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Finite-difference Approximations To Parabolic Free Boundary Value Problems Arising In Modelling of Underground Gas Reservoir. Part I. Algorithms

by

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In the paper some finite-difference approximations to one-dimensional parabolic free boundary value problems arising in modelling of an underground gas reservoir are proposed. In the process of constructing the difference scheme a preparatory transformation of the free boundary problem into a nonlinear problem in an a priori given domain is applied. The proposed schemes are of the homogeneous and balanced type.

1. Introduction

This paper establishes finite-difference approximations for one-dimensional parabolic free boundary value problems suggested by equations modelling flow of gas and water in an underground gas reservoir formed in a water-bearing layer [5, 10]. The algorithms presented in the paper may be useful for computations concerning design and exploitation of underground gas reservoirs as well as for solving optimal control problems of pipeline networks containing such reservoirs.

The mathematical models of underground gas reservoir, formulated in terms of pressure or respectively in terms of filtration velocity, have been presented in [5, 10]. These models belong to class of so called two-layer parabolic free boundary value problems. Their analytical properties such as correctness in the Hadamard sense and the maximum principle have been proved in [10, 11]. We will make use of results presented in those papers in order to demonstrate the convergence of finite-difference approximations to the models considered.

Numerical methods for solving parabolic free boundary value problems have been proposed by many authors [3, 9, 18]. Most of them have investigated multi-phase problems, known also as Stefan problems. In view of the essential difference between multi-phase and multi-layer problems, inherent in form of conditions which hold along free boundary, it was difficult for us to make use of the approaches proposed by these authors.

Finite-difference approximations to multi-layer free boundary value problems have been investigated in [2, 7, 8, 17]. Most of the methods proposed in these works have not been studied from the theoretical point of view. Only numerical results testifying convergence of the methods in the case of particular boundary conditions (for which solution in analytical form is known) have been presented. The only exception is the work [2] by Ciment and Guenther in which theoretical analysis of the convergence of some finite-difference method has been performed as well as numerical results have been discussed. The method proposed in that work generates an uneven grid on each time step what greatly diminishes the efficiency of the method.

In the finite-difference methods which we propose the fixed grid pattern for the whole time interval is used. Moreover, the difference schemes express on the grid continuity flow principle, so they may be treated as a discrete models of filtration phenomena in the underground gas reservoir.

In part I of the paper we present a finite-difference method based on some preparatory transformation of the free boundary value problem into a nonlinear parabolic problem in domain with fixed boundary (Section 4). For the transformed problem we construct in Section 5 a finite-difference scheme of the conservative type, expressing on the grid fundamental physical conservation principles. Two numerical algorithms for solving the free boundary value problem, without iterations and iterative one, are presented (Section 5).

In part II of the paper we will prove the convergence of the finite-difference scheme presented in part I. We also are going to describe a direct finite-difference method in which preparatory transformation of the problem is not used. Comparison between numerical efficiency of the method with transformation and the direct one will be presented.

2. Notations and Conventions

$$D \triangleq \{(x,t) \,|\, x \in (0,l),\ t \in (0,T)\},\quad \Gamma \triangleq \{(x,t) \,|\, x = y(t),\ t \in (0,T)\},$$

$$D_1 \triangleq \{(x,t) \,|\, x \in (0, y(t)),\ t \in (0,T)\},\quad D_2 \triangleq \{(x,t) \,|\, x \in (y(t), l),\ t \in (0,T)\},$$

$$Z_1 \triangleq \{(x,t) \,|\, x \in (0, y(0)),\ t = 0\},\quad Z_2 \triangleq \{(x,t) \,|\, x \in (y(0), l),\ t = 0\}$$

where $l > 0$, $T > 0$; function y describes the free boundary, $y(t) \in (0,l)$ for $t \in [0,T]$, $y(0) = y_0$.

For convenience we denote

$$l_1 = 0,\quad l_2 = 1,$$

$$v(t) \triangleq u_1(y,(t),t) = u_2(y(t),t) \quad \text{for } t \in [0,T],$$

$$u(x,t) \triangleq u_i(x,t) \quad \text{for } (x,t) \in \operatorname{cl} D_i,\ i = 1, 2.$$

By cl Q we denote the closure of the set Q.

For $m \in \mathcal{N}$, $C^m(Q)$ is the class of functions m-times continuously differentiable in Q.

If $(x, t) \in Q \subset R^2$ then, for $m, n \in \mathcal{N}$, $C^{m,n}(Q)$ denotes the class of functions m-times continuously differentiable in Q with respect to x and n-times continuously differentiable in Q with respect to t.

We introduce in D the grid

$$\omega_{h\tau} \triangleq \omega_h \times \omega_\tau = \{(x_i, t_j) \mid x_i \in \omega_h, t_j \in \omega_\tau\}$$

where

$$\omega_h \triangleq \left\{x_i \mid x_i = ih, \ i=0, 1, \dots, N; \ h = \frac{1}{N}\right\},$$

$$\omega_\tau \triangleq \left\{t_j \mid t_j = j\tau, \ j=0, 1, \dots, L; \ \tau = \frac{T}{L}\right\}, \ N, L \in \mathcal{N}.$$

Let Z_i^j denote a grid function defined on the grid $\omega_{h\tau}$. For a fixed $j \in \{0, 1, \dots, L\}$ the following norms in the space of grid functions are used [13]:

$$\|Z^j\|_0 = \max_{i \in \{0, \dots, N\}} |Z_i^j|,$$

$$\|Z^j\|_m = \left[\sum_{i=1}^{N-1} h\,|Z_i^j|^m\right]^{\frac{1}{m}}, \quad m=1, 2,$$

$$\|Z^j\|_3 = \|\chi^j\|_2 \text{ where } \chi_i^j = \sum_{k=1}^{i} hZ_k^j.$$

We also define seminorm $\|\cdot\|_4$ [14]:

$$\|Z^j\|_4 = \|Z^j\|_3 + \left|\sum_{i=1}^{N-1} hZ_i^j\right|.$$

The relationship $W(\delta) = 0(\delta^r)$, $r \in \mathcal{R}$ means that $W(\delta)$ is of the order δ^r when $\delta \to 0+$, i.e.

$$|W(\delta)| \leq M\delta^r$$

where positive constant M is independent of δ.

By $E(x)$ we denote the integer part of the number $x \in \mathcal{R}$.

3. Statement of Two-layer Parabolic Free Boundary Value Problems

The following Dirichlet and Neumann free boundary value problems will be considered.

Problems (B_k^S), $k=1, 2$

Find functions u_1, u_2, y satisfying:

— system of parabolic equations

$$\frac{\partial u_i}{\partial t} - \alpha_i \frac{\partial^2 u_i}{\partial x^2} = 0 \text{ in } D_i, \ i=1, 2; \tag{3.1}$$

— initial conditions

$$y(0)=y_0 \text{ where } y_0 \in (0, l), \tag{3.2}$$

$$u_i(x, 0)=u_{io}(x) \text{ in } Z_i; \tag{3.3}$$

— conditions at the free boundary

$$u_1(y(t), t)=u_2(y(t), t), \quad \gamma_1 \frac{\partial u_1}{\partial x}(y(t), t)=\gamma_2 \frac{\partial u_2}{\partial x}(y(t), t), \quad t \in (0, T]; \tag{3.4}$$

— boundary conditions:

of Dirichlet type in case of Problems (B_1^S)

$$u_i(l_i, t)=F_i(t), \ t \in (0, T]; \tag{3.5}$$

of Neumann type in case of Problem (B_2^S)

$$\gamma_i \frac{\partial u_i}{\partial x}(l_i, t)=\varphi_i(t), \ t \in (0, T]; \tag{3.5'}$$

— ordinary differential equation defining the free boundary

$$\frac{dy}{dt}(t)=\beta u_1(y(t), t), \ t \in (0, T]. \tag{3.6}$$

Here α_i, γ_i, β, y_o, $i=1, 2$, are given positive constants and u_{io}, F_i, φ_i are given functions.

