Control
and Cybernetics
VOL 8 (1979) No. 2

Euler-Lagrange's Conditions For Controls With Bounded Variation

by

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The paper is concerned with the following optimization problem: find the minimum value of the functional $\int_0^1 \Phi(x, u, t)\, dt$ under conditions $\dot{x}=\varphi(x, u, t)$, $x(0)=c$, $x(1)=d$, where u is a function of bounded variation and x is an absolutely continuous function. Under certain assumptions concerning functions φ and Φ the existence of an optimal control has been shown and necessary conditions for the existence have been given in a form that is similar to that of Euler-Lagrange's equations. The final part of the paper presents a physical interpretation of the optimization problem studied.

1. Introduction

One of the basic problems of the optimization theory is the following:

Problem A. Find the minimal value of the functional

$$I(x, u)=\int_0^1 \Phi(x, u, t)\, dt \tag{1}$$

under the conditions:

$$\dot{x}=\varphi(x, u, t),\ x(0)=c,\ x(1)=d,\ u(t) \in P, \tag{2}$$

where x is an absolutely continuous function, $x(t) \in R^n$, u is an essentially bounded function, P is a convex subset of R^r with non-empty interior.

The problems thus formulated can be solved by means of the Pontrygin's maximum principle [18] or the local maximum principle (cf. [6–8]). Problem A can be regarded as one of the basic problems in the optimization theory since, as it turns out, many other more difficult problems can be reduced to Problem A (cf. [6–8, 10]).

Another question which is important from the mathematical point of view is that of the existence of a solution of Problem A.

Many difficulties which appear in this case are due to difficulties in introducing a topology in the set of controls and in specifying (natural and general, if possible) conditions for the function Φ under which the functional (1) would be lower semicontinuous, and the set of controls (or at least, a minimizing sequence) — compact.

As is known, works by Hilbert, Tonelli and McShane are regarded the first and fundamental ones in this field.

Recently their ideas and results have been generalized by Berkovitz [1], Brunovski [2], Cesari [3, 4], Olech [14–16], Polyak [17], Rockafellar [19] and others.

In the paper [16] there has been proved for the first time, after many trials and partial results obtained by some other authors, a necessary and sufficient condition for the lower semicontinuity of the integral functional.

The basic question considered in the present paper differs from Problem A in stronger conditions imposed on the set of controls. Namely, we shall assume additionally that these controls possess a bounded variation, i.e. $\bigvee_0^1 u \leqslant k$, where k is a positive constant.

The investigation of the optimization problems of this type is justified by the following reasons:

(a) under simple assumptions one can prove the existence of an optimal solution;

(b) one can give a necessary condition for the existence, approximate in its form to Euler-Lagrange's equation, which in many cases allows one to determine an optimal control in a direct way;

(c) the problem considered has a natural physical interpretation as well as a geometrical one;

(d) in the limit case, i.e. $k=\infty$, one can obtain new existence theorems for a large class of variational problems, generally non-convex (cf. [20]);

(e) in practice we always use controls with finite variation.

A necessary condition for the existence of the extremum for scalar controls with bounded variation (without constraints on the values of the control) was proved in the papers [11] and [5]. In those papers a number of technical applications of controls with bounded variation were also given.

The present paper is composed of three parts. In the first part (Section2) the optimization problem mentioned above is formulated precisely and the existence of its solutions is shown. The second part (Section 3) contains the proof of a necessary condition for the existence of a solution and the analysis of some of its special cases. In the third part (Section 4) a physical interpretation of the problem under consideration is given.

2. Problem Formulation. Existence Theorem

Let $u=(u^1, u^2, \ldots, u^r)$ be a vector function defined on the interval [0, 1], with values from the space R^r. This function will be said to have a finite variation (oscillation) if each its component is a function of finite variation. Let us recall (cf. e.g.

[13]) that by a variation of a scalar function g on the interval $[a, b]$ we mean the upper bound of the sums $\sum_{i=0}^{n-1} |g(t_{i+1})-g(t_i)|$ over all possible subdivisions of the interval $[a, b]$ by the points $a=t_0<t_1<\ldots<t_n=b$. A variation of the scalar function g will be denoted by $\bigvee_a^b g$.

