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Multilevel Stackelberg Strategies With Many Players On Each Level

by

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This paper deals with hierarchical systems which consist of multiple independent decision makers on each level. Necessary conditions are derived for the open-loop Stackelberg solution in a three level linear-quadratic problem where the open-loop Nash solution is adopted for the interlevel decision problems.

1. Introduction

In the past few years increasing attention has been paid to nonzero-sum differential games. First interest lay in the analysis of the properties of single-level Nash and Stackelberg strategies with different information structures. A relatively complete list of references on these works can be found in [1–3]. The practical motivation for studying these generalized optimal control problems is mainly due to their potential applicability in decentralized economic control problems, see Refs [4–6].

From the point of view of management problem applications, more general organizational configurations are still needed. The formulation where each intermediate decision maker has one leader on the higher level and one follower on the lower describes only very simple hierarchies. Moreover, the coordinator's or leader's role for the upper level decision maker is not the only realistic possibility. In general there can be multiple decision makers on each level. In certain organizational forms the higher level decision maker can adopt the followers' role. Then he wouud represent a passive manager who sets objectives for the independent lower level units and who decides on his control actions only after he has been informed of the lower level decisions. Oligopolistic situations are typical examples of structures with multiple decision makers having the role of a leader on the same level.

In the present paper we shall extend the multilevel Stackelberg formulation to cases where each level may consist of multiple decision makers. This setting covers a wide class of different organizational forms, including those discussed above. The open-loop solutions to a three level case with linear system and quadratic criteria are considered. The players are assumed to employ Nash strategies within each level. The Stackelberg concept is used between the levels, which implies that the higher level players announce their controls to the lower level players in a hierarchical order. The reader should note that the problem description, where the leaders are assumed to be on the higher levels, does not restrict the use of the results in the opposite cases too.

2. Problem Formulation

Consider the following time-invariant linear system

$$\dot{x}=Ax+\sum_{l=1}^{N_1} B_l u_l+\sum_{l=1}^{N_2} C_l v_l+\sum_{l=1}^{N_3} D_l w_l, \tag{1}$$

where x is the state vector and u_i, $i=1, ..., N_1$, v_j, $j=1, ..., N_2$, and w_k, $k=1, ..., N_3$, denote the control vectors of the players on levels 1, 2, and 3, respectively. The numbers of players on the levels are N_1, N_2, and N_3. The performance criterion of player i on level j is given by

$$J_{ji}=\frac{1}{2} x^T(t_f) F_{ji} x(t_f)+\frac{1}{2}\int_{t_0}^{t_f}\Big(x^T Q_{ji} x+\sum_{l=1}^{N_1} u_l^T U_{jil} u_l+ + \sum_{l=1}^{N_2} v_l^T V_{jil} v_l+\sum_{l=1}^{N_3} w_l^T W_{jil} w_l\Big), \tag{2}$$

where the weighting matrices are assumed to be symmetric and satisfy the usual requirements on the positive definiteness and semi-definiteness.

The structure of the decision making problem is defined so that the decisions are made sequentially from level to level in the order 3, 2, 1 but parallelly within each level. Because the players within a level process their decisions at the same time, it is natural to assume the noncooperative Nash equilibrium solution concept in the interlevel optimization problems. Sequential decision making between the levels means that the players on level 3 select their controls first and let the players on level 2 and 1 know their decisions. Correspondingly the players on level 2 select their strategics next and announce the controls to the players on level 1, who make their decisions last. The players are assumed to use open-loop strategies so that the controls are functions only of the initial state.

