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Morrison-type Algorithms For Constrained Optimization *). Part I

by

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This paper discusses the properties, convergence conditions and the speed of convergence of certain penalty-functional techniques. The numerical comparison of these algorithms with the SUMT and the shifted penalty technique is provided.

1. Introduction

In 1968 Morrison [17] observed that the solution to the problem

$$\text{minimize } f(x) \text{ subject to } g_i(x) \leqslant 0, \ i=1, \ldots, n \tag{1}$$

with $f, g_1 : R^m \to R$ can be found by minimizing the functional

$$I(x, M_k) = (f(x) - M_k)^2 + \sum_{i=1}^{n} g_i(x) \max(0, g_i(x)) \tag{2}$$

for a sequence of parameters M_k converging from below to the optimal value of f in (1). This method has been further investigated theoretically and numerically by several authors. In particular, linear convergence of Morrison's algorithm was proven for the convex case [8, 14]. A variant of the method was compared numerically with SUMT [1] algorithm of Fiacco-McCormick in [12, 18], but the results were contradictory. An interesting application is presented in [19] for finding time-optimal control.

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*) This paper was partly prepared during the first author's visit to the Case Western Reserve University, Department of Systems Engineering, Computer Engineering and Information Science, Clevelant, Ohio 44106.

[1]) Through the paper by SUMT we denote the ordinary penalty function method.

Grzegorski [9, 10, 11] described a particular extension of the Morrison algorithm to more general infinite-dimensional problems subject to the constraint

$$g(x) \leqslant 0 \tag{3}$$

with $g \geqslant 0$. Constraints of the type (1) are here included with

$$g(x) = \sum_{i=1}^{n} g_i(x) \max(0, g_i(x)). \tag{4}$$

He applied $I(\cdot)$ of the form

$$I(x, M_k) = p\,(f(x) - M_k)^r + g(x), \quad r, p > 0, \tag{5}$$

and observed that the convergence for large r should be better in the vicinity of optimum, but worse farther from the solution. Also Razumikhin [19] indicated the possibility of tuning r in (5) in order to accelerate the convergence.

The present paper discusses properties of the functional $I(\cdot, \cdot)$ more general than (1) or (5) and convergence of three variants of Morrison's algorithm, also in the absence of convexity.

Convergence estimates are provided for those algorithms without assuming the existence of Kuhn-Tucker vectors. It appears, that the order of convergence for Morrison method based on (4) and (5) is proportional to $r-1$, and thus, might be higher than one. This serves as a basis for a new algorithm with variable exponent r.

Finally, the numerical performance of described algorithms is compared with that of SUMT [6] and the shifted penalty technique [1, 23]. Morrison-type algorithms perform considerably better than SUMT and seem to be comparable with the very efficient shifted penalty method, their logic being less complicated.

Part of the results presented here appeared in the second author's M. Sc. Thesis [16]. The rest has been developed during the other author's stay in the Department of Systems Engineering, Computer Engineering and Information Science at the Case Western Reserve University. The friendly and stimulating atmosphere which accompanied this research, especially the help of Professor Stephen Kahne is here gratefully acknowledged.

2. Properties of the Functional I $(\cdot, \cdot)$

Let X be an arbitrary set, f and g two functionals over X with g nonnegative. For $\gamma \geqslant 0$ denote

$$X_\gamma = \{x \in X : g(x) \leqslant \gamma\}, \quad \hat{f}(\gamma) = \inf_{x \in X_\gamma} f(x). \tag{6}$$

We assume throughout that X_0 is nonempty. Consider the problem

$$\text{minimize } f(x) \text{ subject to } g(x) \leqslant 0. \tag{7}$$

The optimal value of (7) is clearly $\hat{f}(0) < \infty$ since $X_0 \neq \varnothing$.

Problem (7) is below approximated by unconstrained problems of minimizing over $x \in X$ a functional $I(x, M)$ for fixed M, where

$$I(x, M)=\psi(f(x)-M+\mu(g(x)). \tag{8}$$

In regard to the functions $\psi: R \to R_+$, [2] $\mu: R_+ \to R_+$, the following hypotheses are formulated.

(H1) ψ, μ are strictly increasing and forcing [3] on R_+,

(H2) $\psi(t)=\psi(-t)$,

(H3) ψ, μ are differentiable and convex.

