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On Some Properties of Two-phase Parabolic Free Boundary Value Control Problems

by

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In the paper a one-dimensional, two-phase parabolic free boundary value control problem is considered. State equations of the process, boundary conditions and conditions given on free boundary are nonlinear. Equivalent integral representation of the problem is introduced. Local existence of solution of the problem considered is proved as well as uniqueness of this solution and its continuous dependence on data.

1. Introduction

In the paper some two-phase parabolic free boundary value control problems arising in phase change processes [2, 6] are considered. State equations of the process we consider are semi-linear. Boundary conditions as well as conditions on the free boundary are nonlinear. We assume that it is possible to control the process by means of the boundary conditions. In general we take square integrable control functions into account. In Section 3 equivalent integral representations of the investigated problems are derived. In Section 4 correctness in the Hadamard sense of the problems is shown and continuity of an important mapping is discussed.

Problems similar to these introduced in the present paper have been studied only in particular cases. Usually the state equations, boundary conditions and conditions on free boundary have been assumed linear [1, 2, 4, 14]. Free boundary problem with all the nonlinearities mentioned above has been considered only in the one-phase case by Rubinstein [13].

Basic notations:

u_i, $i=1, 2$ — state functions of process,

g — function defining free boundary,

$\gamma \triangleq \dfrac{dg}{dt}$

$$\left.\begin{aligned} p_i &\triangleq \frac{\partial u_i}{\partial x} \\ v_i(t) &\triangleq \frac{\partial u_i}{\partial x}(g(t), t) \\ w_1(t) &\triangleq u_1(0, t) \\ w_2(t) &\triangleq \frac{\partial u_2}{\partial x}(L, t) \end{aligned}\right\} \text{— auxiliary functions,}$$

$E(x, t) \triangleq \begin{cases} (4\pi t)^{-1/2} \exp(-x^2/4t), & x>0, t>0 \\ 0 \quad , & x \leqslant 0, t \leqslant 0 \end{cases}$ — fundamental solution of heat equation,

$Z^i_{t_k}$, $i=1, 2$ — regions defining appropriate phases:

$Z^1_{t_k} = \{(x, t) \mid x \in (0, g(t)),\ t \in (t_o, t_k)\}$,

$Z^2_{t_k} = \{(x, t) \mid x \in (g(t), L),\ t \in (t_o, t_k)\}$;

$\tilde{Z}^i_{t_k} = \bar{Z}^i_{t_k} \cap R \times [t_o, t_k)$,

$C(Z)$ — the space of functions continuous in the set Z,

$C_o(Z)$ — the space of functions continuous and bounded in the set Z,

$C^{2,1}(Z)$ — the space of functions two-times continuously differentiable with respect to x and having first derivative with respect to t continuous in the set Z where $(x, t) \in Z$,

$\mathscr{D}(Z)$ — the space of infinitely differentiable functions with support contained in Z.

2. Problem Statement

We will consider two-phase parabolic free boundary value problem with nonlinear process equations as well as with nonlinear conditions on the fixed and free boundaries.

The semi-linear process equations have the form

$$\frac{\partial u_i}{\partial t}(x, t) - a_i^2 \frac{\partial^2 u_i}{\partial x^2}(x, t) + F_i\left(x, t; u_i, \frac{\partial u_i}{\partial x}, g, \frac{dg}{dt}\right) = 0 \quad \text{for } (x, t) \in Z^i_{t_k}. \tag{2.1}$$

We assume that initial conditions for the process are given:

$$g(t_o) = g_o \text{ where } 0 < g_o < L, \tag{2.2}$$

$$u_1(x, t_o) = \pi_1(x),\ x \in [0, g_o), \tag{2.3}$$

$$u_2(x, t_o) = \pi_2(x),\ x \in (g_o, L]. \tag{2.4}$$

At the fixed parts of the boundaries of $Z^i_{t_k}$ — domains the following nonlinear conditions hold:

$$\frac{\partial u_1}{\partial x}(0, t)=f_1\left(t, u_1(0, t); \varphi_1(t)\right), \quad t \in (t_o, t_k], \tag{2.5}$$

$$u_2(L, t)=f_2\left(t, \frac{\partial u_2}{\partial x}(L, t); \varphi_2(t)\right), \quad t \in (t_o, t_k]. \tag{2.6}$$

On the free boundary

$$u_1\left(g(t), t\right)=h_1\left(g(t), t\right), \quad t \in (t_o, t_k], \tag{2.7}$$

$$u_2\left(g(t), t\right)=h_2\left(g(t), t\right), \quad t \in (t_o, t_k]. \tag{2.8}$$

Dynamics of the free boundary is defined by the ordinary differential equation

$$\frac{dg}{dt}(t)=S\left[t, g(t), u_1\left(g(t), t\right), \frac{\partial u_1}{\partial x}\left(g(t), t\right), u_2\left(g(t), t\right), \frac{\partial u_2}{\partial x}\left(g(t), t\right)\right], \quad t \in (t_o, t_k). \tag{2.9}$$

We take it for granted that:

(A.1) Functions u_1, u_2, g satisfy the following regularity conditions:

$$u_i \in \left\{u_i \in C^{2,1}(Z^i_{t_k}) \,\middle|\, u_i \in C(\bar{Z}^i_{t_k}), \frac{\partial u_i}{\partial x} \in C_0(\tilde{Z}^i_{t_k})\right\},$$

$$g \in \left\{g \in C[t_o, t_k] \,\middle|\, \frac{dg}{dt} \in C(t_o, t_k]\right\}.$$

(A.2) Process equations (2.1), initial conditions (2.2)–(2.4) as well as conditions (2.7)–(2.9) binding at the free boundary are understood in the strong sense (in case of need after continuous extension of appropriate functions) while boundary conditions (2.5), (2.6) are understood in the weak sense, i.e. for any function $\eta \in \mathscr{D}(t_o, t_k)$ the following relationships hold

$$\lim_{\xi \to 0+} \int_{t_o}^{t_k} \left[\frac{\partial u_1}{\partial \xi}(\xi, t)-f_1\left(t, u_1(\xi, t); \varphi_1(t)\right)\right] \eta(t)\, dt=0, \tag{2.10}$$

$$\lim_{\xi \to L-} \int_{t_o}^{t_k} \left[u_2(\xi, t)-f_2\left(t, \frac{\partial u_2}{\partial \xi}(\xi, t); \varphi_2(t)\right)\right] \eta(t)\, dt=0. \tag{2.11}$$

(A.3) Functions φ_1, φ_2 enclosed in boundary conditions (2.5), (2.6) are treated as process controls; these functions are elements of the space $L_2[t_o, t_k]$ and they satisfy the inequalities

$$\varphi_i^m \leqslant \varphi_i(t) \leqslant \varphi_i^M \text{ almost everywhere for } t \in [t_o, t_k]. \tag{2.12}$$

(A.4) Functions π_i enclosed in initial conditions (2.3), (2.4) are two-times continuously differentiable and

$$|\pi_i(x)| \leqslant B_{1i}, \quad \left| \frac{d^2 \pi_i}{dx^2}(x) \right| \leqslant B_i^1 .$$

(A.5) Functions F_1, F_2, f_1, S have continuous first derivatives with respect to all arguments in their domains, functions f_2, h_1, h_2 have continuous all the second derivatives in their domains.