By a variation of the vector function u we shall mean the vector $\nabla_0^1 u=(\bigvee_0^1 u^1, \ldots \ldots, \bigvee_0^1 u^r)$.

Let φ be a vector function defined on the space R^{n+r+1}, with values in the space R^n, and Φ — a scalar function defined on R^{n+r+1}.

Let P stand for a compact set in R^r. Let ϑ be a compact subset of a positive cone in the space R^r. We shall assume that the set ϑ is normal with respect to each of the axes of the coordinates system, i.e. that for any $\lambda \in [0, 1]$ and $i=1, 2, \ldots, r$, if a point $(v^1, \ldots, v^i, \ldots, v^r)$ is in the set ϑ then the point $(v^1, \ldots, \lambda v^i, \ldots, v^r)$ is also in ϑ. It is easy to see that the set will be normal if and only if for an arbitrary point $a=(a^1, \ldots, a^r)$ in ϑ a perpendicular parallelepiped $Q=\{x \in R^r; 0 \leqslant x^i \leqslant a^i,\ i=1, \ldots, r\}$, is also contained in ϑ. Let U denote the set of vector functions $u=(u^1, \ldots, u^r)$ of bounded oscillation, such that $u(t) \in P$, $\nabla_0^1 u \in \vartheta$ and a fixed number k (e.g. the first k) $(0 \leqslant k \leqslant r)$ of the components of u are monotone functions.

U will be called a set of admissible controls, and its elements — admissible controls. (The case of $k=0$ indicates that we do not impose the condition of monotonicity upon any of the components).

Consider the following

Problem 1. Find the minimal value of the functional

$$I(x, u)=\int_0^1 \Phi\big(x(t), u(t), t\big)\, dt \tag{3}$$

under the conditions

$$\dot{x}(t)=\varphi\big(x(t), u(t), t\big), \tag{4}$$

$$x(0)=c, \tag{5}$$

$$x(1)=d, \tag{6}$$

$$u \in U, \tag{7}$$

where x is an absolutely continuous function on $[0, 1]$, $x(t) \in R^n$, c and d are fixed points in R^n.

The space of absolutely continuous functions on $[0, 1]$ with norm $\|x\|=\max|x(t)|$ is denoted by $C^n(0, 1)$.

As for the function Φ and φ, we shall assume that

$$\left.\begin{array}{l}\Phi \text{ satisfies Caratheodory's condition (i.e. is continuous}\\ \text{with respect to } (x, u) \text{ and measuravle with respect to t)}\\ \text{and there exists a function } \alpha\in L_1 \text{ such that}\\ \qquad\qquad \Phi\big(x(t), u(t), t\big)\geqslant\alpha(t)\\ \text{for any } (x, u) \text{ satisfying conditions (4)–(7) and for almost}\\ \text{all } t\in[0, 1];\end{array}\right\}\tag{8}$$

$$\left.\begin{array}{l}\text{there exists an integrable function } \psi \text{ such that}\\ |\varphi(x(t), u(t), t)|\leqslant\psi(t) \text{ for any pair } (x, u) \text{ satisfying}\\ \text{condition (4)–(7).}\end{array}\right\}\tag{9}$$

Let U_0 denote a subset of the set U composed of those elements u for which there exists a solution of equation (4) under conditions (5) and (6).

Now we shall prove the theorem on the existence of a solution of Problem 1. The proof will be preceded by

LEMMA 1. The set of admissible controls U is sequentially compact in the topology of point-wise convergence.

Proof. Let g_n be a sequence of scalar functions, convergent to g_0 and such that $\bigvee\limits_0^1 g_n\leqslant k$, where k is some constant. Note that $\bigvee\limits_0^1 g_0$ is less than k, too. Indeed, let $0=t_0<t_1<\ldots<t_n=1$ be an arbitrary dissection of the interval $[0, 1]$. By the definition of variation of the function g_n we have

$$\sum_{i=1}^{n-1}|g_n(t_{i+1})-g_n(t_i)|\leqslant\bigvee_0^1 g_n\leqslant k.$$

Since the sequence g_n converges to g_0, we obtain

$$\sum_{i=1}^{n-1}|g_0(t_{i+1})-g_0(t_i)|\leqslant k.$$

Owing to the fact that the sequence $\{t_i\}$ was arbitrary, we obtain $\bigvee\limits_0^1 g_0\leqslant k$. Let $\{u_n\}$ be a certain sequence of the set U. Since $\nabla_0^1 u_n\in\vartheta$, and ϑ is a compact set, we may select a subsequence $\{u_{n_k}\}$ such that $\nabla_0^1 u_{n_k}\to a$ and $a\in\vartheta$ $\big(a=(a^1, a^2, \ldots, a^r)\big)$.