It is tacitly assumed that (H1) (H2) are always fulfilled, and (H3) only when explicitly stated.

It has been shown by Grossman [8] that part of the results presented below may be obtained for more general form of $I(\cdot, \cdot)$. Formula (8), however, covers the most important cases met in practice: for $\psi(t)=\mu(t)=t^2$ we obtain the functional of Morrison, and for $\mu(t)=t^2$, $\psi(t)=pt^r$ — that of Grzegórski.

LEMMA 1. Let $\bar{x}$ minimize $I(\cdot, M)$. Then,

(i) $f(\bar{x}) \leqslant \hat{f}(0)$ if $M \leqslant \hat{f}(0)$,

(ii) $f(\bar{x})=\hat{f}(0)$, $g(\bar{x})=0$ if $M=\hat{f}(0)$

Proof. Proof is similar to the original proof of Morrison [17].

We shall first discuss the Everett theorem for functional (8). The Everett theorem is known to hold for linear Lagrange functionals, various penalty and augmented Lagrange functionals —see [5, 15, 23], under no assumptions whatsoever on the regularity of problem (7). This is no longer true for the functional $I(\cdot, M)$.

Assume that X is a topological space. In the sequel we shall frequently invoke the following regularity assumptions.

(A1) For all $\gamma \geqslant 0$ the sets X_γ are connected.

(A2) For each $\gamma>0$, $X_\gamma=\overline{\{x \in X: g(x)<\gamma\}}$.

THEOREM 1. Suppose the functions f, g are continuous, assumptions (A1) (A2) are satisfied. If $M \leqslant \hat{f}(0)$ and $\bar{x}$ minimizes $I(\cdot, M)$ over X then

$$f(\bar{x}) \geqslant M \tag{9}$$

and $\bar{x}$ solves the problem:

$$\text{minimize } f(x) \text{ subject to } x \in X_{\bar{\gamma}},\ \bar{\gamma}=g(\bar{x}). \tag{10}$$

Assumptions (A1) (A2) are obviously satisfied when X is a topological vector space and g is convex continuous.

THEOREM 2. Suppose ψ, μ satisfy (H1) (H2) (H3), X is a topological vector space, $M<\hat{f}(0)$ and $\bar{x}$ minimizes $I(\cdot, M)$ over X. If either (i) f, g are Gateaux differentiable

[2]) $R_+=\{t \in R: t \geqslant 0\}$.

[3]) μ is forcing if $\mu(t) \to 0$ if and only if $t \to 0$.

and (A1) (A2) hold, or (ii) f, g are convex continuous then $\lambda_0=\psi'(f(\bar{x})-M)$, $\lambda_1=\mu'(g(\bar{x}))$ are Lagrange multipliers at $\bar{x}$ for the problem (10).

Moreover, $(\lambda_0, \lambda_1)\neq(0, 0)$.

The proofs lean on the following.

LEMMA 2. Under assumptions of Theorem 1, if $f(x)<M$, then there is a point x' with $g(x')<g(x)$, $f(x')=M$.

Proof. X_0 being nonempty, it contains a point $x_0\in X_0$, $f(x_0)\geq f(0)\geq M$. By assumption (A2) and the continuity of f, there is a point x_1 with $\gamma=g(x_1)<g(x)$, $f(x_1)<M$. Since X_γ is connected and contains x_0, x_1, by continuity of f it must contain x' with $f(x')=M$.

Proof of Theorem 1. For $x\in X_{\bar{\gamma}}$, $f(x)\geq M$. Otherwise we could use Lemma 2 to establish the existence of a point x' with $g(x')<\bar{\gamma}$, $f(x')=M$, and (by (H1) (H2)),

$$I(x', M)=0+\mu(g(x'))<\mu(\bar{\gamma})\leq I(\bar{x}, M)$$

which contradicts the optimality of $\bar{x}$. Since for $x\in X_{\bar{\gamma}}$, $g(x)\leq g(\bar{x})$ and $I(x, M)\geq I(\bar{x}, M)$, we must have by (H1)

$$\psi(f(x)-M)\geq\psi(f(\bar{x})-M)$$

or $f(x)\geq f(\bar{x})$, due to (H1) again.