The above problems are particular cases of Problems (B_k) investigated in [10, 11]. The difference is inherent in form of conditions at the free boundary, more general for Problems (B_k). It has been shown in [1, 4] that such a simplification of the conditions at the free boundary is justified for filtration problems involving displacement of one fluid by another in porous media.

Therefore one can consider Problem (B_k^S) as a model of underground gas reservoir, describing filtration velocity distribution and dynamics of the contact boundary between gas and water. Type of the boundary condition at $x=0$ depends on the kind of control of the gas reservoir. It is assumed that at $x=l$ the steady state conditions hold ($F_2 \equiv 0$ or equivalently $\varphi_2 \equiv 0$ [10]).

We will take it for granted that the following regularity and compatibility conditions for the boundary and initial data of Problems (B_k^S) are fulfilled:

(H1) $F_i \in C^2 [0, T]$;

(H2) $F_i(0)=u_{io}(l_i)$;

(H3) $u_{1o} \in C^2 [0, y_o]$, $u_{2o} \in C^2 [y_o, l]$;

(H4) $u_{1o}(y_o)=u_{2o}(y_o)$, $\gamma_1 u'_{10}(y_o)=\gamma'_2 u_{20}(y_o)$;

(H5) $\varphi_i \in C^1 [0, T]$;

(H6) $\gamma_i u'_{io}(l_i)=\varphi_i(0)$.

We assume also that

(H7) $y(t) \in (0, l)$ in the considered time interval $[0, T]$.

It has been proved in [10, 11] that if the conditions (H1)–(H4), (H7) are satisfied for Problem (B_1^S) and respectively the conditions (H3)–(H7) for Problem (B_2^S), then there exist unique classical solutions $\{u_1, u_2, y\}$ of Problems (B_k^S), $k=1, 2$, in the interval $[0, T]$; at the same time $y \in C^2 [0, T]$.

As a first step in investigation of convergence of numerical methods for these problems we are led to consider the following auxiliary problems associated with a given function y.

Problems (b_k), $k=1, 2$

Let y be a given function such that $y \in C^2 [0, T]$, $y(0)=y_0$, $y(t) \in (0, l)$ for $t \in [0, T]$. We seek functions u_1, u_2 satisfying (3.1)–(3.5).

The conditions (H1)–(H4) guarantee existence of a unique classical solution to Problem (b_1) whereas conditions (H3)–(H6) guarantee existence and uniqueness of solution to Problem (b_2) [10, 11].

4. Reformulation of the Problems

Let us assume that

$$0 < y_m \leqslant y(t) \leqslant y_M < l \quad \text{for } t \in [0, T] \tag{4.1}$$

where y_m, y_M are some given constants. Under this assumption we transform the (x, t)-coordinate system in the following way

$$\xi = \begin{cases} \dfrac{l}{2} \dfrac{x}{y(t)} & \text{for } x \in [0, y(t)) \\[2ex] \dfrac{l}{2} + \dfrac{l}{2} \dfrac{(x-y(t))}{(l-y(t))} & \text{for } x \in [y(t), l]. \end{cases} \tag{4.2}$$

In accordance with (4.2) the domains D_i, $i=1, 2$, are transformed into domains $\tilde{D}_1 \triangleq \{(\xi, t) \mid \xi \in (0, l/2),\ t \in (0, T)\}$ and $\tilde{D}_2 \triangleq \{(\xi, t) \mid \xi \in (l/2, l),\ t \in (0, T)\}$ respectively, whereas the curve Γ is transformed into line $\tilde{\Gamma} \triangleq \{(\xi, t) \mid \xi = l/2,\ t \in (0, T)\}$ (Fig. 1).

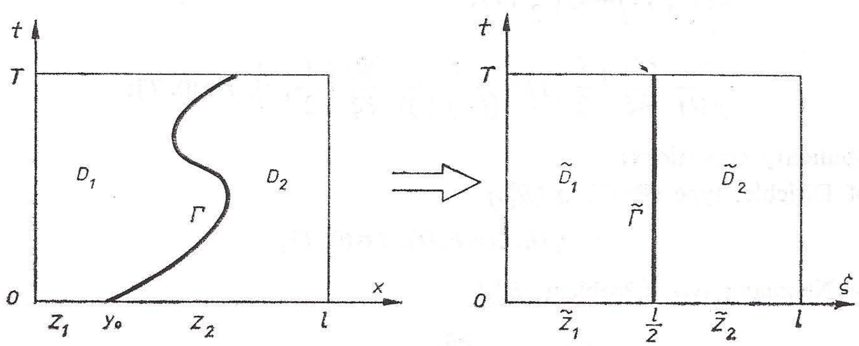

Figure 1

Let us denote $\tilde{D} \triangleq \{(\xi, t) | \xi \in (0, l),\ t \in (0, T)\}$, $\tilde{Z}_1 \triangleq \{(\xi, t) | \xi \in (0, l/2),\ t=0\}$, $Z_2 \triangleq \{(\xi, t) | \xi \in (l/2, l),\ t=0\}$. In the (ξ, t) — coordinate system we define functions $\tilde{u}_i$, $\tilde{u}_{io}$, $\tilde{v}$, $i=1, 2$:

$$\tilde{u}_1 (\xi, t) \triangleq u_1 \left(\frac{2\xi}{l} y(t), t \right) = u_1 (x, t) \text{ for } (\xi, t) \in \text{cl}\, \tilde{D}_1,\ (x, t) \in \text{cl}\, D_1$$

$$\tilde{u}_2 (\xi, t) \triangleq u_2 \left[\left(\frac{2\xi}{l} - 1 \right) (l - y(t)) + y(t), t \right] = u_2 (x, t) \text{ for } (\xi, t) \in \text{cl}\, \tilde{D}_2,\ (x, t) \in \text{cl}\, D_2,$$

$$\tilde{u}_{io} (\xi) \triangleq u_{io} (x) \text{ for } (\xi, 0) \in \text{cl}\, \tilde{Z}_i,\ (x, 0) \in \text{cl}\, Z_i,$$

$$\tilde{v} (t) \triangleq \tilde{u}_1 \left(\frac{l}{2}, t \right) = u_1 (y(t), t) \text{ for } t \in [0, T].$$

Now Problems (B_k^S) formulated in the (ξ, t)—coordinate system take the form:

Transformed Problems (B_k^S), $k=1, 2$

Find functions $\tilde{u}_1$, $\tilde{u}_2$, y satisfying:

— system of nonlinear parabolic equations

$$\frac{\partial \tilde{u}_1}{\partial t} (\xi, t) = \alpha_1 \left(\frac{l}{2y(t)} \right)^2 \frac{\partial^2 \tilde{u}_1}{\partial \xi^2} (\xi, t) + \frac{y'(t)}{y(t)} \xi \frac{\partial \tilde{u}_1}{\partial \xi} (\xi, t) \text{ for } (\xi, t) \in \tilde{D}_1,$$

$$\frac{\partial \tilde{u}_2}{\partial t} (\xi, t) = \alpha_2 \left(\frac{l}{2(l - y(t))} \right)^2 \frac{\partial^2 \tilde{u}_2}{\partial \xi^2} (\xi, t) + \frac{y'(t)}{(l - y(t))} (l - \xi) \frac{\partial \tilde{u}_2}{\partial \xi} (\xi, t)$$

$$\text{for } (\xi, t) \in \tilde{D}_2; \tag{4.3}$$

— initial conditions

$$y(0) = y_0 \text{ where } y_0 \in [y_m, y_M],$$
$$\tilde{u}_i (\xi, 0) = \tilde{u}_{io} (\xi) \text{ in } \tilde{Z}_i; \tag{4.4}$$

