vacuous. Observe that

$$\int_0^1 \varphi(\bar{z}(t), t)\, dt \leqslant \mathcal{L}_0(p, \lambda, \dot{v})$$

$$\leqslant \inf\left\{\int_0^1 h(z(t), \eta(t), t)\, dt: z \in C^\infty\right\}$$

$$=\int_0^1 h(\bar{z}(t), \eta(t), t)\, dt,$$

where the last equality is (2.19). Hence the inequalities above are equalities and the proof is complete. ∎

To summarize, the extremand in (2.11) consists of three terms:

(1) $p(0)^T x_0$,

(2) $\mathcal{L}_0(p, \lambda, \dot{v})$, which is computed by integrating the pointwise minimum of $h(\cdot, \eta(t), t)$ over $t \in [0, 1]$,

(3) $\mathcal{L}_1(p(1))$, the value of the program (2.9) for $\gamma=p(1)$.

As noted earlier, the convergence rate of finite element approximations depends on dual variable regularity. We now examine some changes of variables in the dual functional that improve the regulatriy. First let us consider affine state constraints of the form:

$$K_s(x(t), t)=S(t)\, x(t)+s(t) \leqslant 0. \tag{2.28}$$

Defining the variable

$$\begin{cases} q(t)=S(t)^T v(t)-p(t) \text{ for } t\in(0,1) \\ q(1)=0 \\ q(0)=q(0^+), \end{cases} \tag{2.29}$$

recall the following special case of results in [2]:

LEMMA 2.5. *If* (1.2)–(1.4) *hold and* (p, λ, v) *are feasible in* (D), *then* $\mathscr{L}(p, \lambda, v) > -\infty$ *only if* $q\in A$.

Replacing $p(\cdot)$ by $S(\cdot)^T v(\cdot)-q(\cdot)$ in (1.5), and integrating by parts using the boundary condition $q(1^-)=0$, we get the equivalent dual functional:

$$\begin{aligned} L(q,\lambda,v) &= \mathscr{L}(S^T v-q,\lambda,v) \\ &= \inf\{J(x,u)=J_0(x,u)+[v,s]+ \\ &\quad +(q(0)-S(0)^T v(0))^T x_0 : x\in A, u\in L^\infty\}, \end{aligned} \tag{2.30}$$

where

$$J_0(x,u)=C(x,u)+\langle\lambda, K_c(u)\rangle+\langle q-S^T v, Ax+Bu\rangle+\langle\dot q-\dot S^T v, x\rangle .$$

Observe that J_0 can be expressed in the form

$$J_0(z)=\int_0^1 h(z(t),\eta(t),t)\,dt$$

where

$$\begin{cases} z(t)^T=(x(t)^T, u(t)^T) \\ \eta(t)^T=(\dot q(t)^T, q(t)^T, \lambda(t)^T, v(t)^T) \\ h(z(t),\eta(t),t)=f(z(t),t)+\eta(t)^T g(z(t),t) \\ g(z(t),t)=\begin{bmatrix} x(t) \\ A(t)x(t)+B(t)u(t) \\ K_c(u(t),t) \\ -S(t)(B(t)u(t)+A(t)x(t))-\dot S(t)x(t) \end{bmatrix} \end{cases} \tag{2.31}$$

Defining $\tilde h$ by (2.26), we have the following analogue of Corollary 2.4:

COROLLARY 2.6. *If* (1.2)–(1.4) *hold,* (p,λ,v) *are feasible in* (D), *and* $\mathscr{L}(p,\lambda,v)>-\infty$, *then there exists* $\tilde z\in L^1$ *such that*

$$\begin{cases} \tilde h(t)=h(\tilde z(t),\eta(t),t) \text{ for almost every } t\in[0,1] \text{ and} \\ \displaystyle\int_0^1 h(\tilde z(t),\eta(t),t)\,dt=\inf\left\{\int_0^1 h(z(t),\eta(t),t)\,dt : z\in A\times L^\infty\right\}. \end{cases} \tag{2.32}$$