3. Development of the Necessary Conditions

Let us initiate the solution procedure by considering the controls u_i, $i=1, .., N_1$, on level 1. The players on level 1 know the open-loop strategies of the players on levels 2 and 3 and thus the problem reduces to an N_1-player Nash differential game problem with given v_j, $j=1, \ldots, N_2$, and w_k, $k=1, \ldots, N_3$. The Hamiltonian for players i on level 1 is defined by

$$H_{1i}(x, t, u_1, \ldots, u_N, p_i)=\frac{1}{2}\left(x^T Q_{1i} x+\sum_{l=1}^{N_1} u_l^T U_{1il} u_l+\sum_{l=1}^{N_2} v_l^T V_{1il} v_l+\right.$$

$$\left.+\sum_{l=1}^{N_3} w_l^T W_{1il} w_l\right)+p_{1i}^T\left(Ax+\sum_{l=1}^{N_1} B_l u_l+\sum_{l=1}^{N_2} c_l v_l+\sum_{l=1}^{N_3} D_l w_l\right),$$

$$i=1, \ldots, N_1. \quad (3)$$

Necessary conditions for the open-loop Nash solution can be obtained by variational techniques and they result in the following set of equations:

$$\dot{x}=Ax+\sum_{l=1}^{N_1} B_l u_l+\sum_{l=1}^{N_2} C_l v_l+\sum_{l=1}^{N_3} D_l w_l, \qquad x(t_0)=x_0, \quad (4)$$

$$\dot{p}_{1i}=-Q_{1i} x-A^T p_{1i}, \; p_{1i}(t_f)=F_{1i} x(t_f), \qquad i=1, \ldots, N_1, \quad (5)$$

$$u_i=-U_{1ii}^{-1} B_i p_{1i}, \qquad i=1, \ldots, N_1. \quad (6)$$

In the solution each open-loop control u_i depends implicitly on the higher level controls v_j and w_k through the Lagrange multipliers p_i in (6).

The players on level 2 know the decisions of players on level 3 and they have to take into account the reactions of the players on level 1 to given functions v_j, $j=1, \ldots, N_2$, and w_k, $k=1, \ldots, N_3$. Thus (4)–(6) are additional constraints to the interlevel game on level 2. Substituting (6) into the system equation and in the players' performance criteria, there only remain N_1 extra differential constraints (5) to be taken into account. After the substitutions the Hamiltonian function for player j on level 2 can be written as follows

$$H_{2j}(x, p_{11}, \ldots, p_{1N_1}, t, v_1, \ldots, v_{N_2}, p_{2j}, q_{1j1}, \ldots, q_{1N_1})=$$

$$=\frac{1}{2}\left(x^T Q_{2j} x+\sum_{l=1}^{N_1} p_{1l}^T S_{2jl} p_{1l}+\sum_{i=1}^{N_2} v_l^T V_{2jl} v_l+\sum_{l=1}^{N_3} w_l^T W_{2jl} w_l\right)+$$

$$+p_{2j}^T\left(Ax-\sum_{l=1}^{N_1} S_l p_{1l}+\sum_{l=1}^{N_2} C_l v_l+\sum_{l=1}^{N_3} D_l w_l\right)+$$

$$+\sum_{l=1}^{N_1} q_{1jl}^T(-Q_{1l} x-A^T p_{1l}), \qquad j=1, \ldots, N_2, \quad (7)$$

where

$$S_{2ji}=B_i U_{1ii}^{-1} U_{2ji} U_{1ii}^{-1} B_i^T, \quad (8)$$

$$S_i=B_i U_{1ii}^{-1} B_i^T. \quad (9)$$

The necessary conditions characterizing the open-loop Nash solutions v_j of the N_2-player differential game on level 2 are again obtained by variational methods:

$$\dot{x}=Ax-\sum_{l=1}^{N_1} S_l p_{1l}+\sum_{l=1}^{N_2} C_l v_l+\sum_{l=1}^{N_3} D_l w_l, \; x(t_0)=x_0, \tag{10}$$

$$\dot{p}_{1i}=-Q_{1i}x-A^T p_{1i}, \; p_{1i}(t_f)=F_{1i}x(t_f), \; i=1,..,N_1, \tag{11}$$

$$\dot{p}_{2j}=-Q_{2j}x-A^T p_{2j}+\sum_{l=1}^{N_1} Q_{1l}q_{1jl}, \; p_{2j}(t_f)=$$