(A.6) For admissible controls φ_i and functions u_i, g bounded with derivatives in closures of their domains in the following way

$$|u_i(x,t)| \leqslant A_{1i}^*, \quad |p_i(x,t)| \leqslant A_{2i}^*,$$

$$g(t) \in (0, L), \quad |\gamma(t)| \leqslant A_3^*,$$

mentioned below estimates concerning functions F_i, h_i, S, f_i hold (arguments of functions are everywhere ommitted)

$$\begin{aligned}
&|F_i| \leqslant B_{2i}, && |h_i| \leqslant B_{3i}, \\
&|S| \leqslant B_{4i} && |f_i| \leqslant B_{5i}, \\
&\left|\frac{\partial F_i}{\partial x}\right|, \left|\frac{\partial F_i}{\partial t}\right|, \left|\frac{\partial F_i}{\partial u_i}\right|, \left|\frac{\partial F_i}{\partial p_i}\right|, \left|\frac{\partial F_i}{\partial g}\right|, \left|\frac{\partial F_i}{\partial \gamma}\right| \leqslant B_i^2, \\
&\left|\frac{\partial h_i}{\partial g}\right|, \left|\frac{\partial^2 h_i}{\partial t^2}\right| \leqslant B_i^3, \\
&\left|\frac{\partial S}{\partial t}\right|, \left|\frac{\partial S}{\partial x}\right|, \left|\frac{\partial S}{\partial u_i}\right|, \left|\frac{\partial S}{\partial p_i}\right| \leqslant B^4, \\
&\left|\frac{\partial f_1}{\partial w_1}\right|, \left|\frac{\partial f_1}{\partial \varphi_1}\right| \leqslant B_1^5, \\
&\left|\frac{\partial f_2}{\partial x}\right|, \left|\frac{\partial f_2}{\partial w_2}\right|, \left|\frac{\partial f_2}{\partial \varphi_2}\right| \leqslant B_2^5,
\end{aligned} \tag{2.13}$$

where all constants in (2.13) are positive, finite and depend only on φ_i^m, φ_i^M, A_{1i}^*, A_{2i}^*, A_3^*.

(A.7) The following compatibility conditions are satisfied:

$$\begin{aligned}
\pi_1(g_o) &= h_1(g(t_o), t_o), \\
\pi_2(g_o) &= h_2(g(t_o), t_o).
\end{aligned} \tag{2.14}$$

Solution (u_1, u_2, g) of the free boundary value problem (2.1)–(2.9) is understood in the classical sense, i.e. assumptions (A.1), (A.2) are fulfilled.

3. Equivalent Integral Representation of the Free Boundary Problem (2.1)–(2.9)

Due to analytical properties of thermal potentials of single and double layer [8, 9], applying the Gevrey's method [5] we obtain the following equivalence theorem.

THEOREM 1. Let us suppose assumptions (A.1)–(A.7) to be satisfied.

(i) If (u_1, u_2, g) is a solution of the free boundary value problem (2.1)–(2.9) then $(u_1, u_2, w_1, w_2, p_1, p_2, v_1, v_2, g, \gamma)$ is a solution of the following system of nonlinear integral equations

$$\begin{aligned} u_1(x,t) = & -a_1^2 \int_{t_0}^{t} f_1(\tau, w_1(\tau); \varphi_1(\tau))\, G_{20}(x, 0, t-\tau)\, d\tau + \\ & + \int_0^{g_0} \pi_1(\xi)\, G_{20}(x, \xi, t-t_0)\, d\xi + \\ & + \int_{t_0}^{t} \int_0^{g(\tau)} F_1(\xi, \tau; u_1, p_1, g, \gamma)\, G_{20}(x, \xi, t-\tau)\, d\xi\, d\tau + \\ & + \int_{t_0}^{t} [a_1^2 v_1(\tau) + h_1(g(\tau), \tau)\, \gamma(\tau)]\, G_{20}(x, g(\tau), t-\tau)\, d\tau - \\ & - a_1^2 \int_{t_0}^{t} h_1(g(\tau), \tau) \frac{\partial G_{20}}{\partial \xi}(x, g(\tau), t-\tau)\, d\tau = \\ & \triangleq U_1(x, t; u_1, w_1, p_1, v_1, g, \gamma), \end{aligned} \tag{3.1}$$

$$\begin{aligned} u_2(x,t) = & -a_2^2 \int_{t_0}^{t} f_2(\tau, w_2(\tau); \varphi_2(\tau)) \frac{\partial G_{1L}}{\partial \xi}(x, L, t-\tau)\, d\tau + \\ & + \int_{g_0}^{L} \pi_2(\xi)\, G_{1L}(x, \xi, t-t_0)\, d\xi + \\ & + \int_{t_0}^{t} \int_{g(\tau)}^{L} F_2(\xi, \tau; u_2, p_2, g, \gamma)\, G_{1L}(x, \xi, t-\tau)\, d\xi\, d\tau - \\ & - \int_{t_0}^{t} [a_2^2 v_2(\tau) + h_2(g(\tau), \tau)\, \gamma(\tau)]\, G_{1L}(x, g(\tau), t-\tau)\, d\tau + \\ & + a_2^2 \int_{t_0}^{t} h_2(g(\tau), \tau) \frac{\partial G_{1L}}{\partial \xi}(x, g(\tau), t-\tau) d\tau = \\ & \triangleq U_2(x, t; u_2, w_2, p_2, v_2, g, \gamma), \end{aligned} \tag{3.2}$$

$$w_1(t) = U_1(0, t; u_1, w_1, p_1, v_1, g, \gamma) \triangleq W_1(t; u_1, w_1, p_1, v_1, g, \gamma), \tag{3.3}$$

$$w_2(t) = P_2(L, t; u_2, w_2, p_2, v_2, g, \gamma) \triangleq W_2(t; u_2, w_2, p_2, v_2, g, \gamma), \tag{3.4}$$

$$p_1(x,t)=a_1^2\int_{t_0}^{t} f_1(\tau, w_1(\tau); \varphi_1(\tau))\frac{\partial G_{10}}{\partial \xi}(x,0,t-\tau)\,d\tau+$$
$$+\int_0^{g_0}\frac{d\pi_1}{d\xi}(\xi)\,G_{10}(x,\xi,t-t_o)\,d\xi-$$
$$-\int_{t_0}^{t}\int_0^{g(\tau)} F_1(\xi,\tau;u_1,p_1,g,\gamma)\frac{\partial G_{10}}{\partial \xi}(x,\xi,t-\tau)\,d\xi\,d\tau-$$
$$-a_1^2\int_{t_0}^{t} v_1(\tau)\frac{\partial G_{10}}{\partial \xi}(x,g(\tau),t-\tau)\,d\tau+$$
$$+\int_{t_0}^{t}\frac{\partial h_1}{\partial g}(g(\tau),\tau)\,\gamma(\tau)\,G_{10}(x,g(\tau),t-\tau)\,d\tau=$$
$$\triangleq P_1(x,t;u_1,w_1,p_1,v_1,g,\gamma), \tag{3.5}$$