— conditions at the line $\tilde{\Gamma}$

$$\tilde{u}_1 \left(\frac{l}{2}, t \right) = \tilde{u}_2 \left(\frac{l}{2}, t \right),$$

$$\frac{\gamma_1}{y(t)} \frac{\partial \tilde{u}_1}{\partial \xi} \left(\frac{l}{2}, t \right) = \frac{\gamma_2}{(l - y(t))} \frac{\partial \tilde{u}_2}{\partial \xi} \left(\frac{l}{2}, t \right),\ t \in (0, T]; \tag{4.5}$$

— boundary conditions:

of Dirichlet type (Problem (B_1^S))

$$u_i (l_i, t) = F_i (t),\ t \in (0, T], \tag{4.6}$$

of Neumann type (Problem (B_2^S))

$$\frac{\gamma_i l}{2 [l_i - (-1)^i y(t)]} \frac{\partial \tilde{u}_i}{\partial \xi} (l_i, t) = \varphi_i (t),\ t \in (0, T]; \tag{4.6'}$$

— ordinary differential equation defining the coefficients of equations (4.3)

$$\frac{dy}{dt}(t)=\beta\,\tilde{u}_1\left(\frac{l}{2},t\right),\quad t\in(0,T]. \tag{4.7}$$

For the further considerations it will be convenient to rewrite problems (4.3)–(4.7) in the following way

$$b(\xi,t)\frac{\partial\tilde{u}}{\partial t}(\xi,t)=\frac{\partial}{\partial\xi}\left[a(\xi,t)\frac{\partial\tilde{u}}{\partial\xi}(\xi,t)\right]+c(\xi,t)\,a(\xi,t)\frac{\partial\tilde{u}}{\partial\xi}(\xi,t),$$

$$(\xi,t)\in\tilde{D}_i,\ i=1,2 \tag{4.8}$$

where $\tilde{u}(\xi,t)\triangleq\tilde{u}_i(\xi,t)$ for $(\xi,t)\in\mathrm{cl}\,\tilde{D}_i$,

$$a(\xi,t)=\begin{cases}a_1(\xi,t)\triangleq\dfrac{\gamma_1 l}{2y(t)} & \text{for } (\xi,t)\in\mathrm{cl}\,\tilde{D}_1\setminus\mathrm{cl}\,\tilde{\Gamma}\\[2ex] a_2(\xi,t)\triangleq\dfrac{\gamma_2 l}{2(l-y(t))} & \text{for } (\xi,t)\in\mathrm{cl}\,\tilde{D}_2,\end{cases}$$

$$b(\xi,t)=\begin{cases}b_1(\xi,t)\triangleq\dfrac{2\gamma_1}{\alpha_1 l}y(t) & \text{for } (\xi,t)\in\mathrm{cl}\,\tilde{D}_1\setminus\mathrm{cl}\,\tilde{\Gamma}\\[2ex] b_2(\xi,t)\triangleq\dfrac{2\gamma_2}{\alpha_2 l}(l-y(t)) & \text{for } (\xi,t)\in\mathrm{cl}\,\tilde{D}_2,\end{cases} \tag{4.9}$$

$$c(\xi,t)=\begin{cases}c_1(\xi,t)\triangleq\dfrac{4}{\alpha_1 l^2}y(t)\,y'(t)\,\xi & \text{for } (\xi,t)\in\mathrm{cl}\,\tilde{D}_1\setminus\mathrm{cl}\,\tilde{\Gamma}\\[2ex] c_2(\xi,t)\triangleq\dfrac{4}{\alpha_2 l^2}(l-y(t))\,y'(t)\,(l-\xi) & \text{for } (\xi,t)\in\mathrm{cl}\,\tilde{D}_2\end{cases}$$

$$y(0)=y_0, \tag{4.10}$$

$\tilde{u}(\xi,0)=\tilde{u}_0(\xi)$ where $\tilde{u}_o(\xi)\triangleq\tilde{u}_{io}(\xi)$ for $(\xi,0)\in\mathrm{cl}\,\tilde{Z}_i$;

$$\tilde{u}\left(\frac{l}{2}-,t\right)=\tilde{u}\left(\frac{l}{2}+,t\right), \tag{4.11}$$

$$a\left(\frac{l}{2}-,t\right)\tilde{u}\left(\frac{l}{2}-,t\right)=a\left(\frac{l}{2}+,t\right)\tilde{u}\left(\frac{l}{2}+,t\right),\quad t\in(0,T];$$

$$\tilde{u}(l_i,t)=F_i(t),\quad t\in(0,T] \tag{4.12}$$

or

$$a(l_i,t)\frac{\partial\tilde{u}}{\partial\xi}(l_i,t)=\varphi_i(t),\quad t\in(0,T],\ i=1,2; \tag{4.12'}$$

$$\frac{dy}{dt}(t)=\beta\tilde{u}\left(\frac{l}{2}-,t\right),\quad t\in(0,T]. \tag{4.13}$$

The above boundary value problems formulated in terms of $\tilde{u}$ are respectively Dirichlet or Neumann nonlinear problems for parabolic equation with coefficients discontinuous on given line. Nonlinearity follows from the fact that the coefficients depend upon the solution of an auxiliary ordinary differential equation with the right-hand side dependent in turn on $\tilde{u}$.

5. Finite-difference Approximations to Problems (B_k^S)

In this section we are going to present finite-difference approximations to Problems (B_k^S), $k=1, 2$. The transformed forms of Problems (B_k^S), introduced in the previous section, will be used as a basis in the process of constructing the difference schemes.

In the domain $\tilde{D}$ we introduce the regular grid $\omega_{h\tau}$. We can assume without loss of generality that N is an even number. In view of this there are nodes of the grid $\omega_{h\tau}$ at the line $\tilde{\Gamma}$. According to the definition of the transformation (4.2) the irregular grid $\Omega_{h\tau} \triangleq \Omega_h \times \omega_\tau$ defined in the domain D corresponds to the grid $\omega_{h\tau}$ (Fig. 2), where

$$\Omega_h \triangleq \left\{ x_i \mid x_i = \begin{cases} \dfrac{2y(t_j)}{l}\, ih, & i=0, 1, \ldots, \dfrac{N}{2}; \\ 2\left(1-\dfrac{y(t_j)}{l}\right) ih - l + 2y(t_j), & i=\dfrac{N}{2}+1, \ldots, N; \end{cases} \right\}$$

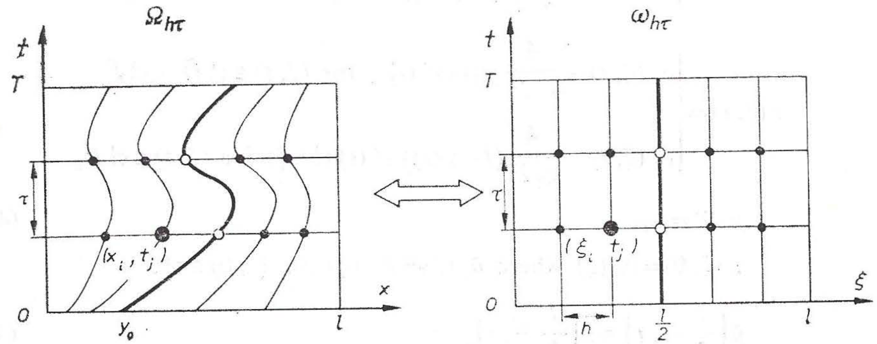

Figure 2

Now we are in a position to begin the construction of difference approximations to Problems (B_k^S).