Subsequences will further be denoted by the same symbols as initial sequences. Let ε be any positive number. From Helly's principle (cf. e.g. [13]) and from the argument quoted at the beginning of the proof it follows that there exists a subsequence such that $u_n\to u_0$ and $\bigvee\limits_0^1 u_0^i\leqslant a^i+\varepsilon$, $i=1, 2, \ldots, r$. As ε is an arbitrary positive number, the set ϑ is normal with respect to each of the coordinates of the system and P is a compact set, we deduce that $u_0 \in U$. We shall prove the following:

THEOREM 1 (on the existence of a solution). If the functions Φ and φ satisfy conditions (8) and (9), and the set U_0 is non-empty, then there exist a solution of Problem 1.

Proof. Denote by m the lower bound of the functional I for $u \in U_0$. From condition (8) it follows that $m > -\infty$. Let $\{x_n, u_n\}$, $u_n \in U_0$, be a sequence minimizing the functional I, that is, $\lim I(x_n u_n) = m$. In the further part of the proof we shall denote subsequences by the same symbols as initial sequences.

From Lemma 1 it follows that out of the sequence $\{u_n\}$ one may extract a subsequence convergent to $u_0 \in U$. From condition (9) it follows that the sequence $\{\dot{x}_n\}$ is uniformly integrable, so the sequence $\{x_n\}$ is a family of uniformly continuous and commonly bounded functions (cf. e.g. [10] § 9.1.2). Consequently, there exists a subsequence uniformly convergent to a certain function x_0. We have shown that out of the minimizing sequence one may extract a subsequence $\{x_n, u_n\}$ such that x_n converges uniformly to some continuous function x_0, while u_n is convergent to $u_0 \in U$ in the topology of point-wise convergence.

From condition (4)–(6) is follows that

$$x_n(t) = c + \int_0^t \varphi(x_n(\tau), u_n(\tau), \tau)\, d\tau, \; x_n(1) = d.$$

As $n \to \infty$, we get

$$x_0(t) = c + \int_0^t \varphi(x_0(\tau), u_0(\tau), \tau)\, d\tau, \; x_0(1) = d.$$

Consequently, we have proved that $u_0 \in U_0$. From assumption (8) and Fatou's lemma it follows that the functional I is lower semicontinuous, thus $\underline{\lim}\, I(x_n, u_n) \geqslant I(x_0, u_0) \geqslant m$. Since $\underline{\lim}\, I(x_n, u_n) = m$, therefore $I(x_0, u_0) = m$.

3. Necessary Conditions For Optimality

Now we are proceeding to the proof of a necessary condition for the existence of an optimal control in Problem 1. This condition has a form similar to Euler-Lagrange's equations known in variational calculus.

In the sequel we shall assume that

P is a rectangular parallelepiped in R^r, and ϑ is a convex set satisfying the conditions given in Section 2. (10)

The functions φ and Ω do not depend explicitly on t, are continuous together with their derivatives φ_x, φ_u, Φ_x, Φ_u with respect to the variables (x, u), and φ_u and Φ_u satisfy Lipschitz's condition with respect to (x, u), i.e. (11)

$$|\varphi_u(x_1, u_1) - \varphi_u(x_2, u_2)| \leqslant L(|x_1 - x_2| + |u_1 - u_2|)$$

and

$$|\Phi_u(x_1 u_1) - \Phi_u(x_2 u_2)| \leqslant L'(|x_1 - x_2| + |u_1 - u_2|)$$

for some $L > 0$.

First of all we shall prove a few lemmas.