$$p_2(x,t)=-a_2^2\int_{t_0}^{t} f_2(\tau, w_2(\tau); \varphi_2(\tau))\frac{\partial G_{2L}}{\partial \xi}(x,L,t-\tau)\,d\tau+$$
$$+\int_{g_0}^{L}\frac{d\pi_2}{d\xi}(\xi)\,G_{2L}(x,\xi,t-t_o)\,d\xi-$$
$$-\int_{t_0}^{t}\int_{g(\tau)}^{L} F_2(\xi,\tau;u_2,p_2,g,\gamma)\frac{\partial G_{2L}}{\partial \xi}(x,\xi,t-\tau)\,d\xi\,d\tau+$$
$$+a_2^2\int_{t_0}^{t} v_2(\tau)\frac{\partial G_{2L}}{\partial \xi}(x,g(\tau),t-\tau)\,d\tau-$$
$$-\int_{t_0}^{t}\frac{\partial h_2}{\partial g}(g(\tau),\tau)\,\gamma(\tau)\,G_{2L}(x,g(\tau),t-\tau)\,d\tau=$$
$$\triangleq P_2(x,t;u_2,w_2,p_2,v_2,g,\gamma), \tag{3.6}$$

$$v_1(t)=2P_1(g(t),t;u_1,w_1,p_1,v_1,g,\gamma)\triangleq V_1(t;u_1,w_1,p_1,v_1,g,\gamma), \tag{3.7}$$

$$v_2(t)=2P_2(g(t),t;u_2,w_2,p_2,v_2,g,\gamma)\triangleq V_2(t;u_2,w_2,p_2,v_2,g,\gamma), \tag{3.8}$$

$$g(t)=g_o+\int_{t_0}^{t}\gamma(\tau)\,d\tau\triangleq Y(t;\gamma), \tag{3.9}$$

$$\gamma(t)=S(t;g,v_1,v_2,h_1|_g,h_2|_g) \tag{3.10}$$

where G_{ij} denote the Green's functions respectively for the first or second boundary value problem ($i=1,2$) in regions $x>0$ or $x<L$ ($j=0,L$):

$$\begin{aligned} G_{i0}(x,\xi,t)&=E(x-\xi,a_1^2t)+(-1)^iE(x+\xi,a_1^2t),\\ G_{iL}(x,\xi,t)&=E(x-\xi,a_2^2t)+(-1)^iE(x+\xi-2L,a_2^2t). \end{aligned} \tag{3.11}$$

(ii) If $(u_1, u_2, w_1, w_2, p_1, p_2, v_1, v_2, g, \gamma)$ is a solution of the system of integral equations (3.1)–(3.10) and in addition all the above functions are Hölder continuous with respect to t:

$$|\nu(t)-\nu(\tau)|<A\,|\sqrt{t-t_0}-\sqrt{\tau-t_o}|, \tag{3.12}$$

then (u_1, u_2, g) is a solution of the free boundary value problem (2.1)–(2.9).

Proof of this theorem will be passed in two stages.

Stage 1 of the proof. We postulate now that functions φ_1, φ_2 enclosed in boundary conditions (2.5), (2.6) are from the space $C\,[t_o, t_k]$. These boundary conditions are understood now in the strong sense. Now the constraint (2.12) is assumed to be satisfied everywhere in the interval $[t_o, t_k]$.

By employing the Gevrey lemma (see Appendix) we will acquire desirable equivalence. In order to attain it we replace everywhere the source function E by appropriate Green function for the real half-line.

Let (u_1, u_2, g) be a solution of the free boundary problem (2.1)–(2.9) defined in the time interval $[t_o, t_k]$ and additionaly assume there exists such a positive constant ε that

$$\varepsilon\leqslant g(t)\leqslant L-\varepsilon,\ \ t\in[t_o, t_k].$$

For these functions u_1, u_2, g the functions F_i, h_i, f_i, S can be treated as depending respectively only on (x, t) or t, so we can use the Gevrey lemma (see Appendix). From this lemma it follows part (i) of the Theorem 1.

To prove part (ii) observe that if $(u_1, u_2, w_1, w_2, p_1, p_2, v_1, v_2, g, \gamma)$ is a solution of the system of integral equations (3.1)–(3.10) satisfying Hölder continuity condition (3.12) then the functions F_i are Hölder continuous with respect to x and t, namely Gevrey lemma again can be applied.

Stage 2 of the proof. Now we postulate control functions φ_1, φ_2 to be elements of the space $L_2\,[t_o, t_k]$. The boundary conditions (2.5), (2.6) are understood now in the sense of the assumption (A.2).

The idea of the proof consists in approximation of the problem with L_2-controls φ_i by a sequence of problems with continuous controls φ_{in}. To do that we must only show possibility of constructing sequences $(\varphi_{in})\subset C\,[t_o, t_k]$ convergent in the space $L_2\,[t_o, t_k]$ to φ_i such that appropriate sequences of solutions (u_{1n}, u_{2n}, g_n) as well as their first derivatives are almost uniformly convergent to solution (u_1, u_2, g) of the discussed problem and to the respective derivatives.

Let us construct sequences $(\varphi_{in})\subset C\,[t_o, t_k]$ such that φ_{in} is Steklov mean function with radius $1/n$ [10, 11]:

$$\varphi_{in}(t)=\int_{t-1/n}^{t+1/n}\omega_n(|t-\tau|)\,\varphi_i(\tau)\,d\tau \tag{3.13}$$

where

$$\omega_n(|t-\tau|)=\begin{cases}\left[\int\limits_{t-1/n}^{t+1/n}\exp\left\{-\frac{1}{n^2}\Big/\left(\frac{1}{n^2}-(t-\tau)^2\right)\right\}d\tau\right]^{-1}\times \\ \qquad\times\exp\left\{-\frac{1}{n^2}\Big/\left(\frac{1}{n^2}-(t-\tau)^2\right)\right\}, & |t-\tau|<\frac{1}{n}\\ 0, \quad |t-\tau|\geqslant\frac{1}{n}\end{cases}$$

$$\varphi_i(\tau)\triangleq 0,\ \tau\notin[t_o, t_k].$$

These sequences are admissible i.e.

$$\varphi_i^m\leqslant\varphi_{in}(t)\leqslant\varphi_i^M,\ t\in[t_o, t_k] \tag{3.14}$$

and the following convergence takes place [10]

$$\|\varphi_{in}-\varphi_i\|\, L_{2[t_o, t_k]}\xrightarrow[n]{}0.$$

Suppose there exist solutions (u_{1n}, u_{2n}, g_n) of the problem (2.1)–(2.9) corresponding to the controls φ_{in} and denote by (u_1, u_2, g) a solution of the same problem corresponding to controls $(\varphi_i$ (if it exists). Our purpose is to show that

$$\|g_n-g\|_{C^1[t_o, t_k]}\xrightarrow[n]{}0, \tag{3.15}$$

and

$$\|u_{in}-u_i\|_{C^{2,1}(\Omega_i)}\xrightarrow[n]{}0 \tag{3.16}$$

where Ω_i is any arbitrary nonempty compact subset of $Z_{t_k}^i$.