$$=F_{2j}x(t_f)-\sum_{l=1}^{N_1} F_{1l}q_{1jl}(t_f), \; j=1,...,N_2, \tag{12}$$

$$\dot{q}_{1ji}=-S_{2ji}p_{1i}+S_i p_{2j}+Aq_{1ji}, \; q_{1ji}(t_0)=0, \; i=1,...,N_1, \; j=1,...,N_2, \tag{13}$$

$$v_j=-V_{2jj}^{-1}C_j^T p_{2j}, \; j=1,...,N_2. \tag{14}$$

For given controls w_k, $k=1,...,N_3$, these equations constitute the necessary conditions for open-loop Stackelberg strategies between two groups of players. The dependence of v_j on w_k is due to the w_k terms in the system equation and the p_{2j} functions in (14).

The last step in the problem solution is to consider the decisions of the players on level 3. They have to take into account the reactions of the players on both level 1 and level 2 to given controls w_k, $k=1,...,N_3$, and additionally the reactions of the players on level 1 to the controls of the players on level 2. Thus the necessary conditions for the solution to the differential game between levels 1 and 2 have to be considered as constraints to the decision problems on level 3. This kind of constraint is already quite complex, yet tractable. The reactions are most easily taken into account by substituting the open-loop controls (6) and (14) into the system equation (10) and into the performance criteria J_{3k}, $k=1,..,N_3$, (2). In the resulting problem the performance criteria J_{3k} are explicit functions of the controls of the players on level 3 and of the system state x together with the additional constraint vectors p_{1i}, $i=1,...,N_1$, and p_{2j}, $j=1,...,N_2$. Besides the system equation with the related substitutions made there are additional differential constraints to the level 3 problem defined by the boundary value problem (11)–(13) which altogether make a total of $1+N_1+N_2+N_1N_2$ vector differential equations. The open-loop Nash solutions to the resulting differential game on level 3 can again be approached by variational methods by defining the Hamiltonians for each players. The notation of the arguments of H_{3k} is omitted for brevity.

$$H_{3k}=\frac{1}{2}\Big(x^T Q_{3k}x+\sum_{l=1}^{N_1} p_{1l}^T S_{3kl}p_{1l}+\sum_{l=1}^{N_2} p_{2l}^T R_{3kl}p_{2l}+$$

$$+\sum_{l=1}^{N_3} w_l^T W_{3kl}w_l\Big)+p_{3k}^T\Big(Ax-\sum_{l=1}^{N_1} S_l p_{1l}-\sum_{l=1}^{N_2} R_l p_{2l}+\sum_{l=1}^{N_3} D_l w_l\Big)+$$

$$+\sum_{l=1}^{N_1} q_{2kl}^T(-Q_{1l}x-A^Tp_{1l})+\sum_{l=1}^{N_2} q_{3kl}^T\Big(-Q_{2l}x+$$

$$-A^Tp_{2l}+\sum_{i=1}^{N_1} Q_{1i}q_{1li}\Big)+\sum_{l=1}^{N_2}\sum_{i=1}^{N_1} r_{kli}(-S_{2li}p_{1i}+$$

$$+S_ip_{2j}+Aq_{1li}),\ k=1,\ldots,N_3, \tag{15}$$

where

$$S_{3ki}=B_i U_{1ii}^{-1} U_{3ki} U_{1ii}^{-1} B_i^T, \tag{16}$$

$$R_{3kj}=C_j V_{2jj}^{-1} V_{3kj} V_{2jj} C_j^T, \tag{17}$$

$$R_j=C_j V_{2jj}^{-1} C_j^T. \tag{18}$$

The following final necessary conditions are obtained:

$$\dot{x}=Ax-\sum_{l=1}^{N_1} S_l p_{1l}-\sum_{l=1}^{N_2} R_l p_{2l}-\sum_{l=1}^{N_3} T_l p_{3l},\quad x(t_0)=x_0, \tag{19}$$

$$\dot{p}_{1i}=-Q_{1i}x-A^Tp_{1i},\ p_{1i}(t_f)=F_{1i}x(t_f),\ i=1,\ldots,N_1, \tag{20}$$

