

## A Concept of Optimality of Degree $p$ in the Observation Theory and Optimal Observability for Convex Functionals

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A concept of observability optimal of degree  $p$  is introduced and considered for Hölder continuous functionals and operators in linear and nonlinear infinite dimensional systems. There are given also results on optimal observability for convex functionals in linear systems.

### 0. Introduction

A problem of optimal observability was posed by Krasovski [3] for finite dimensional linear systems. Rolewicz [10] considered the problem in infinite dimensional ones. Both authors restricted themselves to examining linear functionals. Main notions used in [3] [10] are as follows. Consider a linear system  $X \xrightarrow{C} Y$ , where  $X$  and  $Y$  are Banach spaces and  $C$  is a linear bounded operator. A functional  $f$  belonging to the class  $P_X = X^*$  is said to be observable if there is a functional  $\varphi \in P_Y = Y^*$  such that  $f(x) = \varphi(Cx)$  for all  $x \in X$ . An observation  $\hat{\varphi}$  for  $f$  is called optimal if

$$\sup_{\|\Delta y\| \leq 1} [\hat{\varphi}(y + \Delta y) - \hat{\varphi}(y)] = \inf_{\varphi} \sup_{\|\Delta y\| \leq 1} [\varphi(y + \Delta y) - \varphi(y)] \quad (1)$$

and the infimum is finite, where the infimum is taken over all observations for  $f$ .

The purpose of the note is to extend the class  $P_X$  (and then also  $P_Y$ ) of considered functionals and the class of optimally observable functionals by discussions about an optimality of degree  $p$ . The primary idea of the generalization was seeing that a necessary condition that the left side of (1) be finite for a nonlinear functional  $\hat{\varphi}$  is that  $\hat{\varphi}$  be Lipschitzian and so  $f$  be Lipschitzian (see (6) for  $p=1$ ), and a conviction that there is another concept of optimality characterizing the best (in a certain sense) observations and following us to investigate a larger class of functionals.

The note contains also an extension to the case of the observation for operators and the case when  $C$  is nonlinear, but at this stage the study is far from being

exhaustive. It will be seen from the note that the optimal observation problem is intrinsically equivalent to the problem of the extension preserving the norm in the case of functionals, but the first problem is broader than the second one in the operator case.

In the second part of the note we examine separately convex functionals because these functionals are very often met (specially in extremal problems)

## 1. Optimality of Degree $p$

### 1.a. Recession functionals of degree $p$

Let  $f \in \bar{R}^X$ , where  $X$  is a Banach space and  $\bar{R} = R \cup \{-\infty, +\infty\}$ , and  $f \not\equiv +\infty$ .

DÉFINITION 1. The recession functional of degree  $p > 0$  of  $f$  is a functional  $f0^p \in \bar{R}^X$  such that its epigraph is

$$\text{epi } (f0^p) = \{(x, v) \in X \times R : \text{epi } f + (\lambda x, \lambda^p v) \subset \text{epi } f, \forall \lambda > 0\}$$

This is a direct generalization of the recession functional  $f0^+$  (see [7], [4] chapter 6, in [4]  $f0^+$  is called an asymptotic functional and denoted by  $f_\infty$ ), and we have  $f0^1 = f0^+$ .

THEOREM 1.  $f0^p$  is positively homogeneous of degree  $p$ , i.e.  $(f0^p)(\lambda z) = \lambda^p (f0^p)(z)$  for  $\lambda > 0$ , and one has

$$(f0^p)(z) = \sup_{\substack{x \in \text{dom } f \\ \lambda > 0}} \frac{f(x + \lambda z) - f(x)}{\lambda^p}. \quad (2)$$

Moreover, if  $f$  is convex, then  $f0^p$  is convex and from the closeness of  $f$  follows closeness of  $f0^p$ .

Proof. By definition  $(z, v) \in \text{epi } (f0^p)$  if and only if  $(x, \mu) + (\lambda z, \lambda^p v) = (x + \lambda z, \mu + \lambda^p v) \in \text{epi } f$  for every  $(x, \mu) \in \text{epi } f$  and  $\lambda > 0$ , which means that  $f(x + \lambda z) \leq f(x) + \lambda^p v$ . Hence  $(z, v_1) \in \text{epi } (f0^p)$  for all  $v_1 \geq v$  if  $(z, v) \in \text{epi } (f0^p)$ , i.e.  $\text{epi } (f0^p)$  is epigraph and we obtain formula (2). From this formula it easily follows the positive homogeneity of  $f0^p$ .