By assumption (A.1) the following estimates hold:

$$|u_{in}(x, t)|\leqslant A_{1i},\qquad (x, t)\in Z_{n_k}^i, \tag{3.17}$$

$$|\gamma_n(t)|\leqslant A_3,\qquad t\in[t_o, t_k], \tag{3.18}$$

where A_{1i}, A_3 are positive constants dependent only on final time moment t_k and on bounds of functions π_i, φ_i. The constants A_{1i}, A_3 are independent of n.

Taking into account the equality $g_n(t_o)=g_o$ and the estimate (3.18) we can conclude that the set $\{g_n\,|n\in N\}$ is compact in the space $C[t_o, t_k]$.

Hence it is possible to select some subsequence of (g_n) convergent uniformly in the time interval $[t_o, t_k]$ to a function g satisfying the condition $g(t_o)=g_o$. We denote this subsequence again by (g_n).

By (3.18) the functions g_n are Lipschitz continuous with the constant A_3. Then the function g is also Lipschitz continuous with the same constant. Furthermore, we can assume that after renumeration of the sequence (g_n) if necessary, for all $n\in N$

$$|g_n(g)-g(t)|<\alpha,\ t\in[t_o, t_k] \tag{3.19}$$

where α is any given positive number. Define the following regions

$$\begin{aligned} \Omega^1_{\alpha,\beta} &= \{(x,t)\,|\alpha<x<g(t)-\alpha,\ t_o+\beta<t<t_k\}, \\ \Omega^2_{\alpha,\beta} &= \{(x,t)\,|g(t)+\alpha<x<L-\alpha,\ t_o+\beta<t<t_k\}. \end{aligned} \tag{3.20}$$

The parameters α, β must only secure that

$$\Omega^i_{\alpha,\beta} \neq \emptyset,\ i=1,2. \tag{3.21}$$

Making use of the Bernstein method [7, 11] we can estimate values of all the derivatives $\frac{\partial u_{in}}{\partial x}$, $n \in N$ in the regions $\Omega^i_{\alpha,\beta}$ by constants dependent only on bounds for the data and the a priori — estimates (3.17) for the functions u_{in}.

Due to this property we are able to select such a sequence of indices (n_l) for which

$$\|u_{i,n_l}-u_i\|_{C^{2,1}(\bar{\Omega}^1_{\alpha,\beta})} \xrightarrow[n]{} 0,\ i=1,2. \tag{3.22}$$

Taking into account free choice of α, β we obtain that the above convergence is almost uniform in the regions $\bar{Z}^i_{t_k}$, $i=1,2$. Hence the functions u_i satisfy the appropriate parabolic equations (2.1) in the open regions int $Z^i_{t_k}$.

Furthermore we can conclude that u_{i,n_l} and $\frac{\partial u_{i,n_l}}{\partial x}$ are weakly convergent in $L_2(Z^i_{t_k})$ respectively to u_i and $\frac{\partial u_i}{\partial x}$ as well as $u_{1,n_l}(0,\cdot)$, $\frac{\partial u_{1,n_l}}{\partial x}(0,\cdot)$, $u_{2,n_l}(L,\cdot)$, $\frac{\partial u_{2,n_l}}{\partial x}(L,\cdot)$ are weakly convergent in $L_2[t_o,t_k]$ to some functions $\kappa_1, \chi_1, \kappa_2, \chi_2$. In that case

$$\begin{aligned} \chi_1(t) &= f_1(t, \kappa_1(t); \varphi_1(t)), \\ \kappa_2(t) &= f_2(t, \chi_2(t); \varphi_2(t)) \end{aligned}$$

almost everywhere in $[t_o, t_k]$.

By employing arguments similar to those used by Yu. V. Egorov [3] we can conclude that in the limit the boundary conditions (2.5), (2.6) are fulfilled.

Indeed for every functions $\eta_i \in C^\infty(Z^i_{t_k})$ equal to zero respectively outside regions

$$\begin{aligned} \Omega^*_1 &\triangleq \{(x,t)|0\leqslant x\leqslant\alpha,\ |t-t^*|\leqslant\beta\}, \\ \Omega^*_2 &\triangleq \{(x,t)|L-\alpha\leqslant x\leqslant L,\ |t-t^*|\leqslant\beta\} \end{aligned}$$

where $\alpha>0$, $\beta>0$, $t^* \in [t_o, t_k]$ are some arbitrary constants, the following identities hold

$$\begin{aligned} \int_{t^*-\beta}^{t^*+\beta} [u_{in_l}(L_i,t)-u_i(L_i,t)]\,\eta_i(L_i,t)\,dt &= \iint_{\Omega^*_i} \frac{\partial}{\partial x}[u_{in_l}(x,t)-u_i(x,t)]\times \\ &\times\eta_i(x,t)\,dx\,dt + \iint_{\Omega^*_i} [u_{in_l}(x,t)-u_i(x,t)]\frac{\partial\eta_i}{\partial x}(x,t)\,dx\,dt \end{aligned} \tag{3.23}$$

where $L_1=0$, $L_2=L$.

Since $\frac{\partial u_i}{\partial x} \in L_2(Z^i_{t_k})$ the values $u_i(L_i, t)$ are well defined.

Furthermore, by the convergence of the right hand side of (3.23) to zero when $l \to \infty$ we have

$\kappa_i(t) = u_i(L_i, t)$ almost everywhere for $t \in [t_o, t_k]$.

By (3.23) for every finite functions $\eta \in \mathscr{D}(t_o, t_k)$ the following convergences take place

$$\lim_{\alpha \to 0+} \int_{t_o}^{t_k} \frac{\partial u_1}{\partial x}(\alpha, t)\,\eta(t)\,dt = \int_{t_o}^{t_k} \chi_1(t)\,\eta(t)\,dt,$$

$$\lim_{\alpha \to 0+} \int_{t_o}^{t_k} \frac{\partial u_2}{\partial x}(L-\alpha, t)\,\eta(t)\,dt = \int_{t_o}^{t_k} \chi_2(t)\,\eta(t)\,dt$$

hence the boundary conditions (2.5), (2.6) are fulfilled in the sense of the assumption (A.2).

From the a priori — estimates for the functions $\frac{\partial u_{in}}{\partial x}$ [10] and the Hölder continuity of $\frac{\partial u_{in}}{\partial x}$ [8, 14] it follows that similar properties have functions $\frac{\partial u_i}{\partial x}$. Hence we can conclude that

$$\lim_{x \to g(t)-} \frac{\partial u_1}{\partial x}(x, t) = \frac{\partial u_1}{\partial x}(g(t), t), \quad t \in (t_o, t_k]$$

$$\lim_{x \to g(t)+} \frac{\partial u_2}{\partial x}(x, t) = \frac{\partial u_2}{\partial x}(g(t), t), \quad t \in (t_o, t_k].$$

In this connection functions $\frac{\partial u_i}{\partial x} \in C_0(Z^i_{t_k})$.