■

Now let us consider the general convex state constraint:

LEMMA 2.7. *Suppose that* (1.2)–(1.4) *hold and that* $(\tilde x,\tilde u)\in A\times L^\infty$, $\tilde x(0)=x_0$, (p,λ,v) *are feasible in* (D), *and define*:

$$\begin{cases} G(t)=\nabla_1 K_s(\tilde x(t),t) \quad \text{for all } t\in[0,1] \\ q(t)=G(t)^T v(t)-p(t) \text{ for all } t\in(0,1) \\ q(1)=0 \\ q(0)=q(0^+). \end{cases} \tag{2.33}$$

Then $(\tilde{x}, \tilde{u})$ *achieve the minimum in* (1.5) *if and only if* $q \in A$, *and the following equations are satisfied for almost every* $t \in [0, 1]$:

$$0 = \nabla_2 f(\tilde{x}(t), \tilde{u}(t), t) + B(t)^T (q(t) - G(t)^T v(t)) + \nabla_1 K_c(\tilde{u}(t), t)^T \lambda(t) \quad (3.34)$$

and

$$0 = \nabla_1 f(\tilde{x}(t), \tilde{u}(t), t) + \dot{q}(t) + A(t)^T (q(t) - G(t)^T v(t)) - \dot{G}(t)^T v(t). \quad (2.35)$$

Proof. The necessary conditions for a pair $(\tilde{x}, \tilde{u})$ achieving the minimum in (1.5) was established in [2] while the sufficient condition is a classical result for convex optimization [11, p. 12]. ∎

COROLLARY 2.8. *Suppose that* (1.2)–(1.4) *hold,* (p, λ, v) *are feasible in* (D), p *and* v *are absolutely continuous on* [0, 1] *and* [0, 1) *respectively, and* $(\tilde{x}, \tilde{u})$ *achieve the minimum in* (1.5). *Then for the* $\eta(\cdot)$ *and* $h(\cdot, \cdot, \cdot)$ *defined in* (2.4)–(2.5), *we have*:

$$h(\tilde{z}(t), \eta(t), t) = \inf \{h(z, \eta(t), t) : z \in R^{n+m}\} \quad (2.36)$$

for almost every $t \in [0, 1]$ *where* $\tilde{z}^T = (\tilde{x}^T, \tilde{u}^T)$.

Proof. Integrating by parts in (1.5), we observe that

$$\mathscr{L}(p, \lambda, v) = \int_0^1 h(\tilde{z}(t), \eta(t), t)\, dt + p(1)^T \tilde{x}(1) - p(0)^T x_0 - \\ - K_s(\tilde{x}(1), 1)^T v(1^-). \quad (2.37)$$

From Lemma 2.7, we obtain:

$$q(1^-) = 0 = \nabla_1 K_s(\tilde{x}(1), 1)^T v(1^-) - p(1). \quad (2.38)$$

Since $v(1^-) \leqslant 0$ and the components of $K_s(\cdot, 1)$ are convex, we conclude that:

$$p(1)^T \tilde{x}(1) - K_s(\tilde{x}(1), 1)^T v(1^-) = \mathscr{L}_1(p(1), v(1^-)). \quad (2.39)$$

Combining (2.37), (2.39), and the decomposition (2.6), we get:

$$\mathscr{L}_0(p, \lambda, v) = \int_0^1 h(\tilde{z}(t), \eta(t), t)\, dt. \quad (2.40)$$

On the other hand, $\mathscr{L}_0$ is defined by

$$\mathscr{L}_0(p, \lambda, v) = \inf \left\{ \int_0^1 h(z(t), \eta(t), t)\, dt : z \in A \times L^\infty \right\}. \quad (2.41)$$

Finally, (2.40), (2.41), and classical arguments give us (2.36) (See [2]). ∎

Since q defined in (2.33) depends on $\tilde{x}$, the associated minimizer in the dual functional (1.5), the transformation (2.30) only makes sense when $K_c(\cdot, t)$ is affine. A new transformation valid for general convex constraints is now described:

First observe that for given $(\tilde{x}(t), q(t), v(t), \lambda(t))$ with $\lambda(t) \geq 0$, there exists a unique $\tilde{u}(t)$ satisfying (2.34). Since $K_c(\cdot, t)^T \lambda(t)$ is convex and $f(\tilde{x}(t), \cdot, t)$ is strictly convex, $\tilde{u}(t)$ is the unique solution of the program:

$$\text{minimize}\left\{f(\tilde{x}(t), u, t)+(q(t)-G(t)^T v(t))^T B(t) u + K_c(u, t)^T \lambda(t): u \in R^m\right\}. \tag{2.42}$$

Let $(\tilde{x}, v, \lambda)$ be given. Imposing the initial condition $q(1)=0$, the system (2.35) reduces to an initial value problem for q with $\tilde{u}(t)$ determined by (2.34). Let $(q, \tilde{u})$ denote a solution to the system (2.34)–(2.35) that satisfies the initial condition $q(1)=0$, and define $p=G^T v-q$ where G is given in (2.33). Since Lemma 2.7 is both necessary and sufficient, we see that the given $\tilde{x}$ and the computed $\tilde{u}$ achieve the minimum in the dual function (1.5) for $(p=G^T v-q, \lambda, v)$. Let us choose $(\tilde{x}, \lambda, v)$ for the "dual" variables. The dual function value associated with the triple $(\tilde{x}, \lambda, v)$ is given by

$$l(\tilde{x}, \lambda, v)=\mathscr{L}(G^T v-q, \lambda, v). \tag{2.43}$$

Since $(\tilde{x}, \tilde{u})$ achieves the minimum in (1.5), the right side of (2.43) can be evaluated by inserting $(x, u)=(\tilde{x}, \tilde{u})$ into the extremand of (1.5).

Since the initial value problem (2.34)–(2.35) is nonlinear ($\tilde{u}(t)$ generally depends nonlinearly on $q(t)$), the solution may not exist on the entire interval $[0, 1]$; however, for $(\tilde{x}, \lambda, v)=(x^*, \lambda^*, v^*)$, the solution $(q, \tilde{u})=(q^*, u^*)$ does exist [2]. Hence we can establish a local existence result.

Define the set

$$\Omega=\{\mu=(\tilde{x}, \lambda, v):(\dot{x}, \lambda, v) \in L^{\infty}, \lambda \geq 0\}, \tag{2.44}$$

and for given $\mu \in \Omega$, let $(q, \tilde{u})=(q(\mu), u(\mu))$ denote the solution to the system (2.34)–(2.35) satisfying the initial condition $q(1)=0$ (if this solution exists). Also define the sets:

$$\Omega_\varepsilon=\{\mu \in \Omega: |\mu-\mu^*| \leq \varepsilon\} \tag{2.45}$$

$$\dot{\Omega}_\varepsilon=\{\mu=(x, \lambda, v) \in \Omega_\varepsilon: |\dot{x}-\dot{x}^*| \leq \varepsilon\} \tag{2.46}$$

where $\mu^*=(x^*, \lambda^*, v^*)$.

LEMMA 2.9. *Suppose that* (1.2) *holds,* $K_s \in C^3$, $K_c(\cdot, t)$ *is convex for all* $t \in [0, 1]$, *and there exists* $\alpha>0$ *such that*

$$\nabla_2^2 f(x, u, t) \geq \alpha I$$

for all $x \in R^n$, $u \in R^m$, *and* $t \in [0, 1]$. *If we are given* $\mu^* \in \Omega$ *such that* $q(\mu^*) \in A$ *exists, then there exists* $c<\infty$ *and* $\varepsilon>0$ *such that* $q(\mu) \in A$ *exists for all* $\mu \in \dot{\Omega}_\varepsilon$ *and the following Lipschitz estimate holds*:

$$\|u(\mu_1)-u(\mu_2)\| \leq c\{\|\mu_1-\mu_2\|+\|\dot{x}_1-\dot{x}_2\|\} \tag{2.47}$$

for all $\mu_1, \mu_2 \in \dot{\Omega}_\varepsilon$ *where* $\mu_k = (x_k, v_k, \lambda_k)$, $k = 1, 2$. *Moreover, if the state constraints are affine, there exists* $c < \infty$ *and* $\varepsilon > 0$ *such that*:

$$\|u(\mu_1) - u(\mu_2)\| \leqslant c\, \|\mu_1 - \mu_2\| \tag{2.48}$$

for all $\mu_1, \mu_2 \in \Omega_\varepsilon$.

Proof. By Lemma 2.2, the solution $\tilde{u}(t)$ to (2.42) depends Lipschitz continuously on $(\tilde{x}(t), q(t), \lambda(t), v(t))$. Also observe that for affine state constraints $\dot{G}(t) = \dot{S}(t)$ while for more general state constraints, $\dot{G}(t)$ depends linearly on $\tilde{x}(t)$. Combining these observations with Lemma Al of the appendix completes the proof. ∎

3. Dual Approximations

Using the following three choices for dual variables, we develop finite element approximations to (C):

(1) (p, λ, w),
(2) (q, λ, v),
(3) $(\tilde{x}, \lambda, v)$.

Associated with these dual variables, we have the following feasible sets:

$$\mathfrak{M} = \{\theta = (p, \lambda, w) : p \in A, (\lambda, w) \in L^1, (\lambda, w) \geqslant 0, \ \mathscr{L}_1(p(1)) > -\infty\}, \tag{3.1}$$

$$M = \{\mu = (q, \lambda. v) : q \in A, \lambda \in L^1, v \in BV, \quad q(1) = v(1) = 0, \lambda \geqslant 0, v \text{ nondecreasing}\}, \tag{3.2}$$

and

$$F = \{\sigma = (\tilde{x}, v, \lambda) : \tilde{x} \in A, \lambda \in L^1, v \in BV, \tilde{x}(0) = x_0, \quad \lambda \geqslant 0, v(1) = 0, v \text{ nondecreasing, (2.34)–(2.35) has a solution } q(\sigma) \in A \text{ satisfying } q(\sigma)(1) = 0\} \tag{3.3}$$

The corresponding dual problems are:

$$\sup \{\mathscr{L}_1(p(1)) + \mathscr{L}_0(p, \lambda, w) - p(0)^T x_0 : (p, \lambda, w) \in \mathfrak{M}\}, \tag{3.4}$$

$$\sup \{L(\mu) : \mu \in M\}, \tag{3.5}$$

and

$$\sup \{l(\sigma) : \sigma \in F\}. \tag{3.6}$$

As seen later, the advantages and disadvantages of approximations based on (3.4)–(3.6) are the following:

Advantages:

(3.4) — applies to general convex state constraints, $\mathfrak{M}$ contains few constraints,
(3.5) — high convergence rate for error,
(3.6) — achieves same convergence rate as (3.5) using possibly lower dimensional spaces.

Disadvantages:
(3.4) — low convergence rate for error,
(3.5) — limited to affine state constraints,
(3.6) — the evaluation of the dual function involves integrating a differential equation.

REMARK 3.1. If the state constraints of (C) are vacuous, then $p=q$, $\mathfrak{M}=M$, and programs (3.4) and (3.5) are equivalent.

The dual variables are approximated using finite element subspaces. Let P_k^h denote a piecewise polynomial space consisting of polynomials of degree at most $k-1$ and maximum grid interval h (no continuity restrictions at the grid points). To reduce the dimension of P_k^h, additional continuity restrictions may be imposed. That is, a dual variable could be approximated by the subspace $S_k^h=C^r\cap P_k^h$ where $r\leqslant k-2$.