Let g_0 be any scalar function such that $g_0(t) \in [\alpha, \beta] \subset R$ and $\bigvee_0^1 g_0 \leqslant k$. Let μ stand for either of the numbers: m, M, where m, M denote respectively the lower and the upper bound of the function g_0 on the interval $(t_1, t_2) \subset [0, 1]$. We shall prove the following

LEMMA 2. The function g_1 defined by the formula

$$g_1(t) = \begin{cases} g_0(t) & \text{for } t \notin (t_1, t_2) \\ \mu & \text{for } t \in (t_1, t_2) \end{cases} \tag{12}$$

satisfies the conditions: $g_1(t) \in [\alpha, \beta]$ for $t \in [0, 1]$ and $\bigvee_0^1 g_1(t) \leqslant k$. Besides, if g^0 is a monotone function, then g_1 is monotone, too.

Proof. Only the inequality $\bigvee_0^1 g_1 \leqslant k$ needs to be proved. We have

$$\bigvee_0^1 g_1 = \bigvee_0^{t_1} g_1 + \bigvee_{t_1}^{t_2} g_1 + \bigvee_{t_2}^1 g_1 = \bigvee_0^{t_1} g_0 + \bigvee_{t_1}^{t_2} g_1 + \bigvee_{t_2}^1 g_0 . \tag{13}$$

Let $t_1 = \tau_0 < \tau_1 < \ldots < \tau_n = t_2$ be an arbitrary subdivision of the interval $[t_1, t_2]$. From the definition of the function g_1 we obtain

$$\bigvee_{t_1}^{t_2} g_1 = \sup \sum_{i=1}^{n} |g_i(\tau_i) - g_1(\tau_{i-1})| = |\mu - g_0(t_1)| + |g_0(t_2) - \mu| .$$

Let $\{v_k\} \subset (t_1, t_2)$ be any sequence such that $g_0(v_k) \to \mu$. We have

$$|g_0(v_k) - g_0(t_1)| + |g_0(t_2) - g_0(v_k)| \leqslant \bigvee_{t_1}^{t_2} g_0 .$$

As $k \to \infty$, we get

$$|\mu - g_0(t_1)| + |g_0(t_2) - \mu| \leqslant \bigvee_{t_1}^{t_2} g_0 .$$

Returning to equality (13), we obtain the inequality

$$\bigvee_0^1 g_1 \leqslant \bigvee_0^1 g_0 \leqslant k ,$$

which completes the proof of the Lemma.

LEMMA 3. If an integral functional

$$f(g) = \int_0^1 l(t)\, g(t)\, dt$$

is a functional supporting the set $Q = \{g \in L_\infty;\, g(t) \in [\alpha, \beta] \text{ and } \bigvee_0^1 g \leqslant k\}$ at a point $g_0 \in Q$, and $l(t) > 0$ or $l(t) < 0$ on some interval $(t_1, t_2) \subset [0, 1]$, then there exists a constant c_0 such that $g_0(t) \equiv c_0$ for $t \in (t_1, t_2)$.

Proof. By the definition of a supporting functional we have

$$\int_0^1 l(t)\,g(t)\,dt \geqslant \int_0^1 l(t)\,g_0(t)\,dt\,.$$

Let us put $g=g_1$, where g_1 is a function defined by formula (12). After simple transformations we obtain

$$f(g_1)-f(g_0)=\int_{t_1}^{t_2} l(t)\,(\mu-g_0(t))\,dt\geqslant 0\,.$$

Suppose that $l(t)>0$ for $t\in(t_1,t_2)$ and set $\mu=\inf g_0(t)=c_0$, $t\in(t_1,t_2)$. If $g_0(t)\neq c_0$ on a set of a positive measure, then

$$f(g_1)-f(g_0)<0\,.$$

This contradicts the assumption that f is a functional supporting the set Q. In the case when $l(t)<0$ on the interval (t_1,t_2), it is necessary to adopt $\sup g_0(t)$ instead of c_0. Thus we have obtained the proposition of Lemma 7.

The perpendicular parallelepiped P can be represented in the form

$$P=\{u\in R^r;\ u^i\in[\alpha_i,\beta_i],\ i=1,2,\ldots,r\}$$

where α_i, β_i are some constants.