We shall do it in two stages.

Stage I

First we are going to approximate the auxiliary problems (4.8)–(4.12′) associated with a given function $y \in C^2[0, T]$ satisfying (4.1). We will assume boundary conditions in the more general form

$$a_i(l_i, t)\frac{\partial \tilde{u}}{\partial \xi}(l_i, t) + (-1)^i \sigma_i(t)\, \tilde{u}(l_i, t) = g_i(t), \quad t \in (0, T], \quad i=1, 2 \qquad (5.1)$$

instead of the conditions (4.12) and (4.12′).

Later on to prove the convergence of finite-difference scheme related to problem (4.8)–(4.11), (5.1) we will be under the necessity of assuming some regularity conditions for the coefficients a_i, b_i, c_i, σ_i, $i=1, 2$, as well as for the solution $\tilde{u}$ of problem (4.8)–(4.11), (5.1). Namely, we will assume that:

(H8) $a_i \in C^{2,1}(\mathrm{cl}\, \tilde{D}_i)$, $b_i \in C^{2,1}(\mathrm{cl}\, \tilde{D}_i)$, $c_i \in C^{1,1}(\mathrm{cl}\, \tilde{D}_i)$, $i=1, 2$;

(H9) $\sigma_i \in C^1[0, T]$, $\sigma_i(t) \geqslant 0$, $i=1, 2$,
$0 < \sigma_* \leqslant \sigma_1(t) + \sigma_2(t)$, $t \in (0, T]$ where σ_* is a given constant;

(H10) $\tilde{u}_i \in C^{2,1}(\mathrm{cl}\, \tilde{D}_i)$, $i=1, 2$; $a_i \dfrac{\partial \tilde{u}_i}{\partial \xi}$ satisfies Lipschitz continuity condition in $\mathrm{cl}\, \tilde{D}_i$ with respect to t; $\dfrac{\partial \tilde{u}_i}{\partial t}$ satisfies Lipschitz continuity condition in $\mathrm{cl}\, \tilde{D}_i$ with respect to ξ.

In order to obtain finite-difference approximation to problem (4.8)–(4.11), (5.1) we use the integral-interpolation method [13, 14]. Integrating the equation (4.8) in the elementary grid domain

$$R_{ij} \triangleq \{(\xi, t) | \xi \in [\xi_{i-\frac{1}{2}}, \xi_{i+\frac{1}{2}}],\ t \in [t_{j-1}, t_j]\} \tag{5.2}$$

where $\xi_{i-\frac{1}{2}} = \xi_i - \dfrac{h}{2}$, $\xi_{i+\frac{1}{2}} = \xi_i + \dfrac{h}{2}$, we obtain so called balance equation [13]

$$\frac{1}{h\tau} \iint_{R_{ij}} b(\xi, t) \frac{\partial \tilde{u}}{\partial t}(\xi, t)\, d\xi\, dt = \frac{1}{h\tau} \iint_{R_{ij}} \frac{\partial}{\partial \xi}\left[a(\xi, t) \frac{\partial \tilde{u}}{\partial \xi}(\xi, t)\right] d\xi\, dt + \\ + \frac{1}{h\tau} \iint_{R_{ij}} c(\xi, t)\, a(\xi, t) \frac{\partial \tilde{u}}{\partial \xi}(\xi, t)\, d\xi\, dt. \tag{5.3}$$

The above integral identity is used as a basis for obtaining the difference equations. To this end we rewrite (5.3) in the following way

$$\frac{1}{h\tau} \int_{t_{j-1}}^{t_j} \int_{\xi_{i-\frac{1}{2}}}^{\xi_{i+\frac{1}{2}}} b(\xi, t) \frac{\partial \tilde{u}}{\partial t}(\xi, t)\, d\xi\, dt - \frac{1}{h\tau} \int_{t_{j-1}}^{t_j} [w(\xi_{i+\frac{1}{2}}, t) - w(\xi_{i-\frac{1}{2}}, t)]\, dt + \\ - \frac{1}{h\tau} \int_{t_{j-1}}^{t_j} \int_{\xi_{i-\frac{1}{2}}}^{\xi_{i+\frac{1}{2}}} c(\xi, t)\, w(\xi, t)\, d\xi\, dt = 0 \tag{5.4}$$

where

$$w(\xi, t) = a(\xi, t) \frac{\partial \tilde{u}}{\partial \xi}(\xi, t). \tag{5.5}$$

Note that by (4.11) the relationship (5.4) holds for every R_{ij}, $i=1, \ldots, N-1$; $j=1, \ldots, L$.

Next we approximate the expressions in (5.4) by linear combinations of values of $\tilde{u}$ in the grid points, postulating together with this some interpolations of functions $\tilde{u}$ and w in neighbourhoods of these points.

As a result we get the following homogeneous conservative system of implicit finite-difference equations for U_i^j, $(\xi_i, t_j) \in \omega_{h\tau}$ where U_i^j denote approximate values of a solution to problem (4.8)–(4.11), (5.1) (see [10]):

$$\mathscr{F}_1\,[U_i^j] \triangleq B_i^j \frac{U_i^j - U_i^{j-1}}{\tau} - \frac{1}{h}\left[A_{i+1}^j \frac{U_{i+1}^j - U_i^j}{h} - A_i^j \frac{U_i^j - U_{i-1}^j}{h}\right] -$$

$$- C_i^j\left[\kappa A_{i+1}^j \frac{U_{i+1}^j - U_i^j}{h} + (1-\kappa)\, A_i^j \frac{U_i^j - U_{i-1}^j}{h}\right] = 0,$$

$$i=1, \ldots, N-1; \; j=1, \ldots, L, \qquad (5.6)$$

$$\mathscr{F}_2\,[U_i^j] \triangleq (A_1^j - D_1^j)\frac{U_1^j - U_0^j}{h} - E_1^j \frac{U_0^j - U_0^{j-1}}{\tau} - \sigma_1^j U_0^j = g_1^j, \quad j=1, \ldots, L, \qquad (5.7)$$

$$\mathscr{F}_3\,[U_i^j] \triangleq (A_N^j + D_2^j)\frac{U_N^j - U_{N-1}^j}{h} + E_2^j \frac{U_N^j - U_N^{j-1}}{\tau} + \sigma_2^j U_N^j = g_2^j, \quad j=1, \ldots, L, \qquad (5.8)$$

$$\mathscr{F}_4\,[U_i^j] \triangleq U_i^0 = \tilde{u}_o(\xi_i), \; i=0, 1, \ldots, N \qquad (5.9)$$

where κ is a constant from the interval [0, 1],

$$E_k^j = \frac{1}{2} hb\,(l_k, t_j), \quad D_k^j = \frac{1}{2} hc\,(l_k, t_j)\, a\,(l_k, t_j),$$
$$\sigma_k^j = \sigma_k(t_j), \quad g_k^j = g_k(t_j), \quad k=1, 2; \quad j=1, \ldots, L; \qquad (5.10)$$

coefficients A_i^j, B_i^j, C_i^j are defined by the following expressions

$$A_i^j = \left[\frac{1}{h}\int\limits_{\xi_{i-1}}^{\xi_i} \frac{d\xi}{a(\xi, t_j)}\right]^{-1}, \quad B_i^j = \frac{1}{h}\int\limits_{\xi_{i-\frac{1}{2}}}^{\xi_{i+\frac{1}{2}}} b\,(\xi, t_j)\, d\xi,$$

$$C_i^j = \frac{1}{h}\int\limits_{\xi_{i-\frac{1}{2}}}^{\xi_{i+\frac{1}{2}}} c\,(\xi, t_j)\, d\xi \qquad (5.11)$$

and a, b, c are defined by (4.9).