Proof of Theorem 2. Introduce the Lagrange functional

$$L(x, \lambda_0, \lambda_1)=\lambda_0 f(x)+\lambda_1(g(x)-g(\bar{x})).$$

In the case (i),

$$0=I'_x(\bar{x}, M)=L'_x(\bar{x}, \lambda_0, \lambda_1)=\lambda_0 f'(\bar{x})-\lambda_1 g'(\bar{x})$$

and since by Theorem 1, $\bar{x}$ solves (10), (λ_0, λ_1) are the Lagrange multipliers for (10) at $\bar{x}$.

For the convex case denote

$$F(\varphi, \gamma)=\psi(\varphi-M)+\mu(\gamma),\quad \gamma\geq 0,$$

$$A=\{(\varphi, \gamma): \exists x\in X,\ \varphi\geq f(x),\ \gamma\geq g(x)\},$$

$$B=\{(\varphi, \gamma): \exists x\in X,\ f(x)\geq M,\ \varphi\geq f(x),\ \gamma\geq g(x)\}.$$

We shall show that

$$F(\varphi, \gamma)\geq F(\bar{\varphi}, \bar{\gamma})\quad \forall x\in A \tag{11}$$

where $\bar{\varphi}=f(\bar{x})$. Observe that for $(\varphi, \gamma)\in B$, by (H1)

$$F(\varphi, \gamma)\geq F(f(x), g(x))=I(x, M)\geq I(\bar{x}, M)=F(\bar{\varphi}, \bar{\gamma})$$

where $M\leq f(x)\leq\varphi$, $g(x)\leq\gamma$. If $(\varphi, \gamma)\in A\setminus B$, then $\varphi\geq f(x)$, $\gamma\geq g(x)$ with $f(x)<M$. Apply now Lemma 2:

$$F(\varphi, \gamma)=\psi(\varphi-M)+\mu(\gamma)>0+\mu(g(x'))=I(x', M)\geq F(\bar{\varphi}, \bar{\gamma}),$$

and (11) follows.

F is convex and differentiable on the convex A. A well known necessary condition for (11) to hold is

$$\frac{\partial F(\bar{\varphi}, \bar{\gamma})}{\partial \varphi}(\varphi-\bar{\varphi})+\frac{\partial F(\bar{\varphi}, \bar{\gamma})}{\partial \gamma}(\gamma-\bar{\gamma}) \geqslant 0 \quad \forall(\varphi, \gamma) \in A. \tag{12}$$

But,

$$\lambda_0=\frac{\partial F(\bar{\varphi}, \bar{\gamma})}{\partial \varphi}, \quad \gamma_1=\frac{\partial F(\bar{\varphi}, \bar{\gamma})}{\partial \gamma}$$

and $(f(x), g(x)) \in A,\ x \in X$. So that (12) gives

$$\lambda_0 f(x)+\lambda_1 g(x) \geqslant \lambda_0 f(\bar{x})+\lambda_1 g(\bar{x})$$

i.e., $\bar{x}$ minimizes the Lagrange functional $L(\cdot, \lambda_0, \lambda_1)$ over X.

Since ψ, φ are increasing on R_+ and $f(\bar{x}) \geqslant M$, then $\lambda_0, \lambda_1 \geqslant 0$. Suppose $\lambda_0 = =\lambda_1=0$. Since ψ, μ are convex, with minimum at zero, it obtains $f(\bar{x})=M$, $g(\bar{x})=0$ which is impossible since $M<\hat{f}(0)$.

The following examples show that the assumptions of Theorem 1 cannot be relaxed.

Example 1 (see Figure 1). Let $X=R$, $f(x)=-x$ and

$$g(x)=\begin{cases} 0 & x \leqslant 0 \\ x & 0 \leqslant x \leqslant 0.6 \\ 1.2-x & 0.6 \leqslant x \leqslant 1 \\ x-0.8 & x \leqslant 1 \end{cases}.$$

Then

$$\hat{f}(\gamma)=\begin{cases} -\gamma & 0 \leqslant \gamma<0.2 \\ -0.8-\gamma & \gamma \geqslant 0.2 \end{cases}. \tag{13}$$