Making use of the barrier function method [7] we immediately come to the conclusion that

$$\lim_{t \to 0+} u_1(x, t) = \pi_1(x), \quad x \in [0, g_o)$$

$$\lim_{t \to 0+} u_2(x, t) = \pi_2(x), \quad x \in (g_o, L].$$

From the convergence of the sequence (g_n), by the estimate (3.18) we obtain existence of the derivative $\frac{dg}{dt}$, equal to γ and satisfying the condition (2.9) in the classical sense.

By the uniform convergence

$$u_{in_l} \xrightarrow[n_l]{} u_i \text{ in } \bar{\Omega}^i_{\alpha,\beta}$$

and

$$\frac{\partial u_{in_l}}{\partial x} \xrightarrow[n_l]{} \frac{\partial u_i}{\partial x} \text{ in } \bar{\Omega}^i_{\alpha,\beta} \quad (\alpha > 0)$$

the sequences (u_{in_l}) and $\left(\frac{\partial u_{in_l}}{\partial x}\right)$ are almost uniformly convergent respectively to u_i and $\frac{\partial u_i}{\partial x}$ in $\bar{Z}^i_{t_k}$.

In this way we have shown that it is possible to approximate solution of the free boundary problem (2.1)–(2.9) with controls being elements of the space $L_2\,[t_o, t_k]$ by sequence of solutions of problems (2.1)–(2.9) with some continuous control functions.

Making use of known properties of thermal potentials of single and double layer [8, 9] we can easily observe that operators transforming $\varphi_i \in L_2\,[t_o, t_k]$ into $u_i \in C\,(Z^i_{t_k})$ and into $g \in C\,[t_o, t_k]$, constructed on the basis of appropriate integrals (3.1)–(3.10) are continuous.

So we have shown the equivalence of the introduced differential and integral representations of the two-phase free boundary problem. Q.E.D.

From the above proof it follows immediately

COROLLARY 1. If the free boundary problems (2.1)–(2.9) with continuous controls φ_{in} have solutions (u_{1n}, u_{2n}, g_n) then the problem (2.1)–(2.9) with controls $\varphi_i \in L_2\,[t_o, t_k]$ has a solution (u_1, u_2, g).

4. Correctness in the Hadamard Sense of the Two-phase Free Boundary Problem (2.1)–(2.9)

In view of Corollary 1 we can restrict ourselves to the free boundary problem with continuous control functions φ_i. In this connection we will assume everywhere further that $\varphi_i \in C\,[t_o, t_k]$.

4.1. Existence and Uniqueness of the Solution

By Theorem 1 it is enough to prove existence and uniqueness of the solution to the system of Volterra integral equations (3.1)–(3.10). For the problem considered we are able to show only local existence theorem. Proof of the theorem will be based on employing Piccard's method od successive approximations. For time interval short enough the constructed sequence of approximate solutions will be convergent to the exact solution.

THEOREM 2. Let

— $\varphi_i \in C\,[t_o, t_k]$, $i=1, 2$;

— $\varphi_i^m \leqslant \varphi_i\,(t) \leqslant \varphi_i^M$, $t \in [t_o, t_k]$;

— the assumptions (A.1), (A.4)–(A.7) are fulfilled.

Then in some nontrivial time interval $[t_o, t_f]$ the system of integral equations (3.1)–(3.10) has unique solution $(u_1, u_2, w_1, w_2, p_1, p_2, v_1, v_2, g, \gamma)$ with all the func-

tions uniformly bounded in their domains and satisfying Hölder continuity condition with respect to t:

$$|v(t)-v(\tau)|<A|\sqrt{t-t_o}-\sqrt{\tau-t_o}|.$$

Outline of the proof. To construct the sequence of successive approximations let us select first functions $u_{io}, w_{io}, p_{io}, v_{io}$ $(i=1, 2)$ and g_0, γ_0 such that:

— For the first derivatives of these functions the following inequalities hold

$$\left|\frac{dg_0}{dt}(t)\right|<C_0/\sqrt{t-t_o}, \quad \left|\frac{\partial u_{io}}{\partial t}(x,t)\right|<C_{1i}/\sqrt{t-t_o},$$

$$\left|\frac{\partial p_{io}}{\partial x}(x,t)<C_{2i}/\sqrt{t-t_o}, \quad \left|\frac{\partial p_{io}}{\partial t}(x,t)\right|<C_{3i}/\sqrt{t-t_o}, \tag{4.1}$$

$$\left|\frac{dw_{io}}{dt}(t)\right|<C_{4i}/\sqrt{t-t_o}, \quad \left|\frac{dv_{io}}{dt}(t)\right|<C_{5i}/\sqrt{t-t_o}$$

where $C_0, C_{1i}, \ldots, C_{5i}$ are arbitrarily chosen positive constants.

— The following relationships are satisfied

$$\begin{aligned}
&\frac{dg_0}{dt}(t)=\gamma_0(t), && g_0(t_o)=g_o,\\
&u_{10}(0,t)=w_{10}(t), && u_{20}(L,t)=w_{20}(t),\\
&w_{10}(t_o)=\pi_1(0), && w_{20}(t_o)=\pi_2(L),\\
&u_{io}(x,t_o)=\pi_i(x), && p_{io}(g_0(t),t)=v_{io}(t),\\
&\frac{d\pi_i}{dx}(x)=p_{io}(x,t_o), && \frac{d\pi_i}{dx}(g_o,t_o)=p_{io}(g_o,t_o),\\
&\frac{\gamma\pi_1}{dx}(0)=f_1\big(t_o, w_{10}(t_o); \varphi_1(t_o)\big), &&\\
&\pi_2(L)=f_2\big(t_o, w_{20}(t_o); \varphi_2(t_o)\big) &&
\end{aligned} \tag{4.2}$$

in the appropriate closed intervals.

The Piccard's process of iterations has then the form

$$\begin{aligned}
&w_{i,n+1}(t)=W_i(t; u_{in}, w_{in}, p_{in}, v_{in}, g_n, \gamma_n),\\
&u_{i,n+1}(x,t)=U_i(x,t; u_{in}, w_{i,n+1}, p_{in}, v_{in}\, g_n, \gamma_n),\\
&v_{i,n+1}(t)=V_i(t; u_{i,n+1}, w_{i,n+1}, p_{in}, v_{in}, g_n, \gamma_n),\\
&p_{i,n+1}(x,t)=P_i(x,t; u_{i,n+1}, w_{i,n+1}, p_{in}, v_{i,n+1}, g_n, \gamma_n),\\
&\gamma_{n+1}(t)=S(t; g_n, v_{1,n+1}, v_{2,n+1}, h_1|_{g_n}, h_2|_{g_n}),\\
&g_{n+1}(t)=Y(t; \gamma_{n+1}).
\end{aligned} \tag{4.3}$$

The proof of the convergence of this process will be derived in a way similar to that carried out by Rubinstein [13] for one-phase free boundary problem.