$$\dot{p}_{2j}=-Q_{2j}x-A^Tp_{2j}+\sum_{l=1}^{N_1} Q_{1l}q_{1jl},\ p_{2j}(t_f)=F_{2j}x(t_f)+$$

$$-\sum_{l=1}^{N_1} F_{1l}q_{1jl}(t_f),\ j=1,\ldots,N_2, \tag{21}$$

$$\dot{q}_{1ji}=-S_{2ji}p_{1i}+S_ip_{2j}+Aq_{1ji},\ q_{1ji}(t_0)=0,\ j=1,\ldots,N_2,\ i=1,\ldots,N_1, \tag{22}$$

$$\dot{p}_{3k}=-Q_{3k}x-A^Tp_{3k}+\sum_{l=1}^{N_1} Q_{1l}q_{2kl}+\sum_{l=1}^{N_2} Q_{2l}q_{3kl},$$

$$p_{3k}(t_f)=F_{3k}x(t_f)-\sum_{l=1}^{N_1} F_{1l}q_{2kl}(t_f)-\sum_{l=1}^{N_2} F_{2l}q_{3kl}(t_f),\ k=1,\ldots,N_3 \tag{23}$$

$$\dot{q}_{2ki}=-S_{3ki}p_{1i}+S_ip_{3k}+Aq_{2ki}+\sum_{l=1}^{N_2} S_{2li}r_{kli},$$

$$q_{2ki}(t_0)=0,\ k=1,..,N_3,\ i=1,\ldots,N_1, \tag{24}$$

$$\dot{q}_{3kj}=-R_{3kj}p_{2j}+R_jp_{3k}+Aq_{3kj}-\sum_{l=1}^{N_1} S_l r_{kjl},$$

$$q_{3kj}(t_0)=0,\ k=1,\ldots,N_3,\ j=1,\ldots,N_2, \tag{25}$$

$$\dot{r}_{kji}=-Q_1q_{3kj}-A^Tr_{kji},\ r_{kji}(t_f)=F_{1i}q_{3kj}(t_f),$$

$$k=1,\ldots,N_3,\ j=1,\ldots,N_2,\ i=1,\ldots,N_1, \tag{26}$$

$$w_k=-T_kp_{3k},\ k=1,\ldots,N_3, \tag{27}$$

where

$$T_k=D_k W_{3kk}^{-1} D_k^T. \tag{28}$$

Thus we have a linear two-point boundary value problem for a system of $1+N_1+$ $+N_2+N_3+N_1 N_2+N_1 N_3+N_2 N_3+N_1 N_2 N_3$ vector differential equations. Assuming that all the Lagrange multiplier vectors are linear functions of the system state, the problem can be converted into a nonlinear matrix boundary value problem for the related coefficient matrices. It is clear that the dimensionality of this problem grows very rapidly as the number of players on each level increases. Thus even for very simple configurations the computational difficulties in the solution procedure are likely to be considerable.

A comparison between the necessary conditions obtained for the present system with multiple interlevel players and those developed in [11] for the simpler case with one decision maker on each level shows that the principal boundary value problem structure of the set of equations is not changed. However, the interconnections between the equations and the boundary conditions are more complicated and the number of equations is here essentially greater.

4. A game With Four Players

To get a clearer picture of the structure of the necessary conditions in question and of the solution of the resulting problem let us consider a special case where $N_1=1$, $N_2=2$, and $N_3=1$. In this three-level system there are two decision makers on the intermediate level and only one on the top and bottom levels. Now by omitting the unnecessary indices the set of equations (19)–(26) reduces to the following form

$$\dot{x}=Ax-S_1 p_1-R_1 p_{21}-R_2 p_{22}-T_1 p_3, \quad x(t_0)=x_0, \tag{29}$$

$$\dot{p}_1=-Q_1 x-A^T p_1, \quad p_1(t_f)=F_1 x(t_f), \tag{30}$$

$$\dot{p}_{21}=-Q_{21} x-A^T p_{21}+Q_1 q_{11}, \quad p_{21}(t_f)=F_{21} x(t_f)-F_1 q_{11}(t_f), \tag{31}$$