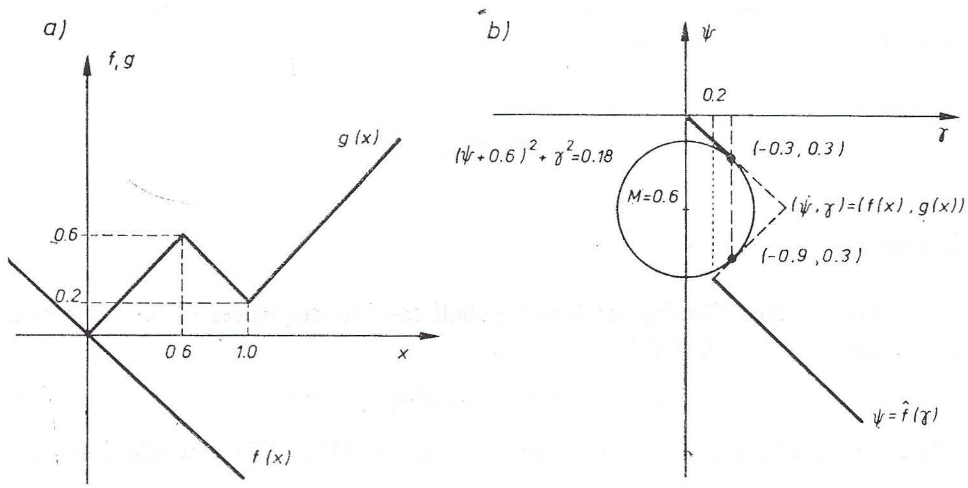

Figure 1

Take $M=-0.6$. There are two minimal points of the Morrison functional

$$I(x, M)=(f(x)-M)^2+g^2(x), \tag{14}$$

namely $x_1=0.3$ and $x_2=0.9$. We have $g(x_1)=g(x_2)=0.3$, $f(x_1)=-0.3>M>f(x_2)=-0.9$, $\hat{f}(0.3)=-1.1<f(x_2)<f(x_1)$.

Observe that the sets X_γ have two components for $0.2<\gamma<0.6$ so that (A1) is not fulfilled.

Example 2 (Figure 2). Let X, f as above and

$$g(x)=\begin{cases} 0 & x\leqslant 0 \\ x & 0\leqslant x\leqslant 0.2 \\ 0.2 & 0.2\leqslant x\leqslant 1 \\ x-0.8 & x\geqslant 1 \end{cases}$$

$\hat{f}(\gamma)$ is again given by (13), while

$$\operatorname{int} X_{0.2}=(-\infty, 1)\neq(-\infty, 0.2)=\{x: g(x)<0.2\},$$

thus (A2) does not hold. For $-1<M<-0.4$ functional (14) attains its only minimum at $\bar{x}=-M$, $f(\bar{x})=M$, $g(\bar{x})=0.2$ and $\hat{f}(0.2)=-1<f(\bar{x})$.

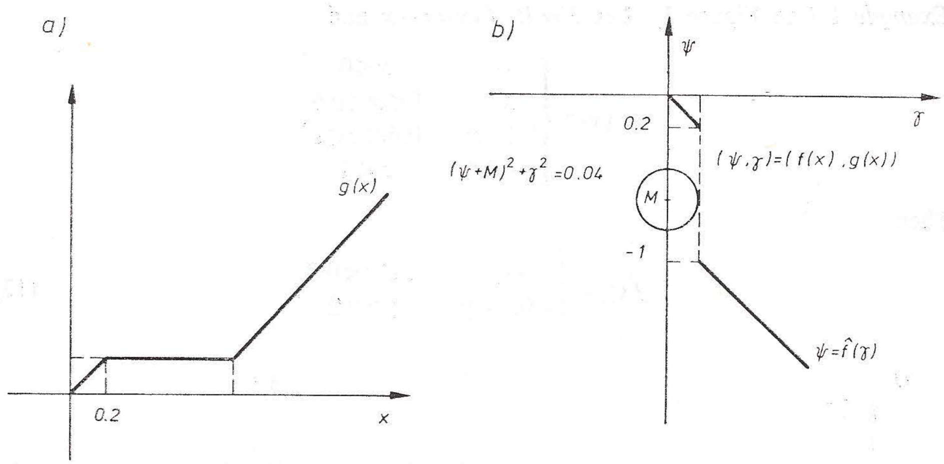

Figure 2

3. Convergence

In this and the following sections we shall consider sequences of the functional of the type (8). For $k\in N$ let

$$I_k(x, M)=\psi_k(f(x)-M)+\mu_k(g(x)) \tag{15}$$

where the functions ψ_k, μ_k are assumed to satisfy (H1), (H2) and additionally

(H4) $\psi_{k+1}(t)\leqslant\psi_k(t)$, $\mu_{k+1}(t)\geqslant\mu_k(t)$.