If  $f$  is convex and  $(z_1, v_1), (z_2, v_2)$  belong to  $\text{epi } (f0^p)$ , then we have for  $0 \leq \alpha \leq 1$ ,  $\lambda > 0$

$$\alpha [\text{epi } f + (\lambda z_1, \lambda^p v_1)] + (1 - \alpha) [\text{epi } f + (\lambda z_2, \lambda^p v_2)] \subset \text{epi } f$$

and hence

$$\text{epi } f + (\lambda [\alpha z_1 + (1 - \alpha) z_2], \lambda^p [\alpha v_1 + (1 - \alpha) v_2]) \subset \text{epi } f.$$

Thus  $\text{epi } (f0^p)$  is convex. If  $f$  is closed, i.e. lower semicontinuous, then

$$\begin{aligned} \lim_{z_n \rightarrow z_0} \inf (f0^p)(z_n) &= \lim_{z_n \rightarrow z_0} \inf \sup_{\substack{x \in \text{dom } f \\ \lambda > 0}} \frac{f(x + \lambda z_n) - f(x)}{\lambda^p} \geq \sup_{\substack{x \in \text{dom } f \\ \lambda < 0}} \frac{f(x + \lambda z_0) - f(x)}{\lambda^p} = \\ &= (f0^p)(z_0) \end{aligned}$$

so  $f0^p$  is closed.

Q.E.D.

THEOREM 2. We have

$$\sup_{\|z\| \leq 1} (f0^p)(z) = \alpha < +\infty \quad (3)$$

if and only if  $f$  satisfies the Hölder condition

$$f(x_2) - f(x_1) \leq \alpha \|x_2 - x_1\|^p \quad (4)$$

where  $\alpha$  is the minimal Hölder constant.

If, in addition,  $f$  is convex, then either  $f = \text{const.}$  or  $p = 1$ .

PROOF. In view of (2) and the positive homogeneity of degree  $p$  of  $f0^p$  we see that (4) holds if and only if

$$\begin{aligned} +\infty > \alpha &= \sup_{x_1 \neq x_2} \frac{f(x_2) - f(x_1)}{\|x_2 - x_1\|^p} = \sup_{\substack{\|x_2 - x_1\| = \lambda \\ \lambda > 0}} \frac{f(x_2) - f(x_1)}{\lambda^p} = \\ &= \sup_{\substack{\|x_2 - x_1\| = \lambda \\ \lambda > 0}} (f0^p) \frac{(x_2 - x_1)}{\lambda} = \sup_{\|z\| = 1} (f0^p)(z) = \sup_{\|z\| \leq 1} (f0^p)(z). \end{aligned}$$

If  $p > 1$ , then (4) implies that  $f = \text{const.}$  (as the Fréchet differential is equal to zero everywhere). Now suppose that  $f$  is convex, not constant and satisfies (4) for  $p < 1$  and so  $f0^p$  is convex (positively homogeneous of degree  $p$ ). We show that this is impossible. In fact, let  $x_1$  be a point such that  $(f0^p)(x_1) > 0$  (the existence of such a  $x_1$  is checked by (3), (4)),  $x_2 > 0$  and  $0 < \lambda < 1$ . Then

$$(f0^p)(\lambda x_1 + (1 - \lambda)x_2) = \lambda^p (f0^p)(x_1) > \lambda (f0^p)(x_1) = \lambda (f0^p)(x_1) + (1 - \lambda)(f0^p)(x_2)$$

which contradicts the convexity of  $f0^p$ . Thus  $p = 1$ .

Q.E.D.

### 1.b. Optimality of Degree $p$

Consider a system (not necessarily linear)  $W^c \rightarrow Y$ . As a natural generalization of (1) we introduce the

DEFINITION 2. An observable functional  $f \in P_X$  is said to be optimally observable of degree  $p > 0$  if there is an observation  $\hat{\phi} \in P_Y$  such that

$$\sup_{\substack{y \in \text{dom } \hat{\phi} \\ \|\Delta y\| \leq 1, \lambda > 0}} \frac{\hat{\phi}(y + \lambda \Delta y) - \hat{\phi}(y)}{\lambda^p} = \inf_{\phi} \sup_{\phi} \frac{\phi(y + \lambda \Delta y) - \phi(y)}{\lambda^p} \quad (5)$$

and that the infimum is finite, where the infimum is taken over all observations for  $f$ .

The point is, that in the definition  $P_X$  (and then  $P_Y$ ) may not be the same for various problems. It will be exactly defined separately for each problem.

Denote the left side of (5) by  $\|\hat{\phi}\|_p$ . Following Theorems 1 and 2

$$\|\hat{\phi}\|_p = \sup_{\|\Delta y\| \leq 1} (\hat{\phi}0^p)(\Delta y) = \text{minimal Hölder constant,}$$

and we see that  $\hat{\varphi}$  is an observation optimal of degree  $p$  only if it is Hölder continuous with exponent  $p$ .

If  $C$  is linear, then it is easy to verify that

$$\sup_{\|y\| \leq 1} (\varphi 0^p)(y) \geq \frac{1}{\|C\|^p} \sup_{\|x\| \leq 1} (f 0^p)(x) \quad (6)$$

whenever  $\varphi$  is an observation for  $f$ . Thus we often define in the natural way, for the general case of nonlinear  $C$ ,  $P_X$  as the class of all functionals Hölder continuous with exponent  $p$ :  $0 < p \leq 1$  ( $P_Y$  is similar).