$$\dot{p}_{22}=-Q_{22} x-A^T p_{22}+Q_1 q_{12}, \quad p_{22}(t_f)=F_{22} x(t_f)-F_1 q_{12}(t_f), \tag{32}$$

$$\dot{q}_{11}=-S_{211} p_1+S_1 p_{21}+Aq_{11}, \quad q_{11}(t_0)=0, \tag{33}$$

$$\dot{q}_{12}=-S_{221} p_1+S_1 p_{22}+Aq_{12}, \quad q_{12}(t_0)=0, \tag{34}$$

$$\begin{aligned}\dot{p}_3=-Q_3 x-A^T p_3+Q_1 q_2+Q_{21} q_{31}+Q_{22} q_{32},&\\ p_3(t_f)=F_3 x(t_f)-F_1 q_2(t_f)-F_{21} q_{31}(t_f)-F_{22} q_{32}(t_f),&\end{aligned} \tag{35}$$

$$\dot{q}_2=-S_{311} p_1+S_1 p_3+Aq_2+S_{211} r_1+S_{221} r_2, \quad q_2(t_0)=0, \tag{36}$$

$$\dot{q}_{31}=-R_{311} p_{21}+R_1 p_3+Aq_{31}-S_1 r_1, \quad q_{31}(t_0)=0, \tag{37}$$

$$\dot{q}_{32}=-R_{312} p_{22}+R_2 p_3+Aq_{32}-S_1 r_2, \quad q_{32}(t_0)=0, \tag{38}$$

$$\dot{r}_1=-Q_1 q_{31}-A^T r_1, \quad r_1(t_f)=F_1 q_{31}(t_f), \tag{39}$$

$$\dot{r}_2=-Q_1 q_{32}-A^T r_2, \quad r_2(t_f)=F_1 q_{32}(t_f). \tag{40}$$

The solution to this linear TPBVP can be sought by employing the usual method of assuming linear relationships between the Lagrange multipliers and the state vector.

$$p_1=K_1\,x,\ p_{21}=K_{21}\,x,\ p_{22}=K_{22}\,x,\ p_3=K_3\,x, \tag{41}$$

$$q_{11}=L_{11}\,x,\ q_{12}=L_{12}\,x,\ q_2=L_2\,x,\ q_{31}=L_{31}\,x,\ q_{32}=L_{32}\,x \tag{42}$$

$$r_1=M_1\,x,\ r_2=M_2\,x. \tag{43}$$

Using these notations (41)–(43) the necessary conditions (29)–(40) transform into the following nonlinear matrix two-point boundary value problem

$$\dot K_1=-K_1\,A-A^T\,K_1+K_1\,S_1\,K_1+K_1\,R_1\,K_{21}+K_1\,R_2\,K_{22}+K_1\,T_1\,K_3,\qquad K_1\,(t_f)=F_1, \tag{44}$$

$$\dot K_{2j}=-K_{2j}\,A-A^T\,K_{2j}+K_{2j}\,S_1\,K_1+K_{2j}\,R_1\,K_{21}+K_{2j}\,R_2\,K_{22}+ +K_{2j}\,T_1\,K_3-Q_{2j}+Q_1\,L_{1j},\ K_{2j}\,(t_f)=F_{2j}-F_1\,L_{1j}\,(t_f),\ j=1,2, \tag{45}$$

$$\dot K_3=-K_3\,A-A^T\,K_3+K_3\,S_1\,K_1+K_3\,R_1\,K_{21}+K_3\,R_2\,K_{22}+K_3\,T_1\,K_3- -Q_3+Q_1\,L_2+Q_{21}\,L_{31}+Q_{21}\,L_{32},\ K_3\,(t_f)=F_3-F_1\,L_2\,(t_f)- -F_{21}\,L_{31}\,(t_f)-F_{22}\,L_{32}\,(t_f), \tag{46}$$