The method used by Rubinstein avails analytical properties of thermal potentials of single and double layer [8, 9]. Following this method we can prove equi-boundedness and equi-continuity of the sequences

$$(u_{in}), (w_{in}), (p_{in}), (v_{in}), (\gamma_n), (g_n), \quad i=1, 2.$$

Making use of the Ascoli-Arzela theorem and of compactness of the operators U_i, W_i, P_i, V_i, S we obtain existence of the solution of the system of integral equations (3.1)–(3.10) in some nontrivial time interval $[t_o, t_f]$.

To prove uniqueness of the solution we may again follow the Rubinstein's method [13].

From estimates derived by means of the Rubinstein's method it follows

REMARK 1. Value of the difference $t_f - t_o$ depends only on

— bounds for the data,

— a priori-estimates for the solution,

— estimates of the functions F_i, h_i, f_i, S and their derivatives (see assumption (A.6)),

— value of $\varepsilon \triangleq \min \{ \inf_{t \in t_o, t_f)} g(t), \inf_{t \in (t_o, t_f)} [L - g(t)]\}$.

If $\varepsilon \to 0$ and maximal value of the bounds for π_i, F_i, h_i, f_i, S tends to infinity then $t_f \to t_o$.

We should note that we were not able to show existence of the solution in any given time interval $[t_o, t_k]$. To prove global existence of the solution it is necessary to assume something more about the problem (2.1)–(2.9) [1, 4, 10, 12, 14]. In particular one of the possible sufficient conditions for global existence is nonnegativeness of the function S in the condition (2.9) [14].

4.2. Continuous Dependence on Data

Let us call the value g_o and functions π_i, φ_i the input data for the free boundary problem (2.1)–(2.9). Our purpose is to show continuous dependence of the solution of the problem (2.1)–(2.9) on the input data.

We denote by (u_1, u_2, g) the solution corresponding to the input data (g_o, π_i, φ_i), existing for $t \in [t_o, t_f]$. The sign $\hat{}$ will correspond everywhere to the perturbed input data $(\hat{g}_o, \hat{\pi}_i, \hat{\varphi}_i)$. We will also use the following notations:

$$t_f' = \min \{t_f, \hat{t}_f\},$$

$$g_o' = \min \{g_o, \hat{g}_o\}, \; g_o'' = \max \{g_o, \hat{g}_o\},$$

$$Q_1 = (0, g_o'), \; Q_2 = (g_o'', L),$$

$$Z_i' = Z_{t_f}^i \cap \hat{Z}_{t_f}^i.$$

Further we define:

— the neighbourhood of the solution (u_1, u_2, g):

$$\begin{aligned}\mathscr{U}_\delta(u_1, u_2, g) \triangleq \{(\hat{u}_1, \hat{u}_2, \hat{g}) \| u_i - \hat{u}_i \|_{C(\bar{Z}_i')} < \delta, \\ \|p_i - \hat{p}_i\|_{C(\bar{Z}_i')} < \delta, \ \|v_i - \hat{v}_i\|_{C[t_o, t_f']} < \delta, \\ \|w_i - \hat{w}_i\|_{C[t_o, t_f']} < \delta, \ \|g - \hat{g}\|_{C[t_o, t_f']} < \delta, \\ \|\gamma - \hat{\gamma}\|_{C[t_o, t_f']} < \delta\}. \end{aligned} \tag{4.4}$$

— the neighbourhood of the input data (g_o, π_i, φ_i):

$$\begin{aligned}\mathscr{W}_\eta(g_o, \pi_i, \varphi_i) \triangleq \{(\hat{g}_o, \hat{\pi}_i, \hat{\varphi}_i) | \ |g_o - \hat{g}_o| < \eta; \\ \left\| \frac{d^j \pi_i}{dx^j} - \frac{d^j \hat{\pi}_i}{dx^j} \right\|_{C(\bar{Q}_i)} < \eta, \ j = 0, 1, 2; \\ \|\varphi_i - \hat{\varphi}_i\|_{C[t_o, t_f']} < \eta\}. \end{aligned} \tag{4.5}$$

We assume in addition that the free boundary problem (2.1)–(2.9) has the following property:

(A.8) If for $(x, t) \in \bar{Z}_i'$ or respectively for $t \in [t_o, t_f']$

$$\begin{aligned} |u_i(x, t)| < A, \quad & |\hat{u}_i(x, t)| < A, \\ |p_i(x, t)| < A, \quad & |\hat{p}_i(x, t)| < A, \\ |v_i(t)| < A, \quad & |\hat{v}_i(t)| < A, \\ |w_i(t)| < A, \quad & |\hat{w}_i(t)| < A, \\ |\gamma(t)| < A, \quad & |\hat{\gamma}(t)| < A, \\ 0 < g(t) < L, \quad & 0 < \hat{g}(t) < L, \end{aligned} \tag{4.6}$$

where A denotes a positive constant dependent only on the bounds for the input data then

$$\begin{aligned} |F_i - \hat{F}_i| < B, \quad & |h_i - \hat{h}_i| < B, \\ |f_i - \hat{f}_i| < B, \quad & |S - \hat{S}| < B, \end{aligned} \tag{4.7}$$

where B is a positive constant dependent only on A and relationship $|\nu - \hat{\nu}| < B$ denotes the system of inequalities

$$|\nu(\alpha_1, \ldots, \alpha_k) - \hat{\nu}(\alpha_1, \ldots, \alpha_k)| < B,$$

$$\left| \frac{\partial \nu}{\partial \alpha_j}(\alpha_1, \ldots, \alpha_k) - \frac{\partial \hat{\nu}}{\partial \alpha_j}(\alpha_1, \ldots, \alpha_k) \right| < B, \ j = 1, \ldots, k$$

fulfilled for all the arguments satisfying (4.6).

Under the above assumption solution of the problem (2.1)–(2.9) is continuously dependent on the input data.

THEOREM 3. Let

— the solution (u_1, u_2, g) of the free boundary problem (2.1)–(2.9) corresponding to the input data (g_o, π_i, φ_i) exist for $t \in [t_o, t_f]$,

— the solution $(\hat{u}_1, \hat{u}_2, \hat{g})$ corresponding to $(\hat{g}_o, \hat{\pi}_i, \hat{\varphi}_i)$ exist for $t \in [t_o, \hat{t}_f]$,

— assumption (A.8) be satisfied.

Then for every $\delta > 0$ there is such a number $\eta > 0$ that

$$(\hat{g}_o, \hat{\pi}_i, \hat{\varphi}_i) \in \mathscr{W}_\eta (g_o, \pi_i, \varphi_i) \Rightarrow (\hat{u}_1, \hat{u}_2, \hat{g}) \in \mathscr{U}_\delta (u_1, u_2, g).$$

Proof of this theorem can be derived by employing a modification of Rubinstein's method [13] proposed for one-phase problem.