Let us define the space

$$W^s=\{f\colon [0,1]\to R\colon f^{(s)}\in L^\infty\}$$

and let $|\cdot|_T$ be the L^∞ norm on the set $T\subset[0,1]$. Recall [16] that for spaces S_k^h defined by local nodal parameters, there exist an interpolation operator $I\colon W^s\to S_k^h$ such that for all $f\in W^s$ and all grid intervals T, we have:

$$\left|\frac{d^j}{dt^j}(f-f^I)\right|_T\leqslant ch^{s-j}|f^{(s)}|_T \tag{3.7}$$

whenever $j\leqslant s\leqslant k$ and $j\leqslant r+1$ where c is a constant that is independent of h and T. (For the remainder of this paper, $\infty>c\geqslant 0$ denotes a generic constant whose value is independent of h and T, and may change in different relations).

For the control problem above, let S_k^h be the Cartesian product of piecewise polynomial spaces, and set

$$\mathfrak{M}_k^h=\mathfrak{M}\cap S_k^h, \tag{3.8}$$

$$M_k^h=M\cap S_k^h, \tag{3.9}$$

and

$$F_k^h=F\cap S_k^h. \tag{3.10}$$

The following approximations to (3.4)–(3.6) are considered:

$$\sup\{\mathscr{L}_1(p^h(1))+\mathscr{L}_0(p^h,\lambda^h,w^h)-p^h(0)^T x_0\colon (p^h,\lambda^h,w^h)\in\mathfrak{M}_k^h\}, \tag{3.11}$$

$$\sup\{L(\mu^h)\colon \mu^h\in M_k^h\}, \tag{3.12}$$

and

$$\sup\{l(\sigma^h)\colon \sigma^h\in F_k^h\}. \tag{3.13}$$

THEOREM 3.2. *If* (1.2)–(1.4) *hold, there exists a solution to the programs* (3.11)–(3.13).

For brevity, the proof of Theorem 3.2 is omitted—this paper concentrates on error estimates.

In Section 6, we show that the error in finite element approximations can be improved if we treat the grid points as free parameters. That is, we optimize the dual variables in the approximations (3.11)–(3.13) over both the nodal values and the placement of grid points. These free grid point sets are now introduced.

Given an ordered set of nodes $0=t_1<t_2<\ldots<t_N=1$, let $P_k(t_1, \ldots, t_N)$ be the finite element space consisting of polynomials of degree at most $k-1$ on each grid interval (t_j, t_{j+1}) (no continuity requirements). Define the sets

$$\mathfrak{I}^h=\{(t_1, \ldots, t_l): 0=t_1<t_2<\ldots<t_l=1,$$
$$l\leqslant 2/h,\ t_{j+1}-t_j\leqslant h \text{ for all } j\}$$
$$\tilde{S}_k^h=C^r\cap\{\bigcup_{\tau\in\mathfrak{I}^h} P_k(\tau)\}.$$

Fimilarly, sets $\tilde{\mathfrak{M}}_k^h$, $\tilde{M}_k^h$, and $\tilde{F}_k^h$ are obtained by replacing in (3.8)–(3.10), S_k^h by $\tilde{S}_k^h$.

The mathematical descriptions of the set $\tilde{S}_k^h$ may seem a bit complex; however, the computational implementation of free grid points sets is considerably simpler—we first approximate (x^*, u^*) using fixed grid points, and estimate the contact points (where constraints change from binding to nonbinding); next we free those grid points that are in the vacinity of contact points, and optimize over both the nodal values and the placement of the free grid points.