Algorithm. Suppose $M_0 \leqslant \hat{f}(0)$ and consider the sequences x_k, M_k, Δ_k:

$$d_k = \inf_{x \in X} I_k(x, M_k) = I_k(x_k, M_k), \tag{16}$$

$$M_{k+1} = M_k + \Delta_k, \tag{17}$$

where

$$\Delta_k = \Delta_k(d_k, g_k, f_k, M_k), \; g_k = g(x_k), \; f_k = f(x_k).$$

THEOREM 3 (compare [8]). If $\Delta_k \geqslant 0$, $M_k \leqslant \hat{f}(0)$ for all k and $\Delta_k \to 0$ implies $d_k \to 0$, then

$$\lim_{k \to \infty} g_k = 0. \tag{18}$$

If, moreover, $\hat{f}(\cdot)$ is lower semicontinuous at zero, then

$$\lim_{k \to \infty} f_k = \hat{f}(0). \tag{19}$$

Proof. The sequence $\{M_k\}$ is monotone bounded, so converges and $\Delta_k \to 0$. Therefore, $d_k \to 0$ and $\mu_k(g_k) \to 0$.

Since $\mu_k(g_k) \geqslant \mu_0(g_k)$ by (H4), $g_k \to 0$ by (H1). To prove (19) observe that $x_k \in X_{g_k}$, so by Lemma 1 (i) and by (18)

$$\hat{f}(0) \geqslant \limsup_{k \to \infty} f_k \geqslant \liminf_{k \to \infty} \hat{f}(g_k) \geqslant \hat{f}(0).$$

The theorem, being very general, brings no real information about the way in which to pick up the sequence Δ_k. Below, three variants of the algorithm will be specified.

Algorithm A

$$M_{k+1}^A = f_k.$$

Algorithm B (Morrison)

$$M_{k+1}^B = M_k + \psi_k^{-1}(d_k).$$

Algorithm C — assume (H3) holds for all ψ_k, μ_k —

$$M_{k+1}^C = f_k + \frac{\mu_k'(g_k)}{\psi_k'(f_k - M_k)} g_k.$$

All the three updating rules are illustrated by Figure 3.

In the remaining of this section we shall show that under appropriate assumptions, Algorithms A, B and C satisfy the requirements of Theorem 3 and, therefore, converge. Roughly speaking, Morrison Algorithm B converges under virtually no assumptions, (similarly as the SUMT algorithms), Algorithm A requires the validity of the Everett theorem, while C is essentially applicable to convex programs. We start with the simplest case of Algorithm B.

THEOREM 4. For Algorithm B, $\lim_{k\to\infty} g_k=0$ and $\lim_{k\to\infty} f_k=\hat{f}(0)$ if $\hat{f}$ is l.s.c. at zero.

Proof. $\Delta_k=M^B_{k+1}-M_k=\psi_k^{-1}(d_k)$ is nonnegative. By (H4), $\psi_k^{-1}(d_k)\geqslant\psi_0^{-1}(d_k)$, and since ψ_0 satisfies (H1), $\Delta_k\to 0$ entails $d_k\to 0$.

Next,

$$d_k\leqslant\psi_k(\hat{f}(0)-M_k)$$

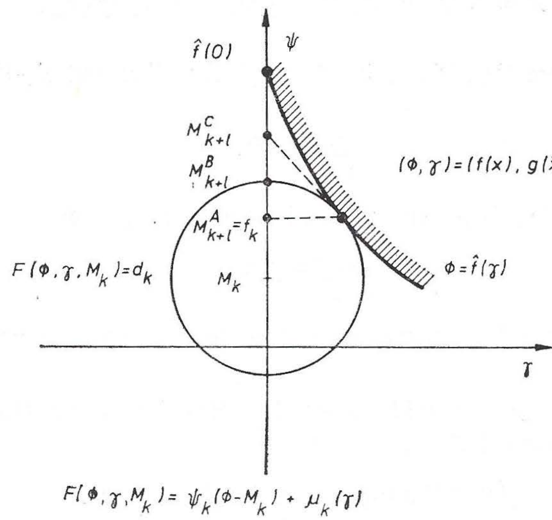

Figure 3

since $\inf_X I\leqslant\inf_{X_0} I$, so that by (H1) (H2)

$$\psi_k^{-1}(d_k)\leqslant|\hat{f}(0)-M_k|$$

and if $M_k\leqslant\hat{f}(0)$, then also $M_{k+1}\leqslant\hat{f}(0)$. It remains to apply Theorem 3.

