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On Measurements in Observations and Optimal Observations for Linear Lumped Parameter Systems

by

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The note contains results concerning observations and optimal observations based on finite numbers of measurements. There are given properties of the sets U_s and $\tilde{U}_s$ of all functionals observable and optimal observable resprectively by s measurements. A modification of Krasovskii's minimax rule is given. Influences of the time interval of the observation on the quality and the number of measurements are investigated. Universal points for the observation are described.

1. Introduction

Consider a linear lumped parameter system

$$\begin{aligned} \dot{x} &= K(t)\,x, \\ y &= G(t)\,x, \qquad 0 \leqslant t \leqslant T, \end{aligned} \tag{1.1}$$

where $K(t)$ is an integrable n by n matrix and $G(t)$ is a continuous m by n one. In pracice the output $y(t)$ is obtained by a measurement, which has an error $\Delta(t): y(t)=G(t)\,x(t)+\Delta(t)$. Supposing that $\Delta(t)$ satisfies the condition $\Delta(.)\in S_\delta$, S_δ being the ball of radius δ in the output space Y, Krasovskii [5] introduced an optimal observation problem with the optimality in the minimum norm sense. Kolmanovskii [4], Chernousko [2] and Solanik [13] considered an optimal observation problem in which the error $\Delta(t)$ is a stochastic variable having the normal distribution with the zero expectation and a known correlation matrix. The optimality is understood in the sense of minimum of an integral functional characterizing the quality of the observation process. Krasovskii's problem was generalized by Kurgianskii [6] as follows. The measurement of the output has now an error of the type $y(t)=G(t)\,x(t)+F(t)\,\zeta(t)$, $F(t)$ being a continuous m by r matrix and $\zeta(t)$ being a unknown function restrited by the condition $\zeta(.)\in Z\subset L_r^2\,[0, T]$, where Z is convex, closed and bounded. The system is also disturbed: $\dot{x}=K(t)x+$ $+C(t)v$, where $C(t)$ is a continuous n by s matrix and v is a disturbance satisfying the condition $v(.)\in V\subset L_s^2[0,T]$, where V is convex, closed and bounded. The optimality of observations is understood in the minimax sense, which is a direct generalization of the minimum norm sense.

Rolewicz [12] extended Krasovskii's problem to infinite dimensional systems consisting of operators acting on Banach spaces. In particular, the system (1.1) is, following Rolewicz, expressed in the form

$$X \xrightarrow{A} \square \xrightarrow{B} Y, \tag{1.2}$$

where continuous linear operators A and B are defined by

$$(Ax)(t)=\varphi(t,t_0)x, \quad 0\leqslant t\leqslant T, \quad x\in X,$$

$$(Bx(\cdot))(t)=G(t)x(t), \quad 0\leqslant t\leqslant T, \quad x(\cdot)\in\square,$$

$\big(\varphi(t,t_0)$ is the transition matrix of (1.1)$\big)$, $X=R^n$ is the space of all initial values, $\square=C_n[0,T]$ is the space of trajectories, $Y=C_E[0,T]$, E being an m-dimensional space, is the output space. Functional $f\in X^*$ is said to be observable if there is $\varphi\in Y^*$ such that $f(x)=\varphi(BAx)$ for all $x\in X$, or in terms of adjoint operators, $f=A^*B^*\varphi$. φ is called an observation for f. Observation φ_0 such that $\|\varphi_0\|=$ $=\inf\{\|\varphi\|:f=A^*B^*\varphi\}$ is called an optimal observation for f. Rolewicz obtained the following fundamental theorem

THEOREM (Rolewicz [12]). If $f\in X^*$ is observable, then there is an optimal observation φ of the form

$$\varphi(y(\cdot))=\sum_{k=1}^{n}(\xi_k, y(t_k)) \tag{1.3}$$

where $\xi_k\in E^*$,, $t_k\in[0,T]$, $(.,.)$ denotes the scalar product in R^m.

Formula (1.3) means that φ is based on $n=\dim X$ measurements.

The aim of this note is trying to answer the questions: "When can one reduce the number of measurements in observations and optimal observations?", "How can one define t_k, ξ_k in formula (1.3)?", "Are there common points of the measurement for all observable functionals?"

A majority of results of the note was announced in [11].

2. Some Elementary Facts

Let P be the set of all observable functionals. Since $\dim BAX=\dim A^*B^*Y^*=$ $=\dim P$, P is a r-dimensional subspace of X^* if and only if the number of linearly independent vector functions among n columns of matrix $G(\cdot)\varphi(\cdot,t_0)$ is equal to r. Moreover, $P=A^*B^*Y^*=((\mathrm{Ker}\ BA)$ (as X is finite dimensional). Hence, as a trivial, consequence of Rolewicz's theorem we obtain

Corollary 2.1. If the number of linearly independent functions within n columns of matrix $G(\cdot)\,\varphi(\cdot, t_0)$ is equal to r, then every observable functional can be optimally observable by r measurements.