The introduced finite-difference scheme is homogeneous in such a sense that the difference operator $\mathscr{F}_1$ has the same form at all nodes of the grid. The difference operators $\mathscr{F}_2$ and $\mathscr{F}_3$ correspond to boundary conditions (5.1) whereas $\mathscr{F}_4$ corresponds to initial condition (4.10).

Let us denote by Z_i^j error of the finite-difference scheme, i.e. the difference

$$Z_i^j \triangleq U_i^j - \tilde{u}_i^j \qquad (5.12)$$

where $\tilde{u}_i^j = \tilde{u}(\xi_i, t_j)$. If we substitute $U_i^j = Z_i^j + \tilde{u}_i^j$ into the expressions (5.6)–(5.9), we obtain the following difference problem for Z_i^j

$$\mathscr{F}_1 [Z_1^j] = \psi_i^j, \quad i=1, \ldots, N-1, \quad j=1, \ldots, L, \tag{5.13}$$

$$\mathscr{F}_2 [Z_i^j] = \nu_1^j, \quad j=1, \ldots, L, \tag{5.14}$$

$$\mathscr{F}_3 [Z_i^j] = \nu_2^j, \quad j=1, \ldots, L, \tag{5.15}$$

$$\mathscr{F}_4 [Z_i^j] = 0, \quad i=0, 1, \ldots, N \tag{5.16}$$

where

$$\psi_i^j = -\mathscr{F}_1 [\tilde{u}_i^j], \tag{5.17}$$

$$\nu_i^j = g_1^j - \mathscr{F}_2 [\tilde{u}_i^j], \quad \nu_2^j = g_2^j + \mathscr{F}_3 [\tilde{u}_i^j]. \tag{5.18}$$

According to the terminology of [13] ψ_i^j is error of approximation of equation (4.8) by the difference equations (5.6); ν_1^j, ν_2^j are errors of approximation of boundary conditions (5.1) by difference equations (5.7), (5.8) (all the errors correspond to the solution of problem (4.8)–(4.11), (5.1)).

Note that if in the difference scheme (5.6)–(5.9) instead of (5.7), (5.8) there are given conditions

$$U_0^j = g_1(t_j), \quad U_N^j = g_2(t_j), \quad j=0, 1, \ldots, L$$

then in problem (5.13)–(5.16) conditions (5.14), (5.15) take the form $Z_0^j = Z_N^j = 0$ and $\nu_1^j = \nu_2^j = 0$, $j=1, \ldots, L$.

To prove the convergence of the difference scheme (5.6)–(5.9) to solution of problem (4.8)–(4.11), (5.1) we make use of works by Samarskii [14, 15, 16] where the finite-difference schemes for partial differential equations with discontinuous coefficients have been investigated.

It follows from [15, Theorem 2, pp. 617–618] that if there are satisfied conditions

(i) $0 < c_1 \leqslant A_i^j \leqslant c_2$

(ii) $0 < c_3 \leqslant B_i^j \leqslant c_4$

(iii) $|C_i^j| \leqslant c_5 \quad i=1, \ldots, N-1;\ j=1, \ldots, L$

(iv) $\left| \dfrac{B_i^j - B_i^{j-1}}{\tau} \right| \leqslant c_6$

(v) $E_k^j \geqslant c_7 h > 0, \qquad \left| \dfrac{E_k^j - E_i^{j-1}}{\tau} \right| \leqslant c_8 E_k^j,$

(vi) $\sigma_k^j \geqslant 0, \quad \sigma_1^j + \sigma_2^j \geqslant \sigma_* > 0 \quad k=1, 2$

where σ_*, c_m $(m=1, \ldots, 8)$ are some given constants independent of h and τ, then for the solution of difference problem (5.13)–(5.16) the following a priori-estimates hold

$$\|Z^j\|_0 \leqslant M_1 \left[\|Z^0\|_0 + \max_{k \in \{1, \ldots, j\}} (\|\psi^k\|_4 + |\nu_1^k| + |\nu_2^k|) \ln^\delta \frac{1}{h} \right] \tag{5.19}$$

$$\text{for } h \in (0, h^*], \quad \tau \in (0, \tau_1^*], \quad j=1, \ldots, L,$$

$$\|Z^j\|_0 \leqslant M_1 \left[\|Z^0\|_0 + \max_{k \in \{1, \ldots, j\}} (\|\psi^k\|_4 + |\nu_1^k| + |\nu_2^k|) \ln^\delta \frac{1}{\tau} \right] \tag{5.20}$$

$$\text{for } h \in (0, h^*], \quad \tau \in (0, \tau_2^*],\ j=1, \ldots, L$$

where $\delta=1+\varepsilon$, ε is any arbitrary positive constant, h^*, τ_1^*, τ_2^* are sufficiently small positive constants; $h^*=h^*(\varepsilon)$, $\tau_2^*=\tau_2^*(\varepsilon)$, $M_1>0$ denotes a constant independent of h and τ.

Now we are going to verify that in the considered problem (4.8)–(4.11), (5.1) the conditions (i)–(vi) are satisfied. From definitions of A_i^j, B_i^j, C_i^j and assumption (4.1) it follows immediately that conditions (i)–(iii) are fulfilled with constants

$$c_1=\frac{l}{2}\min\left\{\frac{\gamma_1}{y_M},\frac{\gamma_2}{l-y_m}\right\},\qquad c_2=\frac{l}{2}\max\left\{\frac{\gamma_1}{y_m},\frac{\gamma_2}{l-y_M}\right\},$$

$$c_3=\frac{2}{l}\min\left\{\frac{\gamma_1}{\alpha_1}y_m,\frac{\gamma_2}{\alpha_2}(l-y_M)\right\},\qquad c_4=\frac{2}{l}\max\left\{\frac{\gamma_1}{\alpha_1}y_M,\frac{\gamma_2}{\alpha_2}(l-y_m)\right\},$$

$$c_5=\frac{4}{l}\bar{c}\max\left\{\frac{y_M}{\alpha_1},\frac{l-y_m}{\alpha_2}\right\}$$

where

$$\bar{c}=\sup_{t\in[0,T]}|y'(t)|. \tag{5.21}$$

To verify condition (iv) observe that

$$\frac{B_i^j-B_i^{j-1}}{\tau}=\frac{B(t_j)-B(t_{j-1})}{\tau}\quad\text{where } B(t)\triangleq\frac{1}{h}\int_{\xi_{i-\frac{1}{2}}}^{\xi_{i+\frac{1}{2}}} b(\xi,t)\,d\xi .$$

If the node $(\xi_i, t_j)\notin\tilde{\Gamma}$ then function b defined by (4.9) is continuous with the first derivative $\frac{\partial b}{\partial t}$ in the domain R_{ij}. Therefore

$$\frac{dB}{dt}(t)=\frac{1}{h}\int_{\xi_{i-\frac{1}{2}}}^{\xi_{i+\frac{1}{2}}}\frac{\partial b}{\partial t}(\xi,t)\,d\xi\ \text{ for } t\in[t_{j-1},t_j]$$

and

$$|B(t_j)-B(t_{j-1})|\leqslant c_6\,\tau \tag{5.22}$$

where

$$c_6=\max\left\{\max_{(\xi,t)\in\operatorname{cl}\tilde{D}_i}\left|\frac{\partial b_i}{\partial t}(\xi,t)\right|,\ i=1,2\right\}=\frac{2\bar{c}}{l}\max\left\{\frac{\gamma_1}{\alpha_1},\frac{\gamma_2}{\alpha_2}\right\}.$$

If $(\xi_i, t_j)\in\tilde{\Gamma}$ then we have

$$\frac{dB}{dt}(t)=\frac{1}{h}\int_{\xi_{i-\frac{1}{2}}}^{l/2}\frac{\partial b_1}{\partial t}(\xi,t)\,d\xi+\frac{1}{h}\int_{l/2}^{\xi_{i+\frac{1}{2}}}\frac{\partial b_2}{\partial t}(\xi,t)\,d\xi\ \text{ for } t\in[t_{j-1},t_j]$$

so that the estimate (5.22) also holds. Taking into account the definitions of E_1^j, E_2^j we can easily check that conditions (v) are fulfilled with constants $c_7=c_3$, $c_8=c_4/c_3$.