Appendix 1. Lipschitz Continuity for Differential Equations

Consider the initial-value problem

$$\dot{x}(t)=f(x(t), \eta(t)),\ x(0)=x_0 \tag{A1}$$

where x: $[0, 1]\to R^n$ and f: $R^n\times R^m\to R^n$. Assume that for $\eta=\eta^*\in L^\infty$, (A1) has an associated solution $x^*\in A$. Let $\mathcal{R}\subset R^n\times R^m$ be compact with $(x^*(t), \eta^*(t))\in\mathcal{R}$ for all $t\in[0, 1]$ and the distance between $(x^*(t), \eta^*(t))$ and the boundary of $\mathcal{R}$ bounded uniformly from zero. Define the set

$$M(\varepsilon)=\{\eta\in L^\infty: |\eta-\eta^*|\leqslant\varepsilon\}. \tag{A2}$$

The following theorem is classical:

Lemma A1. *If $f(\cdot, \cdot)$ is Lipschitz continuous on $\mathcal{R}$, there exists $\varepsilon>0$ such that for all $\eta\in M(\varepsilon)$,* (A1) *has an associated solution $x[\eta](\cdot)$. Moreover, there exists a constant $c<\infty$ satisfying:*

$$\|\dot{x}(\eta_1)-\dot{x}(\eta_2)\|,\ \|x(\eta_1)-x(\eta_2)\|\leqslant c\,\|\eta_1-\eta_2\| \tag{A3}$$

for all $\eta_1, \eta_2\in M(\varepsilon)$.

Appendix 2. Essentially Bounded Controls

THEOREM A2. *If* (1.2)–(1.4) *hold, and* $(x^*, u^*) \in A \times L^2$ *are optimal in* (C), *then* $u^* \in L^\infty$.

Proof. Setting $z^* = (x^*, u^*)$ and $\bar{z} = (\bar{x}, \bar{u})$ where $(\bar{x}, \bar{u})$ were given in (1.4), (1.3) implies that

$$f(z^*(t), t) \geq f(\bar{z}(t), t) + \nabla_1 f(\bar{z}(t), t)\,(z^*(t) - \bar{z}(t)) + \frac{1}{2}\alpha\,|z^*(t) - \bar{z}(t)|^2. \tag{A4}$$

Integrating over $t \in [0, 1]$, we get:

$$C(z^*) \geq C(\bar{z}) - \|\nabla_1 f(\bar{z}(\cdot), \cdot)\|\,\|z^* - \bar{z}\| + \frac{1}{2}\alpha\,\|z^* - \bar{z}\|^2. \tag{A5}$$

Since $f \in C^1$, $\bar{z} \in C^0$, and $z^* \in L^2$, (A5) shows that $C(z^*) > -\infty$. Therefore by [2, Theorem 2], there exists an optimal solution (p^*, λ^*, v^*) to (D), $\mathscr{L}(p^*, \lambda^*, v^*) = C(z^*)$, and $\langle \lambda^*, K_c(u^*) \rangle = 0 = [v^*, K_s(x^*)]$.

Now define the function $\varphi\colon R^m \times [0, 1] \to R$ as follows:

$$\varphi(u, t) = f(x^*(t), u, t) + p^*(t)^T (\dot{x}^*(t) - A(t)\,x^*(t) - B(t)\,u) + \lambda^*(t)^T K_c(u, t). \tag{A6}$$

We shall establish that

$$\varphi(u^*(t), t) = \inf\{\varphi(u, t)\colon u \in R^m\} \tag{A7}$$

for almost every $t \in [0, 1]$. Let E denote the set of measure 1 consisting of those $s \in [0, 1]$ such that $\lambda^*(s)^T K_c(u^*(s), s) = 0$, $\dot{x}^*(s) - A(s)\,x^*(s) - B(s)\,u^*(s) = 0$, and s is a Lebesgue point of the functions $\{p^*(\cdot)^T(\dot{x}^*(\cdot) - A(\cdot)\,x^*(\cdot)), f(z^*(\cdot), \cdot), p^*(\cdot)^T B(\cdot), \lambda^*(\cdot)\}$. Suppose that there exists $s \in E$ and $\hat{u} \in R^m$ with $\varphi(\hat{u}, s) < \varphi(u^*(s), s)$—we show that this is impossible, and hence (A7) holds.