$$\dot L_{1j}=-L_{1j}\,A+AL_{1j}+L_{1j}\,S_1\,K_1+L_{1j}\,R_1\,K_{21}+L_{1j}\,R_2\,K_{22}+L_{1j}\,T_1\,K_3+ -S_{2j1}\,K_1+S_1\,K_{2j},\ L_{1j}\,(t_0)=0,\ j=1,2, \tag{47}$$

$$L_2=-L_2\,A+AL_2+L_2\,S_1\,K_1+L_2\,R_1\,K_{21}+L_2\,R_2\,K_{22}+L_2\,T_1\,K_3- -S_{311}\,K_1+S_1\,K_3+S_{211}\,M_1+S_{221}\,M_2,\ L_2\,(t_0)=0, \tag{48}$$

$$\dot L_{3j}=-L_{3j}\,A+AL_{3j}+L_{3j}\,S_1\,K_1+L_{3j}\,R_1\,K_{21}+L_{3j}\,R_2\,K_{22}+L_{3j}\,T_1\,K_3+ -R_{31j}\,K_{2j}+R_j\,K_3-S_1\,M_j,\ L_{3j}\,(t_0)=0,\ j=1,2, \tag{49}$$

$$\dot M_j=-M_j\,A-A^T\,M_j+M_j\,S_1\,K_1+M_j\,R_1\,K_{21}+M_j\,R_2\,K_{22}+M_j\,T_1\,K_3- -Q_1\,L_{3j},\quad M_j\,(t_f)=F_1\,L_{3j}\,(t_f),\ j=1,2. \tag{50}$$

The open-loop Stackelberg strategies for each player can now be determined in terms of the solution of (44)–(50)

$$u=-M_{11}^{-1}\,B_1^T\,K_1\,\Phi\,(t,t_0)\,x_0, \tag{51}$$

$$v_j=-V_{jj}^{-1}\,C_j^T\,K_{2j}\,\Phi\,(t,t_0)\,x_0,\ j=1,2, \tag{52}$$

$$w=-W_{11}^{-1}\,D_1^T\,K_3\,\Phi\,(t,t_0)\,x_0, \tag{53}$$

where $\Phi\,(t, t_0)$ is the state transition matrix of the system

$$\dot x=(A-S_1\,K_1-R_1\,K_{21}-R_2\,K_{22}-T_1\,K_3)\,x. \tag{54}$$

It is observed that the solution of a multilevel game with more than one player on each level can be brought into a relatively neat two-point boundary system problem which can, in principle, be solved in a straightforward manner. However, the practical iterative computation may often turn out to be cumbersome. The iterative procedure needed in the solution of the boundary value problem can be avoided by making a Riccati-type transformation which decouples the solution to successive initial value problems, see Refs. [11, 13].

5. Conclusion

Hierarchical systems have been subject to considerable interest for many years already. Little attention has been paid to the framework of differentail game theory in this context until quite recently. Generalizations of the methodology in this direction have potential importance from the point of view of the analysis of organizational decision making systems in society and in various kinds of economic units. In this paper necessary conditions have been developed for a three level system with multiple decision makers on each level using a linear system model and quadratic performance measures. A number of interesting organizational forms, including decentralized and oligopolistic structures, can be embedded in this formulation. Using the discrete minimum principle corresponding necessary conditions, such as were obtained in this paper, can also be developed for the open-loop solution in a discrete time system.

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Wielopoziomowe strategie Stackelberga z wieloma graczami na każdym poziomie

Rozpatrzono systemy hirarchiczne złożone z wielu niezależnych decydentów na każdym poziomie. Wyznaczono warunki konieczne dla programowanego rozwiązania Stackelberga w trzypoziomowym zagadnieniu liniowo-kwadratowym przyjmując rozwiązanie programowane Nasha dla międzypoziomowego problemu decyzyjnego.

Многоуровневые стратегии Стакельберга со многими игроками на каждом уровне

В работе рассматриваются иерархические системы состоящие из многих независимыя объектов, принимающих решение, на каждом уровне. Определяются необходимые услових для программируемого решения Стакельберга в трехуровневой линейно-квадратной задаче. принимая программируемое решение Неша для межуровневой задачи прияатия решения,