To satisfy condition (vi) we have been obliged to postulate the mentioned above assumption (H9) relating to functions σ_1, σ_2. Such an assumption excludes the case when both boundary conditions at $x=0$ and $x=l$ are the Neumann conditions. Thus we have verified that conditions (i)–(vi) are actually fulfilled.

In order to make use of the a priori-estimates (5.19), (5.20) we shall estimate $\|\psi^j\|_4$, $|v_1^j|$, $|v_2^j|$ for $j=1, \ldots, L$. The following lemma is true (see Appendix A):

LEMMA 5.1. If conditions (H8)–(H10) are fulfilled then

$$\|\psi^j\|_4 = 0\,(h+\tau), \quad j=1, \ldots, L, \tag{5.23}$$

$$|v_k^j| = 0\,(h^2+\tau), \quad k=1, 2, \; j=1, \ldots, L. \tag{5.24}$$

By (5.23), (5.24) we obtain from (5.19), (5.20) the following result

THEOREM 5.1. Assume that there exists a unique solution of problem (4.8)–(4.11), (5.1) and conditions (H8)–(H10) are satisfied. Then the difference scheme (5.6)–(5.9) is uniformly convergent to the solution of problem (4.8)–(4.11), (5.1) with the rate of convergence $0\left(h \ln^\delta \frac{1}{h} + \tau \ln^\delta \frac{1}{\tau}\right)$ where $\delta=1+\varepsilon$ and ε is any arbitrary positive constant, i.e. for sufficiently small h, τ $(h \leqslant h^*,\ \tau \leqslant \tau^*)$

$$\max_{(\xi_i, t_j) \in \omega_{h\tau}} |U_i^j - \tilde{u}\,(\xi_i, t_j)| \leqslant M \left(h \ln^\delta \frac{1}{h} + \tau \ln^\delta \frac{1}{\tau}\right) \tag{5.25}$$

where M is a positive constant independent of h and τ.

Now we are in a position to pass on to the second stage of constructing finite-difference schemes corresponding to Problems (B_k^S), $k=1, 2$. The estimate (5.25) will be of great importance in our further considerations.

Stage II

We will make use of Problems (B_k^S) in the transformed form (4.8)–(4.13). In order to apply the difference scheme (5.6)–(5.9) to Problems (B_k^S) it is necessary to compute values of function y and its derivative y' at discrete points t_j, $j=1, \ldots, L$, by solving the ordinary differential equation (4.13). We are going to use two different methods of approximating equation (4.13) at each time step: the extrapolation Euler-Cauchy method and the predictor-corrector routines in Euler-Cauchy, Milne or Hamming versions [12].

First we shall describe the algorithm where the extrapolation Euler-Cauchy method is used for solving equation (4.13). Let U_i^j, Y^j, $(Y')^j$ denote respectively approximate values of $\tilde{u}\,(\xi_i, t_j)$, $y\,(t_j)$, $y'\,(t_j)$ and $V^j = U_{N/2}^j$ be corresponding approximation to $\tilde{v}\,(t_j)$.

If the approximate values of solution at t_{j-1} are known then for determining respective values at t_j we compute Y^j on the basis of Euler-Cauchy formula, i.e.

$$Y^j = Y^{j-1} + \tau\,(Y')^{j-1}.$$

To this end first we have to determine $(Y')^{j-1}$.

To do that we will apply a regularization method whose aim is to assure a priori boundedness of the discrete approximations to second derivative of the function y

(see Part II). Idea of this method is the following. For given h and τ ($h=l/N$, $\tau=T/L$) we introduce regularization parameter Q

$$Q \triangleq \min \{\tilde{Q} \in \mathcal{N} \mid \tilde{Q} \geqslant Q_{12} \text{ and } L/\tilde{Q} \in \mathcal{N}\} \tag{5.26}$$

where

$$Q_{12} \triangleq \begin{cases} \max\{Q_1, Q_2\} & \text{if } h \leqslant h^* \text{ and } \tau \leqslant \tau^* \\ 1 & \text{otherwise,} \end{cases}$$

$$Q_1 \triangleq E(h^{1-\varepsilon_1}\tau^{-1})+1, \quad Q_2 \triangleq E(\tau^{-\varepsilon_2})+1$$

and ε_1, ε_2 are arbitrarily chosen constants from the interval (0, 1).

We define

$$K=L/Q. \tag{5.27}$$

Observe that $Q\to\infty$, $K\to\infty$, $Q\tau\to 0$ when $h, \tau\to 0$. $(Y')^j$ will be calculated as follows:

$$(Y')^0=\beta V^0, \quad (Y')^{rQ}=\beta V^{rQ-1} \text{ for } r=1, \dots, K,$$

$$(Y')^j=(Y')^{(r-1)Q} \text{ for } (r-1)Q+1 \leqslant j \leqslant rQ-1.$$

Such a method makes it possible to prove convergence of the algorithm described below.

Finite-difference scheme (1). *Algorithm without iterations*

Given: h, τ, Q;

$$Y^0=y_o, \quad U_i^0=\tilde{u}_o(\xi_i), \quad i=0, 1, \dots, N;$$

$$\text{set } V^0=U^0_{N/2}, \quad (Y')^0=\beta V^0, \quad j=1, \quad r=1.$$

Step 1.

(a) Set $Y^j=Y^{j-1}+\tau(Y')^{j-1}$.

(b) If $y_m<Y^j<y_M$ then go to 1 (c), if not then STOP.

(c) If $j<rQ$ then set $(Y')^j=(Y')^{(r-1)Q}$, otherwise set $(Y')^j=\beta V^{j-1}$ and $r\leftarrow r+1$.

Step 2.