Functionals in X^* are often considered as n-dimensional vectors. Then from the definition of the observability it follows that functional φ of the form (1.3) based on s measurements is an observation for $f \in X^*$ if and only if

$$f=\sum_{k=1}^{s} \varphi^*(t_k, t_0)\, G^*(t_k)\, \xi_k . \tag{2.1}$$

Example 2.1. Consider the system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{2n-1} \\ \dot{x}_{2n} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \ldots 0 \\ 0 & 0 & 1 \ldots 0 \\ \multicolumn{3}{c}{\cdots\cdots\cdots} \\ 0 & 0 & 0 \ldots 1 \\ 0 & 0 & 0 \ldots 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{2n-1} \\ x_{2n} \end{bmatrix} \tag{2.2}$$

$$y=[1, 0, \ldots, 0] \begin{bmatrix} x_1 \\ \vdots \\ x_{2n} \end{bmatrix} = x_1 , \qquad a \leqslant t \leqslant b .$$

Equation (2.1) in this case has the form

$$\begin{cases} f_1 = \displaystyle\sum_{k=1}^{s} \xi_k , \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ f_{2n} = \displaystyle\sum_{k=1}^{s} \xi_k \frac{(t_k - t_0)^{2n-1}}{(2n-1)!} . \end{cases} \tag{2.3}$$

If f has an observation φ^o of the form (1.3) based on p measurements with $\xi_k^o \geqslant 0$, $k=1, \ldots, p$, then φ^o is optimal. Indeed, by (2.3) we have for every observation φ of the form (1.3) (for f)

$$\|\varphi\| = \sum_{k=1}^{s} |\xi_k| \geqslant \sum_{k=1}^{s} \xi_k = f_1 = \sum_{k=1}^{p} \xi_k^o = \|\varphi^o\| .$$

Now consider the integral functional $\tilde{f} \in Y^*$

$$\tilde{f}(x_1(\cdot)) = \int_a^b x_1(t)\, p(t)\, dt , \tag{2.4}$$

$p(t)$ being a nonnegative integrable function vanishing only in a set of measure zero. $\tilde{f}$ corresponds to $f \in X^*$:

$$f = \left[\int_a^b p(t)\, dt, \ldots, \int_a^b p(t) \frac{(t-t_0)^{2n-1}}{(2n-1)} dt \right] . \tag{2.5}$$

$x_1(t)$ satisfying (2.2) is a polynomial of degree $2n-1$. Hence we have the Gauss type quadrature formula

$$\int_a^b x_1(t)\,p(t)\,dt = \sum_{k=1}^{n} \xi_k\, x_1(t_k), \qquad t_k \in [a, b],\ \xi_k > 0.$$

Since $\xi_k > 0$ the formula defines an optimal observation for functional (2.5), i.e. for integral functional (2.4). This optimality was proved in another way in [10].

3. Sets of Functionals Whose Observations or Optimal Observations can be Based on the Same Number of Measurements

By U_s ($\tilde{U}_s$) we denote the set of all $f \in X^*$ whose observations (optimal observations resp.) may be based on s measurements ($U_0 = \tilde{U}_0 = \{0\}$).

THEOREM 3.1:

(a) $$U_0 \subsetneq U_1 \subsetneq \ldots \subsetneq U_{s_1} = U_{s_1+1} = \ldots = U_n = P,$$

(b) $$\tilde{U}_0 \subsetneq \tilde{U}_1 \subsetneq \ldots \subsetneq \tilde{U}_{s_2} = \tilde{U}_{s_2+1} = \ldots = \tilde{U}_n = U_n = P,$$

where s_1 and s_2 depend on the linear system.

The proof is based on the following

LEMMA 3.2:

(a) $$U_{s+r} = U_s + U_r,$$

(b) $$\tilde{U}_{s+r} \subset \tilde{U}_s + \tilde{U}_r.$$

Proof.. (a) Clear, by (2.1).