THEOREM 5. Suppose ψ_k, μ_k satisfy (H1) through (H4), f, g are Gateaux differentiable, (A1) (A2) hold, and for some $\gamma_0>0$, $\|f'(x)\|$ is bounded in X_{γ_0} and $g(x)\to 0$ whenever $\|g'(x)\|\to 0$, $x\in X_{\gamma_0}$. If M_0 is sufficiently close to $\hat{f}(0)$, then the sequences g_k, f_k generated by Algorithm A satisfy $\lim_{k\to\infty} g_0=0$, and $\lim_{k\to\infty} f_k=\hat{f}(0)$ if $\hat{f}$ is l.s.c. at zero.

Proof. By (9), $f_k\geqslant M_k$ and by Lemma 1 (i), $f_k\leqslant\hat{f}(0)$. Thus, for all k, $M_k\leqslant M^A_{k+1}\leqslant\leqslant\hat{f}(0)$. It remains to show that $\Delta_k\to 0$ implies $d_k\to 0$. We have $I'_k(x_k, M_k)=0$, $\Delta_k=f_k-M_k$, so

$$\psi'_k(\Delta_k)f'(x_k)+\mu'_k(g_k)g'(x_k)=0. \tag{20}$$

Since

$$M_0\leqslant M_k\leqslant\hat{f}(0),\ \psi_k(\hat{f}(0)-M_k)\leqslant\psi_k(\hat{f}(0)-M_0)\leqslant\psi_0(\hat{f}(0)-M_0)$$

and

$$\mu_0(g_k)\leqslant\mu_k(g_k)\leqslant d_k\leqslant\psi_k(\hat{f}(0)-M_k)\leqslant\psi_0(\hat{f}(0)-M_0) \tag{21}$$

so that $g_k \leqslant \gamma_0$ if $|M_0 - \hat{f}(0)|$ is sufficiently small. Thus, the sequences $\|f'(x_k)\|$, g_k are bounded. Take a subsequence $g_{k_n} \to g^*$, say. By (H1) (H3) (H4)

$$0 \leqslant \psi'_{k_n}(\Delta_{k_n}) = \lim_{h\to 0} \frac{\psi_{k_n}(\Delta_{k_n}+h)}{h} \leqslant \lim_{h\to 0} \frac{\psi_0(\Delta_{k_n}+h)}{h} = \psi'_0(\Delta_{k_n}) \tag{22}$$

... implies

$$\ldots g'(x_{k_n}) \to 0$$

... ‖ remains bounded.

... implies

over X, and we would have $g_k=0$. Thus, $\lambda_{0k}>0$, and $\eta_k=\frac{\lambda_{1k}}{\lambda_{0k}}$ is finite positive. Hence, $M^C_{k+1}=f_k+\eta_k g_k>f_k=M^A_{k+1}$ and the left hand side of (24a) holds. Again by Theorem 2 (ii), $-\eta_k$ is a Kuhn-Tucker vector to (10) with $\bar{\gamma}=g_k$ and by [20], Theorem 29.1

$$-\eta_k \in \partial \hat{f}(g_k),$$

$$\hat{f}(0)\geqslant \hat{f}(g_k)-\eta_k(\gamma-g_k)\ \forall \gamma \in R.$$

In particular,

$$\hat{f}(0)\geqslant \hat{f}(g_k)+\eta_k g_k=f_k+\eta_k g_k=M^C_{k+1}$$

which ends the proof.