(a) Compute A_i^j, B_i^j, C_i^j, $i=1, \dots, N-1$ on the basis of expressions (5.11) with

$$a(\xi, t_j)=\begin{cases} \dfrac{\gamma_1 l}{2Y^j} & \text{for } \xi\in\left[0, \dfrac{l}{2}\right) \\ \dfrac{\gamma_2 l}{2(l-Y^j)} & \text{for } \xi\in\left[\dfrac{l}{2}, l\right], \end{cases}$$

$$b(\xi, t_j)=\begin{cases} \dfrac{2\gamma_1}{\alpha_1 l} Y^j & \text{for } \xi\in\left[0, \dfrac{l}{2}\right) \\ \dfrac{2\gamma_2}{\alpha_2 l}(l-Y^j) & \text{for } \xi\in\left[\dfrac{l}{2}, l\right], \end{cases} \tag{5.28}$$

$$c(\xi, t_j)=\begin{cases} \dfrac{4}{\alpha_1 l^2} Y^j (Y')^j \xi & \text{for } \xi\in\left[0, \dfrac{l}{2}\right) \\ \dfrac{4}{\alpha_2 l^2}(l-Y^j)(Y')^j(l-\xi) & \text{for } \xi\in\left[\dfrac{l}{2}, l\right]. \end{cases}$$

(b) Compute U_i^j, $i=0, 1, ..., N$, by solving the system of difference equations

$$\mathscr{F}_1 [U_i^j]=0, \quad i=1, ..., N-1 \tag{5.29}$$

with conditions:

for Problem (B_1^S):

$$U_0^j=F_1(t_j), \quad U_N^j=F_2(t_j), \tag{5.30}$$

for Problem (B_2^S):

$$\begin{aligned}(A_1^j-D_1^j)\frac{U_1^j-U_0^j}{h}-E_1^j\frac{U_0^j-U_0^{j-1}}{\tau}&=\varphi_1^j\\ (A_N^j+D_2^j)\frac{U_N^j-U_{N-1}^j}{h}+E_2^j\frac{U_N^j-U_N^{j-1}}{\tau}&=\varphi_2^j\end{aligned} \tag{5.31}$$

where D_k^j, E_k^j, $k=1, 2$, are defined by (5.10), (5.28).

To this end we apply the Gauss elimination method leading for three-diagonal matrices to so called outstrip formulas [13], very efficient and suitable for automatic computation.

(c) Set $V^j=U_{N/2}^j$.

(d) If $j<L$ then go to Step 3, otherwise STOP.

Step 3. $j \leftarrow j+1$ and return to Step 1.

The above algorithm may be used in an iterative version. Then the ordinary differential equation (4.13) is solved by means of a predictor-corrector routine. The modification of the algorithm presented, to obtain its iterative version, is obvious (see [10] for details).

APPENDIX A

Proof of Lemma 5.1.

First we rewrite the expression (5.17) defining ψ_i^j in other form. After adding by the sides (5.17) and (5.4) ψ_i^j can be expressed as follows

$$\psi_i^j=(\mu_x)_i^j+\eta_i^j+\zeta_i^j, \quad i=1, ..., N-1, \quad j=1, ..., L \tag{A.1}$$

where

$$(\mu_x)_i^j=\frac{\mu_{i+1}^j-\mu_i^j}{h},$$

$$\mu_i^j=A_i^j\frac{\tilde{u}_i^j-\tilde{u}_{i-1}^j}{h}-\frac{1}{\tau}\int_{t_{j-1}}^{t_j} w(\xi_{i-\frac{1}{2}}, t)\, dt, \tag{A.2}$$

$$\eta_i^j=\frac{1}{h\tau}\int_{t_{j-1}}^{t_j}\int_{\xi_{i-\frac{1}{2}}}^{\xi_{i+\frac{1}{2}}}\left[b(\xi, t)\frac{\partial\tilde{u}}{\partial t}(\xi, t)-b(\xi, t_j)\frac{\partial\tilde{u}}{\partial t}(\xi_i, t)\right]d\xi\, dt, \tag{A.3}$$

$$\zeta_i^j=\frac{1}{h\tau}\int_{t_{j-1}}^{t_j}\int_{\xi_{i-\frac{1}{2}}}^{\xi_{i+\frac{1}{2}}}\left\{c(\xi,t_j)\left[\kappa A_{i+1}^j\frac{\tilde{u}_{i+1}^j-\tilde{u}_i^j}{h}+(1-\kappa)A_i^j\frac{\tilde{u}_i^j-\tilde{u}_{i-1}^j}{h}\right]-\right.$$
$$\left.-c(\xi,t)\,w(\xi,t)\right\}d\xi\,dt. \qquad (A.4)$$

Now we are going to show that if conditions (H8), (H10) are satisfied then

$$\mu_i^j=\begin{cases}0(h^2+\tau) & \text{for} \quad (i,j)\in\Lambda\\ 0(h+\tau) & \text{for} \quad (i,j)\in\Lambda_\Gamma,\end{cases}$$

$$\eta_i^j=\begin{cases}0(h+\tau) & \text{for} \quad (i,j)\in\Lambda\\ 0(1) & \text{for} \quad (i,j)\in\Lambda_\Gamma,\end{cases} \qquad (A.5)$$

$$\zeta_i^j=\begin{cases}0(h+\tau) & \text{for} \quad (i,j)\in\Lambda\\ 0(1) & \text{for} \quad (i,j)\in\Lambda_\Gamma\end{cases}$$

where

$$\Lambda_\Gamma\triangleq\left\{(i,j)|i=\frac{N}{2},\quad j=1,\ldots,L\right\},$$

$$\Lambda\triangleq\left\{(i,j)|i=1,\ldots,\frac{N}{2}-1,\ \frac{N}{2}+1,\ldots,N-1,\ j=1,\ldots,L\right\}.$$

First let us estimate μ_i^j for $(i,j)\in\Lambda$. In view of the definition of A_i^j we can rewrite the expression (A.2) in the form

$$\mu_i^j=\frac{1}{\tau}\int_{t_{j-1}}^{t_j}\left\{\left[\frac{1}{h}\int_{\xi_{i-1}}^{\xi_i}\frac{d\xi}{a(\xi,t_j)}\right]^{-1}\frac{\tilde{u}_i^j-\tilde{u}_{i-1}^j}{h}-w(\xi_{i-\frac{1}{2}},t)\right\}dt. \qquad (A.6)$$

By the assumed regularity of functions a and $\tilde{u}$ we have

$$A_i^j\triangleq\left[\frac{1}{h}\int_{\xi_{i-1}}^{\xi_i}\frac{d\xi}{a(\xi,t_j)}\right]^{-1}=a(\xi_{i-\frac{1}{2}},t_j)+0(h^2) \quad \text{for} \quad (i,j)\in\Lambda \qquad (A.7)$$

$$\frac{\tilde{u}_i^j-\tilde{u}_{i-1}^j}{h}=\frac{\partial\tilde{u}}{\partial\xi}(\xi_{i-\frac{1}{2}},t_j)+0(h^2) \quad \text{for} \quad (i,j)\in\Lambda. \qquad (A.8)$$

After putting (A.7) and (A.8) into (A.6) we get

$$\mu_i^j=\frac{1}{\tau}\int_{t_{j-1}}^{t_j}[w(\xi_{i-\frac{1}{2}},t_j)-w(\xi_{i-\frac{1}{2}},t)+0(h^2)]\,dt.$$

Hence in view of the assumption (H10) relating to function w we get

$$\mu_i^j=0(h^2+\tau) \quad \text{for} \quad (i,j)\in\Lambda.$$

To estimate μ_i^j for $(i,j)\in \Lambda_\Gamma$ let us rewrite expression (A.2) in the form

$$\mu_i^j=\frac{1}{\tau}\int\limits_{t_{j-1}}^{t_j}\frac{A_i^j}{h}\int\limits_{\xi_{i-1}}^{\xi_i}\left[\frac{\partial\tilde{u}}{\partial\xi}(\xi,t_j)-w(\xi_{i-\frac{1}{2}},t)\right]d\xi\,dt. \tag{A.9}$$

Taking into account the equality

$$\frac{A_i^j}{h}\int\limits_{\xi_{i-1}}^{\xi_i}\frac{w(\xi_{i-\frac{1}{2}},t)}{a(\xi,t_j)}d\xi=w(\xi_{i-\frac{1}{2}},\tau) \tag{A.10}$$

and (A.9) we obtain

$$\mu_i^j=\frac{1}{\tau}\int\limits_{t_{j-1}}^{t_j}\left[\frac{A_i^j}{h}\int\limits_{\xi_{i-1}}^{\xi_i}\frac{w(\xi,t_j)-w(\xi_{i-\frac{1}{2}},t)}{a(\xi,t_j)}d\xi\right]dt. \tag{A.11}$$

Since w is a function continous in $\mathrm{cl}\,\tilde{D}$ as well as it satisfies Lipschitz continuity condition in $\mathrm{cl}\,\tilde{D}_i$, $i=1,2$, with respect to ξ and t, we conclude that

$$|w(\xi,t_j)-w(\xi_{i-\frac{1}{2}},t)|=0(h+\tau) \tag{A.12}$$

for $(\xi,t)\in\{(\xi,t)|\xi\in[\xi_{i-1},\xi_i],\ t\in[t_{j-1},t_j]\}$, $(i,j)\in\Lambda_\Gamma$.