Now let  $C$  be nonlinear. Let  $\mathcal{R}$  be the relation of equivalence:  $x_1 \mathcal{R} x_2$  if and only if  $Cx_1 = Cx_2$ . Then we have

**THEOREM 3.** If there are  $k > 0$  and  $0 < q \leq 1$  such that for any  $y_1, y_2 \in CX$  one can find  $x_1, x_2$ :  $Cx_1 = y_1$ ,  $Cx_2 = y_2$ , satisfying condition

$$\|x_1 - x_2\| \leq k \|y_1 - y_2\|^q. \quad (7)$$

Then each functional  $f$  on  $X$  such that  $f$  is well-defined on  $X/\mathcal{R}$  and is Hölder continuous with exponent  $p$ , is observable optimally of degree  $pq$ .

**Proof.** Put  $\varphi(Cx) = f(x)$ .  $\varphi$  is well-defined on  $CX$  since  $f$  is well-defined on  $X/\mathcal{R}$ . By (7) we have for any  $y_1, y_2 \in CX$ :

$$\varphi(y_1) - \varphi(y_2) = f(x_1) - f(x_2) \leq \alpha \|x_1 - x_2\|^p \leq \alpha k^p \|y_1 - y_2\|^{pq}.$$

Thus  $\varphi$  is Hölder continuous with exponent  $0 < pq \leq 1$ . Consequently,  $\varphi$  can be extended to  $\hat{\varphi}$  defined on the whole  $Y$  preserving this condition ([5], [11], [2] p. 120). Clearly,  $\hat{\varphi}$  is an observation optimal of degree  $pq$ . Q.E.D.

In what follows, without loss of generality, we shall consider only functionals  $f$  such that  $f(0) = 0$ .

**COROLLARY 1.** If  $C$  is linear, bounded and  $CX$  is closed, then each functional  $f$  satisfying the Hölder condition with exponent  $0 < p \leq 1$  and vanishing on  $\text{Ker } C$  is observable optimally of degree  $p$ .

**Proof.** We need only verify condition (7) for  $q = 1$ . Since  $CX$  is closed by the Banach theorem on inverse operators there is  $k > 0$  such that for all  $y \in CX$  there exists  $x$  such that  $Cx = y$  and  $\|x\| \leq k \|y\|$ . Then for given  $y_1, y_2 \in CX$  we can choose  $x_1, x_2$  such that  $Cx_1 = y_1$ ,  $Cx_2 = y_2$  and (7) holds for  $q = 1$ . Q.E.D.

**Example 1.** Let  $X = c_0$  and  $Y$  be any Banach space containing  $c_{op}$  as a closed subspace, where  $c_{op}$  is the space of all sequences  $\{y_n\}$  such that  $n^p y_n \rightarrow 0$  ( $0 < p \leq 1$ ) with the norm  $\|\{y_n\}\| = \sup_n |n^p y_n|$ . Let  $C(\{x_n\}) = \{x_n/n^p\}$ . Let  $A$  be a (arbitrary) subset of  $X$  and let  $f(x) = d_p(x, A) \stackrel{\text{df}}{=} \inf_{z \in A} \|z - x\|^p$ . Then  $f$  satisfies the Hölder condition (on the whole  $X$ )

$$f(x_1) - f(x_2) \leq \|x_1 - x_2\|^p. \quad (8)$$

In fact, one can choose  $z_2 \in A$  such that  $f(x_2) \geq \|x_2 - z_2\|^p - \varepsilon$ . Of course  $f(x_1) \leq \|x_1 - z_2\|^p$ . We have therefore

$$f(x_1) - f(x_2) \leq \|x_1 - z_2\|^p - \|x_2 - z_2\|^p + \varepsilon \leq \|x_1 - x_2\|^p + \varepsilon$$

which implies (8) by the arbitrariness of  $\varepsilon > 0$ .

We shall now verify that the functional  $\hat{\varphi}(\cdot) = d_p(\cdot, CA)$  defined on  $Y$  is an observation optimal of degree  $p$  for  $f$ . The restriction of all observations for  $f$  to  $CX = C_{op}$  is

$$\varphi(\{x/n^p\}) = f(\{x_n\}) = \inf_{\{z_n\} \in A} \sup_n n^p |x_n/n^p - z_n/n^p| = d_p(Cx, CA).$$

$\varphi$  satisfies (8) on  $CX$  and hence  $\varphi$  is its extension preserving the Hölder condition (8).

One easily sees that  $\|f\|_p = \sup_{\|x\| \leq 1} (f(0^p)(x) < +\infty$  is really a norm and the space of all functionals Hölder continuous with exponent  $p$ :  $0 < p \leq 1$  ( $f(0) = 0$ ) is a Banach space with norm  $\|f\|_p$ , which is denoted by  $X^{(p)}$ . Let  $C: X \rightarrow Y$  be linear and bounded. Let  $C^{(p)}: Y^{(p)} \rightarrow X^{(p)}$  be defined by  $\varphi(Cx) = (C^{(p)}\varphi)(x)$ ,  $x \in X$ ,  $\varphi \in Y^{(p)}$ . Then  $C^{(p)}$  is linear and bounded (as trivially checked). It is not hard to show that  $C^{(p)}$  is also a bounded operator of  $Y^{(p)}$  with  $Y$ -topology into  $X^{(p)}$  with  $X$ -topology (compare e.g. [10], Theorem IV.4.2).