Outline of the proof. Let us introduce the notations

$$\begin{aligned}
\Delta u_i &= \max_{(x,t) \in \bar{Z}_i'} |u_i(x,t) - \hat{u}_i(x,t)|, \\
\Delta p_i &= \max_{(x,t) \in \bar{Z}_i'} |p_i(x,t) - \hat{p}_i(x,t)|, \\
\Delta v_i &= \max_{t \in [t_o, t_f']} |v_i(t) - \hat{v}_i(t)|, \\
\Delta w_i &= \max_{t \in [t_o, t_f']} |w_i(t) - \hat{w}_i(t)|, \quad i = 1, 2 \\
\Delta g &= \max_{t \in [t_o, t_f']} |g(t) - \hat{g}(t)|, \\
\Delta \gamma &= \max_{t \in [t_o, t_f']} |\gamma(t) - \hat{\gamma}(t)|.
\end{aligned} \tag{4.8}$$

First we restrict ourselves to the case

$$\hat{g}_o = g_o. \tag{4.9}$$

Let us consider functions $\Phi(x, t; \alpha_1, \ldots, \alpha_k)$, $\hat{\Phi}(x, t; \alpha_1, \ldots, \alpha_k)$ continuously differentiable with respect to all arguments contained in the set

$$\{(x, t; \alpha_1, \ldots, \alpha_k) | (x, t) \in \bar{Z}_i'; |\alpha_j| < A, j = 1, \ldots, k\}$$

and uniformly bounded there. For such functions the following inequality holds

$$\begin{aligned}
|\Phi(x, t; u_i, w_i, p_i, v_i, g, \gamma) - \hat{\Phi}(x, t; \hat{u}_i, \hat{w}_i, \hat{p}_i, \hat{v}_i, \hat{g}, \hat{\gamma})| \leqslant \\
\leqslant \mathfrak{A}(\Phi, \hat{\Phi}) + A_1 (\Delta u_i + \Delta p_i + \Delta v_i + \Delta w_i + \Delta g + \Delta \gamma)
\end{aligned} \tag{4.40}$$

where A_1 is a positive constant independent of

$$\mathfrak{A}(\Phi, \hat{\Phi}) \triangleq \max_{(x, t; \alpha_1, \ldots, \alpha_k)} |\Phi(x, t; \alpha_1, \ldots, \alpha_k) - \hat{\Phi}(x, t; \alpha_1, \ldots, \alpha_k)|.$$

Taking into account the possibility of estimating variations of individual terms in (3.1)–(3.10) by functions of the form $\bar{A} + \bar{\bar{A}}\sqrt{t}$ where $\bar{A}$, $\bar{\bar{A}}$ are uniformly defined constants [8, 9, 14], by (4.10) we get

$$
\begin{aligned}
&\Delta U_i \leqslant A_2\sqrt{t-t_o}\,(\Delta u_i+\Delta p_i+\Delta w_i+\Delta v_i+\Delta\gamma)+A_3\mathfrak{A}_d\,,\\
&\Delta P_i \leqslant A_2\sqrt{t-t_o}\,(\Delta u_i+\Delta p_i+\Delta\gamma)+A_4\,(\Delta w_i+\Delta v_i)+A_3\,\mathfrak{A}_d\,,\\
&\Delta W_i \leqslant A_2\sqrt{t-t_o}\,(\Delta u_i+\Delta p_i+\Delta w_i+\Delta v_i+\Delta\gamma)+A_3\,\mathfrak{A}_d\,,\\
&\Delta V_i \leqslant A_2\sqrt{t-t_o}\,(\Delta u_i+\Delta p_i+\Delta w_i+\Delta v_i+\Delta\gamma)+A_3\,\mathfrak{A}_d\,,\\
&\Delta S \leqslant A_2\sqrt{t-t_o}\,\Delta\gamma+A_4\,(\Delta v_1+\Delta v_2)+A_3\,\mathfrak{A}_d
\end{aligned}
\tag{4.11}
$$

where $\mathfrak{A}_d \triangleq \max\{\mathfrak{A}(F_i, \hat{F}_i), \mathfrak{A}(\pi_i, \hat{\pi}_i), \mathfrak{A}(h_i, \hat{h}_i), \mathfrak{A}(f_i, \hat{f}_i);\ i=1, 2\}$ and A_2, A_3, A_4 are positive constants independent of $\mathfrak{A}_d$.

Both solutions (u_1, u_2, g) and $(\hat{u}_1, \hat{u}_2, \hat{g})$ of the problem (2.1)–(2.9) are defined for $t \in [t_o, t_f']$. It follows from (4.11) that in a nontrivial subinterval $[t_o, t_f'']$

$$
\Delta u_i,\ \Delta p_i,\ \Delta w_i,\ \Delta v_i,\ \Delta g,\ \Delta\gamma \leqslant A(t)\,\mathfrak{A}_d,\quad i=1, 2 \tag{4.12}
$$

where $A(t)$ is a positive increasing function of variable t, independent of variations $\Delta F_i, \Delta\pi_i, \Delta h_i, \Delta f_i$.

Since the value t_f'' depends in fact only on the bounds of the input data [12, 13], the estimates (4.12) can be extended to the whole time interval $[t_o, t_f']$. In this way we have shown that under assumption (4.9) Theorem 3 holds.

Now suppose that the assumption (4.9) is not satisfied.

Let us introduce new coordinates

$$
x^* = \frac{g_o}{\hat{g}_o}\,x, \qquad t^* = \frac{g_0^2}{\hat{g}_0^2}\,t \tag{4.13}
$$

and auxiliary functions

$$
\left.
\begin{aligned}
&u_i^*(x^*, t^*) = \hat{u}_i(x, t), \qquad p_i^*(x^*, t^*) = \frac{\hat{g}_o}{g_o}\,\hat{p}_i(x, t),\\
&v_i^*(t^*) = \frac{\hat{g}_o}{g_o}\,\hat{v}_i(t),\\
&w_1^*(t^*) = \hat{w}_1(t), \qquad w_2^*(t^*) = \frac{\hat{g}_o}{g_o}\,\hat{w}_2(t),\\
&g^*(t^*) = \frac{g_o}{\hat{g}_o}\,\hat{g}(t), \qquad \gamma^*(t^*) = \frac{\hat{g}_o}{g_o}\,\hat{\gamma}(t)
\end{aligned}
\right\}
\tag{4.14}
$$

$$
\left.
\begin{aligned}
&F_i^*(x^*, t^*; u_i^*, \ldots, \gamma^*) = \frac{\hat{g}_o^2}{g_o^2}\,\hat{F}_i(x, t; \hat{u}_i, \ldots, \hat{\gamma}),\\
&\pi_i^*(x^*) = \hat{\pi}_i(x), \qquad h_i^*(x^*, t^*) = \hat{h}_i(x, t),\\
&f_1^*(t^*, w_1^*; \varphi_1) = \frac{\hat{g}_o}{g_o}\,\hat{f}_1(t, \hat{w}_1; \varphi_1),\\
&f_2^*(t^*, w_2^*; \varphi_2) = \hat{f}_2(t, \hat{w}_2; \varphi_2),\\
&S^*(t^*; g^*, \ldots, h_2^*|_{g^*}) = \frac{\hat{g}_o}{g_o}\,\hat{S}(t; \hat{g}, \ldots, \hat{h}_2|_{\hat{g}}).
\end{aligned}
\right\}
\tag{4.15}
$$

Then the system of integral equations (3.1)–(3.10) is satisfied by the functions $u_i^*, p_i^*, w_i^*, v_i^*, g^*, \gamma^*$.