Combining (A.11) and (A.12) as well as the definition of A_i^j we get

$$\mu_i^j=0(h+\tau)\quad \text{for}\quad (i,j)\in\Lambda_\Gamma.$$

Now we are going to estimate η_i^j. Due to the assumed regularity of functions b, $\frac{\partial\tilde{u}}{\partial t}$ in $\mathrm{cl}\,\tilde{D}_i$, $i=1,2$, we conclude that

$$\eta_i^j=0(h+\tau)\ \text{for}\ (i,j)\in\Lambda.$$

From the fact that functions b and $\frac{\partial\tilde{u}}{\partial t}$ have discontinuity of the first kind along the line $\tilde{\Gamma}$ we get

$$b(\xi,t)\frac{\partial\tilde{u}}{\partial t}(\xi,t)-b(\xi,t_j)\frac{\partial\tilde{u}}{\partial t}(\xi_i,t)=0(1) \tag{A.13}$$

in the sets R_{ij}, $(i,j)\in\Lambda_\Gamma$.

From (A.13) and (A.3) it follows that $\eta_i^j=0(1)$ for $(i,j)\in\Lambda_\Gamma$. By discontinuity of the function c on $\tilde{\Gamma}$

$$c(\xi,t_j)\left[\kappa A_{i+1}^j\frac{\tilde{u}_{i+1}^j-\tilde{u}_i^j}{h}+(1-\kappa)A_i^j\frac{\tilde{u}_i^j-\tilde{u}_{i-1}^j}{h}\right]-c(\xi,t)\,w(\xi,t)=0(1)$$

for $(\xi,t)\in R_{ij}$, $(i,j)\in\Lambda_\Gamma$.

That is why $\zeta_i^j=0(1)$ for $(i,j)\in\Lambda_\Gamma$.

There are still to estimate ζ_i^j for $(i,j)\in\Lambda$. Let us observe that by (A.7) and (A.8)

$$A_i^j\frac{\tilde{u}_i^j-\tilde{u}_{i-1}^j}{h}=w(\xi_{i-\frac{1}{2}},t_j)+0(h^2). \tag{A.14}$$

After substituting (A.14) into (A.4) we get $\zeta_i^j=0\,(h+\tau)$ for $(i,j)\in\Lambda$. Thus we have proved that all the estimates (A.5) hold.

Now we are going to show (5.23). First let us estimate $\|\psi^j\|_3$. To this end observe that by (A.5) we immediately get

$$\|(\mu_x)^j\|_3^2=\sum_{i=1}^{N-1} h\Big[\sum_{k=1}^{i} h(\mu_x)_k^j\Big]^2=\sum_{i=1}^{N-1} h(\mu_{i+1}^j-\mu_1^j)^2\leqslant$$

$$\leqslant 2\sum_{i=1}^{N-1} h\,[(\mu_1^j)^2+(\mu_{i+1}^j)^2]=2\Big[\sum_{i=1}^{N-1} h(\mu_1^j)^2+\sum_{\substack{i=2\\ i\neq N/2}}^{N-1} h(\mu_i^j)^2+h(\mu_{N/2}^j)^2\Big]=$$

$$=2\,[(N-1)\,h\,0\,(h^4+\tau^2)+(N-3)\,h\,0\,(h^4+\tau^2)+h\,0\,(h^2+\tau^2)]=0\,(h^3+\tau^2), \quad \text{(A.15)}$$

$$\|\eta^j\|_3\leqslant 2\sqrt{l}\,\|\eta^j\|_1=2\sqrt{l}\Big[\sum_{\substack{i=1\\ i\neq N/2}}^{N-1} h\,|\eta_i^j|+h\,|\eta_{N/2}^j|\Big]=$$

$$=2\sqrt{l}\,[l\,0\,(h^2+\tau)+h\,0\,(1)]=0\,(h+\tau), \quad \text{(A.16)}$$

$$\|\zeta^j\|_3\leqslant 2\sqrt{l}\,\|\zeta^j\|_1=2\sqrt{l}\Big[\sum_{\substack{i=1\\ i\neq N/2}}^{N-1} h\,|\zeta_i^j|+h\,|\zeta_{N/2}^j|\Big]=$$

$$=2\sqrt{l}\,[l\,0\,(h+\tau)+h\,0\,(1)]=0\,(h+\tau). \quad \text{(A.17)}$$

By (A.15)–(A.17) we obtain

$$\|\psi^j\|_3=0\,(h+\tau) \quad \text{for} \quad j=1,\dots,L. \quad \text{(A.18)}$$

Now we shall estimate $\big|\sum_{i=1}^{N-1} h\psi_i^j\big|$. Observe that due to (A.5)

$$\Big|\sum_{i=1}^{N-1} h(\mu_x)_i^j\Big|=|\mu_N^j-\mu_1^j|\leqslant|\mu_N^j|+|\mu_1^j|=0\,(h^2+\tau), \quad \text{(A.19)}$$

$$\Big|\sum_{i=1}^{N-1} h\eta_i^j\Big|\leqslant\sum_{i=1}^{N-1} h\,|\eta_i^j|=\|\eta^j\|_1=0\,(h+\tau), \quad \text{(A.20)}$$

$$\Big|\sum_{i=1}^{N-1} h\zeta_i^j\Big|\leqslant\|\zeta^j\|_1=0\,(h+\tau). \quad \text{(A.21)}$$

Hence

$$\Big|\sum_{i=1}^{N-1} h\psi_i^j\Big|=0\,(h+\tau). \quad \text{(A.22)}$$

From (A.18) and (A.22) it follows (5.23). The estimates (5.24) follow from results of Samarskii [14]. Q.E.D.

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Aproksymacje różnicowe parabolicznych zagadnień brzegowych ze swobodną granicą opisujących dynamikę podziemnego zbiornika gazu. Część I. Algorytmy

W pracy zaproponowano aproksymacje różnicowe jednowymiarowych parabolicznych zagadnień brzegowych ze swobodną granicą, występujących przy modelowaniu podziemnego zbiornika gazu w warstwie wodonośnej. Przy konstruowaniu schematów różnicowych stosowano wstępną transfor-

mację zagadnienia ze swobodną granicą, pozwalającą sformułować równoważne mu nieliniowe zagadnienie paraboliczne w obszarze zadanym a priori. Proponowane schematy różnicowe należą do klasy tzw. schematów jednorodnych, zbilansowanych.

Разностные аппроксимации параболических задач со свободной границей, возникающих при моделировании подземного газохранилища. Часть I. Алгоритмы

В статье введены разностные аппроксимации одномерных параболических задач со свободной границей, возникающих при моделировании подземного газохранилища в водоносном пласте. При построении разностных схем использовано некоторое преобразование исходной задачи со свободной границей в краевую задачу с выпрямленной границей. Представленные разностные схемы принадлежат классу однородных консервативных схем.