Let us verify an analog of the Alaoglu theorem: The unit ball  $S = \{f \in X^{(p)}: \|f\|_p \leq 1\}$  is compact in  $X$ -topology. According to Tichonoff theorem the product  $P = \prod_{x \in X} [-\|x\|^p, \|x\|^p]$  is compact in  $X$ -topology. The fact that  $f(0) = 0$  for all  $f \in X^{(p)}$  implies that  $S \subset P = \{f: X \rightarrow R \mid f(x) \leq \|x\|^p\}$ . On the other hand,  $S$  is closed in  $X$ -topology. Indeed, if  $\{f_n\} \subset S$  and  $f_n \rightarrow f$  in  $X$ -topology, then  $f \in S$  since we have the estimate

$$\begin{aligned} |f(x) - f(z)| &\leq (|f(x) - f_n(x)| + |f_n(x) - f_n(z)| + |f_n(z) - f(z)|) \\ &\leq |f_n(x) - f_n(z)| + 2\varepsilon \leq \|x - z\|^p + x\varepsilon \end{aligned}$$

where  $\varepsilon$  is arbitrarily small. Thus  $S$  is compact.

Now employing a method of Rolewicz [8] we easily get the following formula for the Hölder norm of observations optimal of degree  $p$ .

**THEOREM 4.** If  $f \in X^{(p)}$ ,  $0 < p \leq 1$ , is observable, then

$$\inf \{\|\varphi\|_p: \varphi \in Y^{(p)}, f = C^{(p)}\varphi\} = \sup_{x \in X} \inf \{\|\varphi\|_p: \varphi \in Y^{(p)}, f(x) = \varphi(Cx)\}. \quad (9)$$

**Proof.** Let  $a$  be the left side of (9) and  $b$  be the right one. Clearly  $a \geq b$ . Let  $G_b = C^{(p)}S_b = \{g \in X^{(p)}: g = C^{(p)}\varphi, \|\varphi\|_p \leq b\}$ .  $S_b$  is compact in  $Y$ -topology and  $C^{(p)}$  is a bounded linear operator of  $Y^{(p)}$  with  $Y$ -topology into  $X^{(p)}$  with  $X$ -topology, so  $G_b$  is compact and then closed in  $X$ -topology. Arguing by contradiction we suppose  $a > b$ . Then  $f \notin G_b$ . Therefore, by the Hahn-Banach theorem there is  $x \in X$  such that  $g(x) < \beta - \varepsilon$  for all  $g \in G_b$  and  $f(x) \geq \beta$ . Thus, for all  $\varphi \in S_b$  we have  $\varphi(Cx) = g(x) < f(x) - \varepsilon$ , which contradicts the definition of  $b$ . Finally we obtain  $a = b$ .

Q.E.D.

REMARK 1. In the case of functionals (instead of  $X^{(p)}, Y^{(p)}$  we have  $X^*, Y^*$ ) one can verify the equivalence between (9) and the following classical form of the Krein's problem of moments:

$$\inf \{ \|\varphi\| : \varphi \in Y^*, f = C^* \varphi \} = \frac{1}{\rho}$$

if and only if

$$\inf \{ \|y\| : y = Cx, f(x) = 1 \} = \rho > 0.$$

But the latter is not true in nonlinear cases.

THEOREM 5. Let  $C$  be linear, bounded and  $CX$  be closed. Then from the convergence of functionals observable optimally of degree  $p$   $\{f_n\} \subset X^{(p)} : \|f_n - f\|_p \rightarrow 0$  it follows that there are observations optimal of degree  $p$   $\hat{\phi}_{n_k}$  and  $\hat{\phi}$  (for  $f_{n_k}$  and  $f$  resp.) such that  $\|\hat{\phi}_{n_k}\|_p \rightarrow \|\hat{\phi}\|$  and  $\hat{\phi}_{n_k}(y) \rightarrow \hat{\phi}(y)$  for all  $y \in Y$ .