Let u_0 be a certain admissible control. Denote $\overset{1}{\underset{0}{\nabla}}\, u_0=(k_0^1, k_0^2, \ldots, k_0^r)$. We shall prove the following

Lemma 4. If an integral functional

$$f(u)=\int_0^1 a(t)\,u(t)\,dt=\sum_{i=1}^{r}\int_0^1 a^i(t)\,u^i(t)\,dt$$

is a functional supporting the set U at a point u_0, then each of the functionals

$$f_i(u)=\int_0^1 a^i(t)\,u^i(t)\,dt$$

is a functional supporting the set

$$U_i=\{u^i\in L_\infty;\ u^i(t)\in[\alpha_i,\beta_i]\ \text{and}\ \overset{1}{\underset{0}{\bigvee}}\ u^i\leqslant k_0^i\}$$

at a point u_0^i.

Proof. Suppose that for some k the inequality

$$\int_0^1 a^k(t)\,u^k(t)\,dt<\int_0^1 a^k(t)\,u_0^k(t)\,dt$$

holds, where u^k is an element of the set U_k.

Let $\tilde u=(u_1^o, \ldots, u_{k+1}^o, u_k, u_{k+1}^o, \ldots, u_r^o)$. It is easy to see that $\tilde u$ is an admissible control ($\tilde u\in U$), and $f(\tilde u)<f(u_0)$. This contradicts the assumption.

LEMMA 5. If a function g: $[0, 1]\to R$ has a bounded oscillation, then the interval $[0, 1]$ can be represented in the form

$$[0, 1]=N\cup E_0\cup\bigcup_{n=1}^{\infty} I_n,$$

where N is some denumerable set, $E_0=\{t\in[0, 1]; g(t)=0\}$, I_n, $n=1, 2, \ldots$, are open intervals on which $g(t)>0$ or $g(t)<0$.

Proof. Let us define the following sets N, C_+, C_-:

N — a set of points of discontinuity of the function g, to which we also add the points $t=0$ and $t=1$,

C_+ — a set of points of continuity for which $g(t)>0$,

C_- — a set of points of continuity for which $g(t)<0$.

The set N is known to be denumerable. Let $t_0\in C_+$. Then there exists some neighbourhood $0_{t_0}\subset(0, 1)$ of a point t_0, in which $g(t)>0$. Denote by E_+ a set

$$E_+=\bigcup_{t_0\in C_+} 0_{t_0}.$$

It is easy to see that E_+ is an open set, $E_+\subset(0, 1)$, $C_+\subset E_+$, and $g(t)>0$ for $t\in E_+$.

The set E_+ being open and linear can be represented in the form

$$E_+=\bigcup_{n=1}^{\infty} I_n^+,$$

where I_n^+ are open intervals. Assuming that $t_0\in C_-$, we can construct, in a similar manner, a set

$$E_-=\bigcup_{n=1}^{\infty} I_n^-.$$

Since $[0, 1]=\mathrm{N}\cup E_0\cup E_+\cup E_-$, we obtain the proposition of the Lemma. The right side of the equality $[0, 1]=\mathrm{N}\cup E_0\cup\bigcup_{n=1}^{\infty}\cup I_n$ will be called a distribution of the interval $[0, 1]$, corresponding to the function g.

Let u_0 be an optimal control and x_0 an optimal trajectory in Problem 1. Directly from assumption (11) there follows

LEMMA 6. The function $a^k(t)=\big(\varphi_u^*(x_0(t), u_0(t))\,\psi(t)-\lambda_0\,\Phi(x_0(t), u_0(t))\big)^k$, $k=1, 2, \ldots, r$, has a bounded oscillation for any absolutely continuous function ψ. Let $[0, 1]=N^k\cup E_0^k\cup\bigcup_{n=1}^{\infty} I_n^k$ be a distribution corresponding to the function a^k.

Denote by W_{11}^n a space of absolutely continuous functions with norm $\|x\|=|x(0)|+\int_0^1|\dot{x}(t)|\,dt$. Now we shall prove

THEOREM 2. If

1. (x_0, u_0) is a solution of Problem 1.

2. There is a neighbourhood $V\subset W_{11}^n$ of the point x_0 such that for any $x\in V$, any $u_1, u_2\in U$ and any $\alpha\in[0, 1]$ there exists some $u\in U$ such that $\varphi(x, u)=\alpha\varphi(x, u_1)+(1-\alpha)\,\varphi(x, u_2)$.

3. The function Φ is convex with respect to u for any $x \in V$, then there exists: an absolutely continuous function ψ, a constant $\lambda_0 \geqslant 0$, and a function $c=(c^1, c^2, \ldots, c^r)$ such that:

(i) $\dfrac{d\psi(t)}{dt} = -\varphi_x^*(x_0(t), u_0(t))\,\psi(t) + \lambda_0\,\Phi_x(x_0(t), u_0(t))$.