Define the interval

$$I_\delta = \{t \in [0, 1]\colon |t - s| \leq \delta\}.$$

Since $s \in E$, we see that

$$\begin{aligned}\int_{I_\delta} \varphi(u^*(t), t)\,dt &= \int_{I_\delta} f(z^*(t), t)\,dt \\ &= f(z^*(s), s)\ \text{meas}\,(I_\delta) + o\,(\text{meas}\,(I_\delta)) \\ &= \varphi(z^*(s), s)\ \text{meas}\,(I_\delta) + o\,(\text{meas}\,(I_\delta)).\end{aligned} \tag{A8}$$

Similarly, we have:

$$\int_{I_\delta} \varphi(\hat{u}, t)\,dt = \varphi(\hat{u}, s)\ \text{meas}\,(I_\delta) + o\,(\text{meas}\,(I_\delta)). \tag{A9}$$

Since $\varphi(\hat{u}, s) < \varphi(u^*(s), s)$, (A8) and (A9) imply that

$$\int_{I_\delta} \varphi(\hat{u}, t)\,dt < \int_{I_\delta} \varphi(u^*(t), t)\,dt \tag{A10}$$

for δ sufficiently small.

Define the sets

$$I_0^k=\{t\in[0,1]: t\notin I_\delta,\ |u^*(t)|\leqslant k\},$$
$$I_\infty^k=[0,1]-(I_\delta\cup I_0^k)=\{t\in[0,1]: t\notin I_\delta,\ |u^*(t)|>k\},$$

and the control

$$u_\delta^k=\begin{cases}\hat u & \text{for } t\in I_\delta\\ u^*(t) & \text{for } t\in I_0^k\\ 0 & \text{for } t\in I_\infty^k\end{cases}.$$

By the continuity properties of Lebesgue integrals, we know that

$$\lim_{k\to\infty}\int_{I_0^k\cup I_\infty^k}[\varphi(u_\delta^k(t),t)-\varphi(u^*(t),t)]\,dt=$$
$$=\lim_{k\to\infty}\int_{I_\infty^k}[\varphi(0,t)-\varphi(u^*(t),t)]\,dt=0 \qquad \text{(A11)}$$

since meas $(I_\infty^k)\to 0$ as $k\to\infty$ and $(\varphi(0,\cdot),\varphi(u^*(\cdot),\cdot))\in L^1$. Combining (A10) and (A11), we see that for δ sufficiently small and k sufficiently large

$$\int_0^1\varphi(u_\delta^k(t),t)\,dt<\int_0^1\varphi(u^*(t),t)\,dt=C(z^*), \qquad \text{(A12)}$$

by complementary slackness. Hence (A12) along with the definition of $\mathscr{L}$ implies that for k sufficiently large:

$$\mathscr{L}(p^*,\lambda^*,\nu^*)\leqslant\int_0^1\varphi(u_\delta^k(t),t)\,dt+[\nu^*,K_s(x^*)]<C(z^*). \qquad \text{(A13)}$$

This contradicts the strong duality result that $\mathscr{L}(p^*,\lambda^*,\nu^*)=C(z^*)$, and (A7) has been established.

Define the function

$$\Lambda(u,t)=f(x^*(t),u,t)-p^*(t)^T B(t)\,u.$$

Since $\lambda^*(t)\geqslant 0\geqslant K_c(u^*(t),t)$ and $\lambda^*(t)^T K_c(u^*(t),t)=0$ for almost every $t\in[0,1]$, (A7) and the convexity of $\varphi(\cdot,t)$ give us:

$$\Lambda(u^*(t),t)=\inf\{\Lambda(u,t): u\in R^m,\ K_c(u,t)\leqslant 0\} \qquad \text{(A14)}$$

for almost every $t\in[0,1]$. Applying (2.14), we obtain:

$$\frac{1}{2}\alpha\,|u^*(t)-\bar u(t)|\leqslant|\nabla_1\Lambda(\bar u(t),t)|, \qquad \text{(A15)}$$

for almost every $t\in[0,1]$. Therefore, $u^*\in L^\infty$ since $f\in C^1$, $(x^*,\bar u)\in C^0$, and $p^*\in BV$. ∎

To be continued in next issue.