Proof. Since  $CX$  is closed,  $C^{(p)} Y^{(p)}$  is closed (as easily proved) and then it is a Banach space. Put  $Y_0^{(p)} = Y^{(p)} / \text{Ker } C^{(p)}$ . Then operator  $C^{(p)}$  induces an operator  $C_0^{(p)} : Y_0^{(p)} \rightarrow C^{(p)} Y^{(p)}$  by the formula  $C_0^{(p)}[\varphi] = C^{(p)} \varphi$  for  $[\varphi] \in Y_0^{(p)}$ . In virtue of the Banach theorem on inverse operators  $\|f_n - f\|_p \rightarrow 0$  yields  $\|[\varphi_n] - [\varphi]\|_p \rightarrow 0$ . Clearly optimal (of degree  $p$ ) observations for  $f_n$  are any such  $\hat{\phi}_n \in [\varphi_n]$  that  $\|\hat{\phi}_n\|_p = \|[\varphi_n]\|_p$  (Corollary 1 shows the existence of such  $\hat{\phi}_n$ ). Since  $[\varphi_n]$  converges we have  $\|\hat{\phi}_n\| \leq M$  for all  $n$ . By  $Y$ -compactness of balls in  $Y^{(p)}$  we can extract a subsequence  $\hat{\phi}_{n_k}$  such that  $\hat{\phi}_{n_k}(y) \rightarrow \hat{\phi}(y)$  for all  $y \in Y$ . Observing that  $f_{n_k}(x) = \hat{\phi}_{n_k}(Cx) \rightarrow \hat{\phi}(Cx) = f(x)$  shows that  $\hat{\phi} \in [\varphi]$ . Obviously,  $\|\hat{\phi}\|_p = \|[\varphi]\|_p$ . Thus  $\hat{\phi}$  is an observation optimal of degree  $p$ . Q.E.D.

### 1.c. Observability Optimal of Degree $p$ for Operators

Consider a system (not necessarily linear)  $X \xrightarrow{C} Y$ . Similarly to the functional case, we say that an operator  $F$ , belonging to a class  $R_X$  of operators of  $X$  into a Banach space  $Z$ , is observable if there is  $\Phi \in P_Y$  ( $P_Y$  is defined correspondently to  $P_X$ ) such that the following diagram is commutative

$$\begin{array}{ccc} X & \xrightarrow{C} & Y \\ F \searrow & & \swarrow \Phi \\ & Z & \end{array}$$

On the analogy of (5) we say that  $\hat{\Phi}$  is an observation optimal of degree  $p$  if

$$\sup_{y_1 \neq y_2} \frac{\|\hat{\Phi}(y_2) - \hat{\Phi}(y_1)\|}{\|y_2 - y_1\|^p} = \inf_{\Phi} \sup_{y_1 \neq y_2} \frac{\|\Phi(y_2) - \Phi(y_1)\|}{\|y_2 - y_1\|^p} \quad (10)$$

and the infimum is finite.

Denoting the left side of (10) by  $\|\hat{\Phi}\|_p$  we can easily verify that it is a norm and the space  $(Y, Z)^{(p)}$  of all operators  $\Phi : Y \rightarrow Z$  such that  $\|\Phi\|_p < +\infty$  is a Banach space.

To get an analog of Theorem 3 we need a notion of hyperconvex spaces ([1], [6]). In [6] Nachbin said such spaces to have the binary intersection property).

A norm space  $Z$  is called hyperconvex (or a  $P_1$ -space) if every collection of its closed balls, any two members of which intersect, has a nonvoid intersection. Aronszajn and Panitchpakdi have proved the

**THEOREM ([1]).** Let  $Z$  be a normed space. In order that any operator  $\Phi$  of any normed space  $Y_1$  into  $Z$  with some subadditive modulus of continuity  $\delta(\varepsilon)$  possess, for any normed space  $Y$  containing  $Y_1$  as a subspace, an extension to the whole of  $Y$  with the same modulus  $\delta(\varepsilon)$ , it is necessary and sufficient that  $Z$  be hyperconvex.

Paying attention that for operators  $\Phi$  satisfying the Hölder condition with exponent  $0 < p \leq 1$  the minimal modulus of continuity  $\delta(\varepsilon) = \|\Phi\|_p \varepsilon^p$  is subadditive we obtain

**THEOREM 3'.** Let  $Z$  be hyperconvex. Let  $C$  satisfy the condition in Theorem 3. Then each operator  $F: X \rightarrow Z$  such that  $F$  is well-defined on  $X/\mathcal{R}$  and satisfies the Hölder condition with exponent  $p$  is observable optimally of degree  $pq$ .

Optimal (of degree  $p$ ) observation  $\hat{\Phi}$  obtained by Theorem 3' has the property that  $\|\hat{\Phi}\|_p = \|\hat{\Phi}|_{CX}\|_p$ , i.e.  $\hat{\Phi}$  is "absolutely optimal". However, the hyperconvexity is a very strong condition. For instance, if  $Z$  is finite dimensional, then  $Z$  is hyperconvex if and only if  $Z$  has a norm of type "sup". An optimality "nonabsolute", i.e.  $\|\hat{\Phi}\|_p > \|\hat{\Phi}|_{CX}\|_p$  can be obtained in much more cases, e.g. by Theorem 6 below. Using a method similar to [9] (based on an extension of the Alaoglu theorem) we trivially get.

**THEOREM 6.** Let  $C$  be nonlinear. Let  $P_X = (X, Z)^{(p)}$  and  $P_Y = (Y, Z)^{(q)}$ . If there is a separating topology  $\tau$  in  $Z$  such that the unit ball is  $\tau$ -compact, then each observable operator  $F \in P_X$  is observable optimally of degree  $q$ .