To be continued in next issue.

so that $g_k \leqslant \gamma_0$ if $|M_0 - \hat{f}(0)|$ is sufficiently small. Thus, the sequences $\|f'(x_k)\|$, g_k are bounded. Take a subsequence $g_{k_n} \to g^*$, say. By (H1) (H3) (H4)

$$0 \leqslant \psi'_{k_n}(\Delta_{k_n}) = \lim_{h\to 0} \frac{\psi_{k_n}(\Delta_{k_n}+h)}{h} \leqslant \lim_{h\to 0} \frac{\psi_0(\Delta_{k_n}+h)}{h} = \psi'_0(\Delta_{k_n}) \tag{22}$$

hence, (20) implies

$$\mu'_{k_n}(g_{k_n})\, g'(x_{k_n}) \to 0$$

if $\Delta_{k_n} \to 0$, because then $\psi'_0(\Delta_{k_n}) \to \psi'_0(0) = 0$ by (H2) and $\|f'(x_{k_n})\|$ remains bounded. If $g^* \neq 0$, then by (H1) (H3) (H4) $\psi'_{k_n}(g_{k_n}) \geqslant \delta > 0$ and $g'(x_{k_n}) \to 0$, which implies $g_{k_n} = g(x_{k_n}) \to 0$, yielding a contradiction. Thus, $g^* = 0$ and $\lim_{k\to\infty} g_k = 0$.

THEOREM 6. Suppose f, g are convex continuous. Then the sequences g_k generated by Alforithms A and C converge to zero, and corresponding $\lim_{k\to\infty} f_k = \hat{f}(0)$, if $\hat{f}$ is l.s.c. at zero.

Proof. Consider the Algorithm A first. By Theorem 1 and Lemma 1 (i), $M_k \leqslant \leqslant M^A_{k+1} \leqslant \hat{f}(0)$ for all k, as shown above. By Theorem 2 (ii), $\lambda_{0k} = \psi'_k(f_k - M_k) = = \psi'_k(\Delta^A_k)$, $\lambda_{1k} = \mu'_k(g_k)$ are the Lagrange multipliers for problem (10) with $\bar{y} = g_k$ at x_k, so for all $x \in X$ we have

$$\begin{aligned} L(x, \lambda_{0k}, \lambda_{1k}) &\geqslant L(x_k, \lambda_{0k}, \lambda_{1k}), \\ L(x, \lambda_{0k}, \lambda_{1k}) &= \lambda_{0k} f(x) + \lambda_{1k}(g(x) - g_k) \end{aligned} \tag{23}$$

in particular

$$\lambda_{0k} \hat{f}(0) \geqslant \lambda_{0k} f_k + \lambda_{1k} g_k$$

or

$$\lambda_{0k}(\hat{f}(0) - f_k) \geqslant \lambda_{1k} g_k \geqslant 0 . \tag{24}$$

We shall prove that $\Delta^A_k \to 0$ implies $g_k \to 0$. By (21), $\{g_k\}$ is bounded, let g^* be any of its cluster points with $g_{k_n} \to g^*$. By (22), $\lambda_{0k_n} \to 0$ and since $\{\hat{f}(0) - f_{k_n}\}$ is bounded, (24) yields

$$\lambda_{1k_n} g_{k_n} \to 0 .$$

If $g^* \neq 0$, then by (H1) (H3) (H4), $\lambda_{1k_n} \geqslant \delta > 0$ and $g_{k_n} \to 0$ which is a contradiction. Thus, $g^* = 0$ and $\lim g_k = 0$.

Consider now Algorithm C. We shall show that

$$M^A_{k+1} \leqslant M^C_{k+1} \leqslant \hat{f}(0), \qquad \Delta^C_k \geqslant \Delta^A_k \tag{24a}$$

and this will ensure the convergence of Algorithm C , due to the convergence of Algorithm A.

If $g_k = 0$ then x_k is the desired solution, by Theorem 1. Suppose then $g_k \neq 0$. By (H1) (H3) (H4) $\lambda_{1k} > 0$. If $\lambda_{0k} = 0$, then from (23) x_k would minimize

$$\lambda_{1k}(g(x) - g_k)$$

Wskazówki dla autorów

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Recomendatious for the Authors

"Control and Cybernetics" publishes original papers which have not previously appeared in other journals. The publications of the papers in English is recommended. No paper should exceed in length 20 type written pages (210×297 mm) with lines spaced and a 50 mm margin on the lefthand side. Papers should be submitted in duplicate. The plan and form of the paper should be as follows:

2. The heading should include the title, the full names and surnames of the authors in alphabetic order, the name of the institution he represents and the name of the city or town. This heading should be followed by a brief summary (about 15 typewritten lines).