(b) Let $f \in \tilde{U}_{s+r}$. Its optimal observation φ_{s+r} is defined by t_k, ξ_k, $k = 1, \ldots, s+r$, as follows

$$f = \sum_{k=1}^{s} \varphi^*(t_k, t_0)\, G^*(t_k)\, \xi_k + \sum_{k=s+1}^{s+r} \varphi^*(t_k, t_0)\, G^*(t_k)\, \xi_k \stackrel{\mathrm{df}}{=} f_s + f_r.$$

The first sum defines an optimal observation φ_s for f_s. Indeed, if $\|\varphi_s\| = \sum_{k=1}^{s} \|\xi_k\| >$ inf $\{\|\varphi\| : f_s = A^* B^* \varphi\}$, then by Rolewicz's theorem there are η_k, θ_k, $k = 1, \ldots, n$, such that

$$f_s = \sum_{k=1}^{n} \varphi^*(\theta_k, t_0)\, G^*(\theta_k)\, \eta_k \quad \text{and} \quad \sum_{k=1}^{n} \|\eta_k\| < \sum_{k=1}^{s} \|\xi_k\|.$$

This contradicts the optimality of φ_{s+r}, since $\|\varphi_{s+r}\| = \|\varphi_s\| + \|\varphi_r\|$. Similarly, the second sum defines an optimal observation φ_r for f_r. Q. E. D.

Proof of the theorem. (a) follows from the fact that if $U_{i-1}=U_i$, then $U_{i+1}= =U_i+U_1=U_{i-1}+U_1=U_i$. Now we shall show (b). For this purpose we shall prove that if $\tilde{U}_{i-1}=\tilde{U}_i$, then $\tilde{U}_i=\tilde{U}_{i+1}$. Let $f_{i+1}\in\tilde{U}_{i+1}$ and let φ_{i+1} defined by t_k, ξ_k, $k=1, \ldots, i+1$, be its optimal observation:

$$f_{i+1}=\varphi^*(t_1, t_0)\, G^*(t_1)\, \xi_1+\sum_{k=2}^{i+1} \varphi^*(t_k, t_0)\, G^*(t_k)\, \xi_k \overset{\mathrm{df}}{=} f_1+f_i .$$

Clearly φ_1 defined by t_1, ξ_1 and φ_i based on t_k, ξ_k, $k=2, \ldots, i+1$, are optimal observations for f_1 and f_i respectively. On the other hand, since $f_i\in\tilde{U}_i=\tilde{U}_{i-1}$, there is φ_{i-1} which is an optimal observation for f_i. Thus $\|\varphi_1\|+\|\varphi_{i-1}\|=\|\varphi_1\|+ +\|\varphi_i\|=\|\varphi_{i+1}\|$, i.e. $f_{i+1}\in\tilde{U}_i$. Q. E. D.

From equation (2.1) it follows that

$$U_s=\bigcup_{t_1, \ldots, t_s\in[0, T]} \mathrm{Lin}\,[\varphi^*(t_1, t_0)\, G^*(t_1), \ldots, \varphi^*(t_s, t_0)\, G^*(t_s)]$$

where Lin [ ] indicates the space spanned by the columns of the matrix. Furthermore, U_s and $\tilde{U}_s$ are cones and they are symmetrical with respect to the origin.

Since X is finite dimensional and so $A^*B^*Y^*$ is closed, by the Banach theorem on inverse operators we trivially get

LEMMA 3.3. If $f^n\to f^o$ as $n\to\infty$, then

$$\inf\{\|\varphi\|: f^n=A^*\, B^*\, \varphi\}\to\inf\{\|\varphi\|: f^o=A^*\, B^*\, \varphi\}.$$

In the majority of practical cases, the time interval $[0, T]$ is bounded and closed. In this case we have

THEOREM 3.4. $\tilde{U}_s$ is closed for all $s\geqslant 0$.

Proof. Let $f_n\in\tilde{U}_s$ and $f_n\to f_0$. Of course $f_0\in A^*B^*Y^*$. Hence f_0 is observable. By the definition of $\tilde{U}_s$ there are optimal observations φ_n for f_n of the form

$$\varphi_n(y(\cdot))=\sum_{k=1}^{s}(\xi_k^n, y(t_k^n)).$$

The norms of φ_n are, in view of Lemma 3.3, uniformly bounded. It implies that the norms of ξ_k^n are also uniformly bounded. Using the compactness argument we can extract a subsequence φ_{n_i} such that $\xi_k^{n_i}$ and $t_k^{n_i}$ are convergent to ξ_k and t_k correspondently. Thus

$$\varphi_0(y(\cdot))=\sum_{k=1}^{s}(\xi_k, y(t_k))$$

defines an optimal observation for f_0. Therefore $f_0\in\tilde{U}_s$. Q. E. D.

PROPOSITION 3.5. If for all $t\in[0, T]$ rank $G(t)=m$, then U_1 is closed.