(ii) c^k is a constant function on each interval I_n^k.

(iii) $(\varphi_u^*(x_0(t), u_0(t))\,\psi(t) - \lambda_0\,\Phi_u(x_0(t), u_0(t))^k\,(c^k(t) - u_0^k(t)) = 0$, $k=1, 2, \ldots, r$, for all but at most a denumerable number of points $t \in [0, 1]$.

Proof. Let us adopt $X = W_{11}^n([0, 1])$ and $Y = L_1^n(0, 1) \times R^n \times R^n$. Let $F: X \times U \to Y$ be an operator defined by the equality

$$F(x, u) = \left(\frac{dx}{dt} - \varphi(x, u),\ x(0) - c,\ x(t) - d\right).$$

We may formulate Problem 1 in the following way:
Problem 1'. Find the minimal value of the functional

$$I(x, u) = \int_0^1 \Phi(x(t), u(t))\,dt$$

under the conditions

$$F(x, u) = 0, \quad u \in U.$$

Making use of assumption (1)–(3) stated above, of assumption (11) and Theorem 1 (cf. [10] § 0.4 p. 63), it is not difficult to see that Problem 1' satisfies the conditions of the extremumu principle (cf. [10] § 1.1 Corollary 1 p. 80).

Lagrange's function for Problem 1' is of the form

$$\alpha(x, u, \lambda_0, \lambda_1, \lambda_2, y^*) = \lambda_0 \int_0^1 \Phi(x, u)\,dt + \lambda_1(x(0) - c) + \lambda_2(x(1) - d) + \int_0^1 \left\langle \psi(t), \frac{dx}{dt} - \varphi(x, u) \right\rangle dt,$$

where $\psi \in L_\infty^n(0, 1)$, $\lambda_1, \lambda_2 \in R^n$, $\lambda_0 \in R$.

Applying the extremum principle ([10] § 1.1), we obtain

$$\lambda_0 \int_0^1 \Phi_x(x_0, u_0)\,\bar{x}(t)\,dt + \lambda_1\,\bar{x}(0) + \lambda_2\,\bar{x}(1) + \int_0^1 \left\langle \psi(t), \frac{d\bar{x}}{dt} - \varphi_x(x_0, u_0)\,\bar{x} \right\rangle dt = 0 \text{ for any } \bar{x} \in W_{11}^n, \tag{14}$$

$$\int_0^1 \{\lambda_0\,\Phi(x_0, u_0) - \langle \psi, \varphi(x_0, u_0)\rangle\}\,dt = \min_{u \in U} \int_0^1 \{\lambda_0\,\Phi(x_0, u) - \langle \psi, \varphi(x_0, u_0)\rangle\}\,dt. \tag{15}$$

Denote by H the function

$$H(\lambda_0, \psi, x, u) = \langle \psi, \varphi(x, u)\rangle - \lambda_0 \Phi(x, u).$$

Equalities (14) and (15) can therefore be written in the form

$$\int_0^1 \left\langle \psi, \frac{d\bar{x}}{dt} \right\rangle dt - \int_0^1 \langle H_x, \bar{x}\rangle\, dt + \lambda_1 \bar{x}(0) + \lambda_2 \bar{x}(1) = 0 \tag{16}$$

$$\int_0^1 H(\lambda_0, \psi, x_0, u_0)\, dt = \max_{u \in U} \int_0^1 H(\lambda_0, \psi, x_0, u)\, dt. \tag{17}$$

Integrating, we obtain successively

$$-\int_0^1 \langle H_x, \bar{x}\rangle\, dt = -\int_0^1 \left\langle H_x, \bar{x}(0) + \int_0^t \dot{\bar{x}}(\tau)\, d\tau \right\rangle dt =$$

$$= \left\langle \int_t^1 H_x\, d\tau, \bar{x}(0) + \int_0^t \dot{\bar{x}}(\tau)\, d\tau \right\rangle \Bigg|_{t=0}^{t=1} +$$

$$-\int_0^1 \left\langle \int_t^1 H_x\, d\tau, \dot{\bar{x}}(t) \right\rangle dt = \left\langle \int_0^1 H_x\, d\tau, \bar{x}(0) \right\rangle - \int_0^1 \left\langle \int_t^1 H_x\, d\tau, \dot{\bar{x}}(t) \right\rangle dt.$$