In addition the variations $\Delta u_i, \ldots, \Delta\gamma$ can be represented in the form

$$\Delta u_i = \max_{(x,t)} |u_i(x,t) - u_i^*(x^*, t^*)|,$$

$$\Delta p_i = \max_{(x,t)} |p_i(x,t) - \frac{g_o}{\hat{g}_o} p_i^*(x^*, t^*)|,$$

$$\Delta w_1 = \max_t |w_1(t) - w_1^*(t^*)|,$$

$$\Delta w_2 = \max_t \left| w_2(t) - \frac{g_o}{\hat{g}_o} w_2^*(t^*) \right|,$$

$$\Delta v_i = \max_t \left| v_i(t) - \frac{g_o}{\hat{g}_o} v_i^*(t^*) \right|,$$

$$\Delta g = \max_t \left| g(t) - \frac{\hat{g}_o}{g_o} g^*(t^*) \right|,$$

$$\Delta\gamma = \max_t \left| \gamma(t) - \frac{g_o}{\hat{g}_o} \gamma^*(t^*) \right|.$$

The solutions $(u_i, \ldots, \gamma)$ and $(u_i^*, \ldots, \gamma^*)$ are associated with the same value of initial position of the free boundary. In this connection for the variations

$$\Delta^* u_i = \max_{(x,t)} |u_i(x,t) - u_i^*(x,t)|,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\Delta^* \gamma = \max_t |\gamma(t) - \gamma^*(t)|$$

the estimates (4.12) also hold.

If $|g_o - \hat{g}_o| \to 0$ then $x^* \to x$ and $t^* \to t$. Hence, since the functions $u_i, \ldots, \gamma$ and $\hat{u}_i, \ldots, \hat{\gamma}$ are continuous with respect to (x, t), we get

$$\lim_{|g_o - \hat{g}_o| \to 0} |\Delta u_1 - \Delta^* u_i| = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\lim_{|g_o - \hat{g}_o| \to 0} |\Delta\gamma - \Delta^* \gamma| = 0.$$

That is why taking into account Theorem 1 and known properties of thermal potentials [8, 9] also in the case $\hat{g}_o \neq g_o$ we get continuous dependence of solution of problem (2.1)–(2.9) on the input data.

The continuous dependence takes place in the whole time interval $[t_o, t_f']$ due to the same arguments as previously. Q.E.D.

It is worth to note that for the one-phase free boundary problem one can also show the continuous dependence of the solution on the coefficients of parabolic differential operator [13].

5. Concluding Remarks

(i) The proved properties of two-phase parabolic free boundary problems can be extended to multi-phase problems.

(ii) The integral representations of free boundary problems will be applied in the next paper to solving some control problems.

(iii) The results of the paper can be extended to free boundary problems with coefficients dependent on (x, t).

APPENDIX

Suppose that functions x_1, x_2 are continuously differentiable for $t>t_o$, functions $\sqrt{t-t_o}\, dx_i/dt$ are continuous for $t \geqslant t_o$ and there exists such $\varepsilon>0$ that for every $t \geqslant t_o$

$$x_2(t)-x_1(t) \geqslant \varepsilon.$$

Let us denote

$$\Omega=\{(x, t) \mid x_1(t)<x<x_2(t), \quad t>t_o\}.$$

$$\Gamma=\Gamma_0 \cup \Gamma_1 \cup \Gamma_2,$$

$$\Gamma_i=\{(x_i(t), t) \mid t \geqslant t_o\}, \quad i=1, 2$$

$$\Gamma_0=\{(x, t_o) \mid x_1(t_o) \leqslant x \leqslant x_2(t_o)\}.$$

In the paper [5] it has been derived the following lemma concerning the integral representation of solution to parabolic equation.

LEMMA (Gevrey [5]). Let

— u be function bounded in $\Omega \subset R^2$,

— $u \in C^{2,1}(\Omega)$,

— $u, \dfrac{\partial u}{\partial x}$ be continuous in $\bar{\Omega}$ (at most except the points $(x_1(t_o), t_o)$, $(x_2(t_o), t_o)$,

— F is function satisfying in the region Ω the Hölder condition with respect both to x and to t.

Then

— function u satisfying in Ω the parabolic equation

$$\frac{\partial u}{\partial t}-a^2 \frac{\partial^2 u}{\partial x^2}+F(x, t)=0 \tag{a.1}$$

has the following integral representation

$$u(x,t)=\int\limits_{\Gamma_1\cup\Gamma_2}\left[a^2\frac{\partial u}{\partial\xi}(\xi,\tau)\,E\left(x-\xi,a^2(t-\tau)\right)-\right.$$

$$\left.-u(\xi,\tau)\frac{\partial E}{\partial\xi}\left(x-\xi,a^2(t-\tau)\right)\right]d\tau+\int\limits_{\Gamma_0}u(\xi,t_o)\,E\left(x-\xi,a^2(t-t_o)\right)d\xi+$$

$$+\iint\limits_{\Omega}F(\xi,\tau)\,E\left(x-\xi,a^2(t-\tau)\right)d\xi\,d\tau\,, \qquad \text{(a.2)}$$

where E is fundamental solution of the heat equation in $R\times(t_o,+\infty)$;

— function u having the integral representation (a.2) satisfies in Ω the parabolic equation (a.1).

Observe that the function E can be replaced in (a.2) by the Green's functions of parabolic boundary value problems.

Taking into account possibility of differentiating under the integrals in (a.2) and the relationships

$$\frac{\partial G_{10}}{\partial x}=-\frac{\partial G_{20}}{\partial\xi},\quad \frac{\partial G_{1L}}{\partial\xi}=-\frac{\partial G_{2L}}{\partial x},$$

we get integral representations of $\frac{\partial u}{\partial x}$. On the basis of these representations by the discontinuity of thermal potential of double layer [4, 8] we obtain integral representations of functions $u\left(x_i(\cdot),\cdot\right)$, $\frac{\partial u}{\partial x}\left(x_i(\cdot),\cdot\right)$; $i=1,2$.

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Pewne własności sterowanych dwufazowych parabolicznych zagadnień brzegowych ze swobodną granicą

W artykule rozważana jest pewna klasa sterowanych jednowymiarowych dwufazowych parabolicznych zagadnień brzegowych ze swobodną granicą. Przyjmuje się, że nieliniowe są równania stanu procesu, warunki brzegowe oraz warunki obowiązujące na swobodnej granicy i określające jej dynamikę. W pracy wprowadza się reprezentację całkową zagadnienia. Podany jest dowód lokalnego istnienia rozwiązania rozważanego zagadnienia, jednoznaczności tego rozwiązania i jego ciągłej zależności od danych.

О некоторых свойствах управляемых двухфазных параболических краевых задач со свободной границей

В работе рассмотрена некоторая управляемая одномерная двухфазная параболическая краевая задача со свободной границей. Принято, что нелинейны уравнения состояния процесса, краевые условия и условия заданные на свободной границе. В работе введена эквивалентная интегральная репрезентация проблемы. Доказаны: локальное существование решения рассматриванной проблемы, единственность этого решения и его непрерывная зависимость от данных.