Clearly any reflexive space  $Z$  or any space  $Z$  conjugate to a Banach space satisfies the condition in Theorem 6.

This theorem shows that the optimal observation problem is broader than the problem of the extension without increasing the norm. In the case of functionals they are equivalent to each other.

## 2. Optimal Observability for Convex Functionals

In this section we shall restrict ourselves to discussions about a linear system  $X \xrightarrow{C} Y$ . The class  $P_X(P_Y)$  contains all closed proper convex functionals defined on  $X$  (on  $Y$ ). By Theorem 2, convex functionals are never observable optimally of degree  $p < 1$  (unless they are constants) so we may examine only the ordinary optimality (of degree 1). A glance at (6) shows that being Lipschitzian is a necessary condition for optimal observability.

**PROPOSITION 1.** A closed proper convex functional  $f$  on  $X$  ( $f \in P_X$ ) is observable if and only if there exists a collection of observable affine continuous functionals  $\{h_\tau: \tau \in T\}$  such that  $f(x) = \sup_{\tau} h_\tau(x)$  for all  $x \in X$ .

Proof. Necessary: Let  $\varphi$  be an observation for  $f$ . Since  $\varphi$  is closed and convex, there is a collection of affine continuous functionals  $g_\tau$  such that  $\varphi = \sup g_\tau$ . Let  $h_\tau$  be defined on  $X$  by  $h_\tau(x) = g_\tau(Cx)$ . Then  $f = \sup h_\tau$ . We have  $h_\tau(x) = g_\tau(Cx) = g_\tau^0(Cx) + \alpha_\tau \stackrel{\text{df}}{=} h_\tau^0(x) + \alpha_\tau$ , where  $g_\tau^0$  and then  $h_\tau^0$  are linear.  $h_\tau$  are continuous, inasmuch as  $|h_\tau^0(x)| = |g_\tau^0(Cx)| \leq \|g_\tau^0\| \|C\| \|x\|$ . Finally,  $h_\tau$  are observable (as  $g_\tau$  are observations).

Sufficiency: If  $g_\tau$  are observations for  $h_\tau$ , then  $\varphi = \sup g_\tau$  is an observation for  $f$ .  
Q.E.D.

THEOREM 7. If  $CX$  is closed, than:

(a)  $f \in P_X$  is observable if and it vanishes on  $\text{Ker } C$ .

(b)  $f \in P_X$  is optimally observable if and only if it vanishes on  $\text{Ker } C$  and is Lipschitzian.

Proof.

(a) The "only if" is clear. To show the "if" note that for every expression  $f = \sup h_\tau$  we have  $h_\tau|_{\text{Ker } C} \leq f|_{\text{Ker } C} = 0$ , i.e.  $h_\tau|_{\text{Ker } C}$  is bounded above (on  $\text{Ker } C$ ), and hence it is equal to a constant, say  $\beta_\tau$ . Therefore  $h_\tau$  has the form  $h_\tau = h_\tau^0 + \beta_\tau$ , where  $h_\tau^0$  is linear and then vanishes on  $\text{Ker } C$ . By the closeness of  $CX$ ,  $h_\tau^0$  is observable, which means that  $h_\tau$  is observable.

(b) We prove the "if". Let  $\varphi$  be the (unique) restriction to  $CX$  of all observations for  $f$ . Similarly to Corollary 1 we see that  $\varphi$  is Lipschitzian. Let  $\varphi = \sup g_\tau$ , where  $g_\tau$  are affine continuous functionals. Then  $\varphi 0^+ = \sup g_\tau 0^+$  and hence

$$\|\varphi\|_1 = \sup_{\|y\| \leq 1} (\varphi 0^+)(y) = \sup_{\tau} \sup_{\|y\| \leq 1} (g_\tau 0^+)(y) = \sup_{\tau} \|g_\tau\|.$$

Using Hahn-Banach theorem we extend  $g_\tau$  to  $\tilde{g}_\tau$  defined on the whole  $Y$  preserving the norms. Clearly,  $\tilde{\varphi} = \sup \tilde{g}_\tau$  is an optimal observation for  $f$ .  
Q.E.D.

The above theorem shows an extension with conservation of the Lipschitzian convex functional norm and convexity. Indeed, let  $f$  be a Lipschitzian convex functional on a linear subspace  $X_1$  (not necessarily closed) of a Banach space  $X$ . To extend  $f$  preserving the mentioned conditions, first extend  $f$  preserving the (uniform) continuity to Banach space  $\bar{X}_1$  and next optimally observe  $f$  in linear system  $\bar{X}_1 \xrightarrow{i} X$ , where  $i$  is the identity into.

Example 2. Let  $X, Y, C, f$  and  $A$  be as in Example 1 with  $p=1$ . Suppose furthermore, that  $A$  is a convex subset. Then it is easy to see that  $f(x) = d_1(x, A)$  is convex. We obtain finally an optimal observation  $\hat{\varphi}(\cdot) = d_1(\cdot, CA)$  defined on  $Y$ .