2. Figures, photographs, tables, diagrams should be enclosed to the manuscript. The texts related to the figures should be typed on a separate page.

3. Of possible all mathematical expressions should be typewritten. Particular attention should be paid to differentiation between capital and small letters. Greek letters should as a rule be defined. Indices and exponents should be written with particular care. Round brackets should not be replaced by an inclined fraction line.

4. References should be put on the separate page. Numbers in the text identified by references should be enclosed in brackets. This should contain the surname and the initial of Christian names, of the author (or authors), the complete title of the work (in the original language) and, in addition:

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Author	Tittle	Vol.	Year	No.	Page
NGUYÊN DINH QUYÊT:	A minimizing problem of a quadratic functional for the systems described by right invertible operators in Hilbert space	7	1978	3	27
NGUYÊN DINH QUYÊT:	On linear systems described by right invertible operators acting in a linear space	7	1978	2	33
OLBROT A. W.:	Control of retarted systems with function space constraints. Pt. II. Approximate controllability	6	1977	2	17
PAWŁOW I.:	On some properties of two-layer parabolic free boundary value problems	7	1978	4	19
SAWIK T.:	Stochastic optimal control of a multi-facility, multi-product production scheduling with random times of supplies	6	1977	3/4	21
SOKOŁOWSKI J.:	Remarks on existence of solutions for parametric optimization problems for partial differential equations of parabolic type	7	1978	2	47
TOŁWIŃSKI B.:	Numerical solution of N-person non-zero-sum differential games	7	1978	1	37
TOŁWIŃSKI B.:	On the existence of Nash equilibrium points for differential games with linear and non-linear dynamics	7	1978	3	37
TRYBUS L.:	Nash feedback strategies for a two-person partial differential game with a Neumann-type parabolic system	7	1978	4	5

Control
and Cybernetics

Contents

volumes 6 (1977) and 7 (1978)

Autor tytuł	Tom	Rok	Nr	Str.
LASIECKA I., MALANOWSKI K.: O dyskretnej w czasie aproksymacji Ritza-Galerkina zadań sterowania optymalnego dla układów parabolicznych przy ograniczeniu sterowania	7	1978	1	21
LASIECKA I., MALANOWSKI K.: O regularności tozwiązań wypukłych zadań sterowania optymalnego dla układów parabolicznych przy ograniczeniach sterowania	6	1977	3/4	57
NGUYÉN DINH QUYÉT: Problem minimalizacji funkcjonału kwadratowego dla układów opisanych przez operatory prawostronnie odwracalne w przestrzeni Hilberta	7	1978	3	27
NGUYÉN DINH QUYÉT: Układy liniowe opisane za pomocą prawostronnie odwracalnych operatorów w przestrzeni liniowej	7	1978	2	33
OLBROT A. W.: Sterowanie układów z opóźnieniami przy ograniczeniach w przestrzeni funkcyjnej. Cz. II. Sterowalność aproksymacyjna	6	1977	2	17
PAWŁOW I.: O pewnych wartościach dwuwarstwowych parabolicznych zagadnień brzegowych ze swobodną granicą	7	1978	4	19
SAWIK T.: Stochastyczne sterowanie optymalne harmonogramem produkcji o wielu stanowiskach i wielu wyrobach z przypadkowymi czasami dostaw	6	1977	3/4	21
SOKOŁOWSKI J.: Uwagi na temat istnienia rozwiązań dla zagadnień sterowania optymalnego dla liniowego równania różniczkowego cząstkowego typu parabolicznego	7	1978	2	47
TOŁWIŃSKI B.: Metody numeryczne rozwiązywania wieloosobowych gier różniczkowych o sumie niezerowej	7	1978	1	37
TOŁWIŃSKI B.: O istnieniu punktów równowagi Nasha dla gier różniczkowych z liniowymi i nieliniowymi równaniami stanu	7	1978	3	57
TRYBUS L.: Pozycyjne strategie Nasha dla dwuosobowej cząstkowej gry różniczkowej z układem parabolicznym typu Neumanna	7	1978	4	45

Control and Cybernetics

Spis treści

tomów 6 (1977) i 7 (1978)

Control
and Cybernetics

СОДЕРЖАНИЕ

Том 6 (1977) и 7 (1978)