Taking into consideration the equality

$$\bar{x}(1) = \bar{x}(0) + \int_0^1 \dot{\bar{x}}(t)\, dt$$

and (16), we obtain

$$\int_0^1 \left\langle \psi - \int_t^1 H_x\, d\tau + \lambda_2, \dot{\bar{x}}(t) \right\rangle dt + \left\langle -\int_0^1 H_x\, d\tau + \lambda_1 + \lambda_2, \bar{x}(0) \right\rangle = 0.$$

Since the last equality holds for every absolutely continuous function, we obtain, in particular

$$\psi(t) - \int_t^1 H_x\, d\tau + \lambda_2 = 0$$

for any $t \in [0, 1]$.

Consequently, the function ψ is absolutely continuous and satisfies equation (i) from the proposition of Theorem 2.

Let u_1 be an arbitrary point of the set U. Since U is convex, we get

$$u = u_0 + \lambda(u_1 - u_0) \in U \tag{18}$$

for $\lambda \in [0, 1]$.

Making use of equalities (17) and (18), we obtain

$$\int_0^1 H(\lambda_0, \psi, x_0, u_0)\, dt \geqslant \int_0^1 H(\lambda_0, \psi, u_0)\, dt + \lambda \int_0^1 H_u (u_1 - u_0)\, dt + 0(\lambda).$$

Hence

$$\int_0^1 -H_u(\lambda_0, \psi, x_0, u_0)\, u_0(t)\, dt \leqslant \int_0^1 -H_u(\lambda_0, \psi, x_0, u_0)\, u_1(t)\, dt.$$

We accept $a(t) = -H_u(\lambda_0, \psi(t), x_0(t), u_0(t))$.

Making use of Lemmas 4, 6 and 3, we obtain the proposition of Theorem 2.

To illustrate the application of Theorem 2, let us consider now the linear case. Namely, assume that the function φ is of the form

$$\varphi(x, u, t) = Ax + Bu$$

and

$$\Phi(x, u, t) = ax + bu,$$

where A is a matrix with constant coefficients of the dimension $n \times n$, B is a constant matrix of the dimension $n \times r$, a and b are n- and r-dimensional vectors, respectively. It is easily seen that, in the case, equation (i) is of the form

$$\frac{d\psi}{dt} = -A^* \psi + \lambda_0 a.$$

It is known that a solution of this equation is the function $\psi = (\psi_1, \ldots, \psi_n)$, where ψ_i are the real or imaginary parts of quasipolynomials of the form (cf, e.g. [12])

$$W_i = p_0^i + p_1^i(t) \exp \lambda_1 t + \ldots + p_n^i(t) \exp \lambda_n t, \quad i = 1, 2, \ldots, n,$$

where $p_0^i = \text{const.}$, $\lambda_1, \lambda_2, \ldots, \lambda_n$ — characteristic roots of the matrix A, $p_1^i, \ldots, p_n^i$ — algebraic polynomials of the variable t.

Condition (iii) takes the form

$$(-B^* \psi(t) + \lambda_0 b)^k (c^k(t) - u_0^k(t)) = 0.$$

Assume that the first factor is not identically zero.

It is easy to see that each of the functions

$$(-B^* \psi(t) + \lambda_0 b)^k, \quad k = 1, 2, \ldots, r, \tag{19}$$

is analytic and, consequently, possesses a finite number of zeros on the interval $[0, 1]$. Functions of the form (19) will be called switching functions. Making use of Theorems 1 and 2 we deduce that the optimal control does exist and is a piece-wise constant function. In the case when the matrix A has real eigenvalues, the function (19) is a real quasipolynomial of degree, at most, $(n+1)$ and for that reason the number of its roots does not exceed n. Each component of the optimal control, in this case, will be a piece-wise constant function and the number of its jump points will not exceed n. In the case considered, Problem 1 reduces to a certain problem of linear programming.