Theorems 4 and 5 are still in force (the argument with the convexity of functionals is simple). Formula (9) now has the form

$$\inf \{ \|\varphi\|_1 : \varphi \in Y^o, f = C^{(1)} \varphi \} = \sup_{x \in X} \inf \{ \|\varphi\|_1 : \varphi \in Y^o, f(x) = \varphi(Cx) \} \quad (9')$$

where  $Y^o \subset Y^{(1)}$  is the set of all Lipschitzian convex functionals on  $Y$ .

PROPOSITION 2. If  $f$  is observable, then there is an observation  $\varphi$  such that  $f^*$  is an observation for  $\varphi^*$  in the conjugate system  $X^* \xleftarrow{C^*} Y^*$ . However, the optimal

observability of  $f$  implies that for each observation  $\varphi$ ,  $\varphi^*$  is not optimally observable and that  $f^*$  is not optimally observable in any linear system.

Proof. One can take an observation  $\varphi$  for  $f$  such that

$$\varphi^*(y^*) = \sup_{y_0 \in \text{dom } \varphi} \{ \langle y_0, y^* \rangle - \varphi(y_0) \} = \sup_{y_0 \in CX \cap \text{dom } \varphi} \{ \langle y_0, y^* \rangle - \varphi(y_0) \}$$

(for example,  $\varphi(\cdot) = \text{cl} [\tilde{\varphi}(\cdot) + \delta(\cdot, CX)]$ , where  $\tilde{\varphi}$  is any observation for  $f$  and  $\delta$  is the indicator functional). Hence, the first assertion is proved via the equality

$$\varphi^*(y^*) = \sup_{x \in \text{dom } f} \{ \langle x, C^* y^* \rangle - f(x) \} = f^*(C^* y^*).$$

We have  $(f^* 0^+)(x^*) = \sup_{x \in \text{dom } f} \langle x, x^* \rangle$  ([7], Theorem 13.3), and so

$$\sup_{\|x^*\| \leq 1} (f^* 0^+)(x^*) = \sup_{\substack{\|x^*\| \leq 1 \\ x \in \text{dom } f}} \langle x, x^* \rangle.$$

By the Banach-Steinhaus theorem the right side is finite, and then  $f^*$  is Lipschitzian, if and only if  $\text{dom } f$  is bounded. If  $f$  is optimally observable, then it is Lipschitzian and  $\text{dom } f = X$ . Hence  $f^*$  is not Lipschitzian. At the same time, for each observation  $\varphi$  we have  $\text{dom } \varphi \supset CX$ . Thus  $\varphi^*$  is also not Lipschitzian. Q.E.D.

THEOREM 8. Let either  $X$  or  $Y$  be finite dimensional. Let a sequence of observable Lipschitzian differentiable (Fréchet) convex functionals  $f_n$  satisfy the following conditions

(a)  $f_n(x) \rightarrow f(x)$  for all  $x \in X$  and  $f$  is differentiable (Fréchet).

(1)  $\lim_{\substack{n \rightarrow \infty \\ \lambda \rightarrow \infty}} \frac{1}{\lambda} f_n(\lambda x)$  exists and is finite for all  $x \in X$ .

(c) From  $x \in \text{Ker } C$  and  $x \neq 0$  it follows  $(f_n 0^+)(x) > 0$ .

Then there are optimal observations  $\tilde{\varphi}_{n_k}$  and  $\tilde{\varphi}$  (for  $f_{n_k}$  and  $f$  resp.) such that  $\|\tilde{\varphi}_{n_k}\|_1 \rightarrow \|\tilde{\varphi}\|_1$  and  $\tilde{\varphi}_{n_k}(y) \rightarrow \tilde{\varphi}(y)$  for all  $y \in Y$ .

The proof begins with the following lemma:

LEMMA 1. If  $f_n(x)$  are Lipschitzian convex functionals (not necessarily differentiable) satisfying conditions (a) and (b), then  $f(x)$  is also Lipschitzian and convex.

Proof. We have

$$L \stackrel{\text{df}}{=} \lim_{\substack{n \rightarrow \infty \\ \lambda \rightarrow \infty}} \frac{1}{\lambda} f_n(\lambda x) = \lim_{n \rightarrow \infty} \lim_{\lambda \rightarrow \infty} \frac{1}{\lambda} f_n(\lambda x) = \lim_{n \rightarrow \infty} (f_n 0^+)(x).$$

On the other hand

$$L = \lim_{\lambda \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{\lambda} f_n(\lambda x) = (f 0^+)(x).$$

Consequently for all  $x \in X$  we have  $(f_n 0^+)(x) \rightarrow (f 0^+)(x)$  and hence  $\sup_n (f_n 0^+)(x) \leq \alpha_x < +\infty$ . Paying attention to the argument just before Theorem 4 we see that the proof will be complete if the following analogue of the Banach-Steinhaus theorem holds: If  $f_i, i \in I$ , are Lipschitzian convex functionals such that  $\sup_i (f_i 0^+)(x) \leq \alpha_x < +\infty$ , then  $\sup_i \|f_i\|_1 \leq \beta < +\infty$ . But this can be verified by the same argument as in the classical proof of the mentioned theorem employing the Baire category theorem. Q.E.D.