4. Physical Interpretation

We shall now give a simple example for application of the theorems proved above, which is, at the same time, a physical interpretation of the optimization problem under consideration. Example. Let us consider an object with one axis of symmetry moving in a plane. The object is supplied with three engines. The first of the engines can produce a force vector directed to the right of the axis of the object, the second — to the left, whereas the third engine can increase or decrease the speed of the object along its axis. Let $u_1 = u_1(t)$ and $u_2 = u_2(t)$ denote the quantities of fuel used within the time interval $[0, t]$, by the first and by the second engine, respectively, while $u_3 = u_3(t)$ denotes the difference between quantities of fuel used by the third engine for increasing and decreasing the speed of the object. Suppose that the engines are supplied with fuel from a common tank of capacity 1, and that both the first and the second engines cannot use more fuel than $\frac{1}{4}$ each.

Assume moreover that the motion of the object can be described by the equation

$$\dot{x} = \varphi(x, u, t)$$

and the performance is of the form

$$F(x, u) = \int_0^1 \Phi(x, u, t)\, dt.$$

Note that finding the optimal control (the function u), which carries the object from the state $x(0) = c$ to the state $x(1) = d$, reduces to solving Problem 1, where

$$P = \left\{u \in R^3,\ 0 \leqslant u_1 \leqslant \frac{1}{4};\ 0 \leqslant u_2 \leqslant \frac{1}{4},\ -1 \leqslant u_3 \leqslant 1\right\}.$$

$$V = \left\{v \in R^3;\ 0 \leqslant v_1 \leqslant \frac{1}{4},\ 0 \leqslant v_2 \leqslant \frac{1}{4},\ 0 \leqslant v_3 \leqslant 1 - v_1 - v_2\right\}.$$

Suppose that

$$\varphi(x, u, t) = Ax + Bu \quad \text{and} \quad \Phi(x, u, t) = ax + bu,$$

where

$$A = \begin{vmatrix} 0, & 1 \\ 1, & 0 \end{vmatrix}, \quad B = \begin{vmatrix} 1, & -1, & 0 \\ 0, & 0, & 1 \end{vmatrix}, \quad a = (1, 1),\ b = (1, 1, 1).$$

It is easy to verify that equation (i) in this case takes the form

$$\frac{d\varphi}{dt} = -A^* \psi + \lambda_0 a,$$

and the function $\psi = (\psi_1, \psi_2)$, where

$$\psi_1(t) = -C_1 \exp t + C_2 \exp(-t) + \lambda_0,$$

$$\psi_2(t) = C_1 \exp t + C_2 \exp(-t) + \lambda_0,$$

is its solution.

Switching functions (19) have the form

$$(-B^* \psi+\lambda_0 b)_1=C_1 \exp t-C_2 \exp(-t),$$
$$(-B^* \psi+\lambda_0 b)_2=-C_1 \exp t+C_2 \exp(-t)+2\lambda_0,$$
$$(-B^* \psi+\lambda_0 b)_3=-C \exp t-C_2 \exp(-t).$$

We can see that each of these functions is not identically zero. Besides, the first and the third ones have at most one zero in the interval [0, 1], and the second — at most two. We may eventually state that the optimal control does exist, each of its components is piece-wise constant, the first and the third possessing at most one jump point each, whereas the second — at most two. Therefore we can see that, in this case, the optimal control of the fuel consumption consists in its "explosive" distribution.

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Received, October 1978.

Warunki Eulera-Lagrange'a dla sterowań z ograniczonym wahaniem

Rozważono zadanie sterowania optymalnego w klasie sterowań z ograniczonym wahaniem. Udowodniono warunek konieczny istnienia rozwiązania oraz twierdzenia o istnieniu. Podano także interpretację fizyczną rozważanego problemu.

Условия Эйлера-Лагранжа для управления с ограниченными изменииями

В работе рассматривается следующая задача оптимизации: найти минимальное значение функционала

$$\int_0^1 \Phi(x, u, t)\, dt$$

ирп условиях: $\dot{x}=\varphi(x, u, t)$, $x(0)=c$, $x(1)=d$, где u является функцией с ограниченными изменениями, а x является функцией абсолютно непрерывной. При некоторых предпосылках, касающихся функций φ и Φ, показано существование оптимального управления и даны необходимые условия его существования, аналогичные случаю уравнений Эйлера-Лагранжа. В заключительной части работы представлена физическая интерпретация исследуемой задачи оптимизации.