**Proof of Theorem 8.** We see via the lemma that  $f$  is optimally observable. Assumption (c) implies that the restrictions  $\varphi_n$  and  $\varphi$  of all observations for  $f_n$  and  $f$  resp. to  $CX$  are (Lipschitzian, convex) differentiable on  $CX$  ([7], p. 255). Then for each  $y_0 \in CX$  the gradient  $\nabla \varphi_n(y_0)$  is the unique subgradient of  $\varphi_n$  at  $y_0$ . Putting  $g_{y_0}^n(y) = \langle \nabla \varphi_n(y_0), y - y_0 \rangle + \varphi_n(y_0)$  we obtain the affine functionals  $g_{y_0}^n(y)$  such that  $\varphi_n(y) = \sup_{y_0 \in CX} g_{y_0}^n(y)$ . Since  $CX$  is finite dimensional,  $\nabla \varphi_n(y_0) \rightarrow \nabla \varphi(y_0)$  for all  $y_0 \in CX$  ([7], Theorem 25.7) and then  $g_{y_0}^n(y) \rightarrow g_{y_0}(y) = \langle \nabla \varphi(y_0), y - y_0 \rangle + \varphi(y_0)$  for all  $y \in CX$ . Hence  $\|g_{y_0}^n\| \rightarrow \|g_{y_0}\|$ . We extend  $g_{y_0}^n$  and  $g_{y_0}$  to  $\hat{g}_{y_0}^n$  and  $\hat{g}_{y_0}$  defined on  $Y$  without increasing the norms. Putting  $\hat{\varphi}_n(y) = \sup_{y_0 \in CX} \hat{g}_{y_0}^n(y)$  and  $\hat{\varphi}(y) = \sup_{y_0 \in CX} \hat{g}_{y_0}(y)$  we have  $\|\hat{\varphi}_n\|_1 = \sup_{y_0 \in CX} \|\hat{g}_{y_0}^n\| \rightarrow \|\hat{\varphi}\|_1 = \sup_{y_0 \in CX} \|\hat{g}_{y_0}\|$ . By  $Y$ -compactness of balls in  $Y^{(1)}$  we can select a converging subsequence  $\hat{\varphi}_{n_k}(y) \rightarrow \tilde{\varphi}(y)$  for all  $y \in Y$ . Since  $\tilde{\varphi} = \hat{\varphi}$  on  $CX$  and  $\|\tilde{\varphi}\|_1 = \|\hat{\varphi}\|_1$ ,  $\tilde{\varphi}$  is an optimal observation for  $f$ . Thus  $\tilde{\varphi}_{n_k} = \hat{\varphi}_{n_k}$  and  $\tilde{\varphi}$  are required optimal observations. Q.E.D.

**REMARK 2** We have studied cases when  $Y$  is a Banach space. In practice these cases happen when the accuracy of the measurement of the output  $y$  guaranties the restriction  $\|\Delta y\| \leq \delta$ , where  $\Delta y$  is the error of the measurement and  $\|\cdot\|$  is the norm in  $Y$ . In other words, we have  $\Delta y \in U$ , where  $U$  is a ball in  $Y$ . If  $U$  is not a ball but it is closed, convex, symmetrical and containing 0 in the interior, then the Minkowski functional of  $U: m(y) = \inf \{r > 0: y/r \in U\}$  gives a norm and we meet the same situation. Furthermore, if  $U$  is inbounded but closed, convex, symmetrical and containing 0 in the interior, then the Minkowski functional yields a pseudonorm. In this case by method similar to that presented here, we can obtain nearly all results in this note. Unfortunately, the optimal observation problem becomes much more complicated when  $U$  is unbounded and nonsymmetrical (and convex, closed, containing 0 in the interior), because  $m(y)$  is only a pseudohalfnorm, which does not define a linear topology on  $Y$ .

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## Pojęcie optymalności stopnia $p$ w teorii obserwacji i obserwacja optymalna dla funkcjonalów wypukłych

Wprowadzono pojęcie obserwowalności optymalnej stopnia  $p$  i rozpatrzono je dla funkcjonalów i operatorów ciągłych w sensie Höldera w liniowych i nieliniowych układach nieskończenie wymiarowych. Podano także pewne wyniki dotyczące obserwowalności optymalnej dla funkcjonalów wypukłych w układach liniowych.

## Понятие оптимальности степени $p$ в теории наблюдений и оптимальное наблюдение для выпуклых функционалов

Вводится понятие оптимальной наблюдаемости степени  $p$ , которое рассматривается для функционалов и операторов, непрерывных в смысле Гельдера, в линейных и нелинейных бесконечномерных системах.

Даются также некоторые результаты, касающиеся оптимальной наблюдаемости для выпуклых функционалов в линейных системах.