Proof. Let $f^n = \varphi^*(t^n, t_0)\, G^*(t^n)\xi^n$ and $f^n \to f^o$. Since rank $G(t) = m$ for all $t \in [0, T]$ there is a constant M such that $\|\xi^n\| \leqslant M\|f^n\|$ for all. n. Using the compactness argument we can chose converging subsequences ξ^{n_i} and t^{n_i}. Then $f^o = \varphi^*(t^o, t_0)\, G^*(t^o)\,\xi^o$, where $t^o = \lim\ t^{n_i}$ and $\xi^o = \lim \xi^{n_i}$. Q. E. D.

If the time interval is unbounded, then U_1 may not be closed as is shown by the following

Example 3.1. Consider the system

$$\begin{cases} \dot{x}_1 = \lambda_1 x_1 \\ \dot{x}_2 = \lambda_2 x_2 \end{cases} \quad \lambda_2 < \lambda_1 < 0,$$

$$y = x_1 + x_2, \quad 0 \leqslant t < +\infty, \quad t_0 = 0.$$

Since $\lambda_1 < 0$, $\lambda_2 < 0$ all solutions of the differential equations are continuous and bounded and so they are elements of the Banach space $C\,[0, +\infty)$. Let us observe functionals $f^n = \left(1, \dfrac{1}{n}\right)$. It is easily verified that φ^n based on one measurement $t^n = \dfrac{1}{\lambda_1 - \lambda_2} \ln n$, $\xi^n = n^{\frac{\lambda_1}{\lambda_2 - \lambda_1}}$ is an observation for f^n. On the other hand, $f^n \to f^o = (1, 0)$, which is not observable by one measurement. Thus U_1 is not closed.

Two examples below give answers to the quaestion: whether $\tilde{U}_s = U_s$?

Example 3.2. Consider a stationary two-dimensional system whose matrix has complex eigenvalues $\alpha \pm i\beta$ $(\beta \neq 0)$:

$$\begin{cases} \dot{x}_1 = \alpha x_1 - \beta x_2, \\ \dot{x}_2 = \beta x_1 + \alpha x_2, \end{cases}$$

$$y = x_1 + g x_2, \quad 0 \leqslant t \leqslant T,$$

(taking $G = (1, g)$ does not lose generality). It is not hard to see, by equation (2.1), that $U_1 = X^*$ if $T \geqslant \pi/\beta$. Furthermore, if $\alpha = 0$, then solving equation (2.1) for the considered system relatively ξ_k and comparing the norms of all observations φ_2 (consisting of two measurements) with the norm of φ_1 show that $\tilde{U}_1 = U_1 = X^*$.

Example 3.3. Consider the above system. Let $\alpha < 0$. Then one easily sees that all functionals satisfying the condition $g f_1 = f_2$ belong to $\tilde{U}_1$, i.e. $\{0\} \subsetneq \tilde{U}_1$. Now suppose $\alpha = -10$, $\beta = 1$, $g = 1$ $f = (10{,}0)$. The only φ_1 based on $t_1 = \pi/4$, $\xi_1 = \dfrac{10}{\sqrt{2}} e^{10\pi/4}$ has the norm $\|\varphi_1\| = \dfrac{10}{\sqrt{2}} e^{10\pi/4} > \|\varphi_2\|$, where φ_2 is based on two measurements at 1/10 and 1/9. Thus $\{0\} \subsetneq \tilde{U}_1 \subsetneq U_1$.

Example 3.2 has the following extension

PROPOSITION 3.6. Consider a $2n$-dimensional stationary system with one dimensional output $(g \neq 0)$

$$\dot{x} = Kx,$$

$$y = gx, \quad 0 \leqslant t \leqslant T, \quad t_0 = 0.$$

If matrix K has $2n$ single complex eigenvalues $\alpha_k \pm i\beta_k$ and $T \geqslant \pi/\beta$ ($\beta = \min \beta_k$, $\beta_k > 0$), then $U_n = X^*$.

Proof. Basing on the form of $\varphi(t, 0)$ we obtain

$$y(t) = g\varphi(t, 0)\, x = e^{\alpha_1 t}(\delta_1 \cos \beta_1 t + \sigma_1 \sin \beta_1 t) + \ldots +$$
$$+ e^{\alpha_n t}(\delta_n \cos \beta_n + \sigma_n \sin \beta_n t) \overset{\mathrm{df}}{=} y_1(t) + \ldots + y_n(t),$$

and $y(t) = y_1(t)$ if $x = (x^1, x^2, 0, \ldots, 0) \overset{\mathrm{df}}{=} x_1$. Then we have a corresponding decomposition $X = X_1 \oplus \ldots \oplus X_n$. Applying Example 3.2 to two-dimensional system

$$X_k \xrightarrow{A/X_k} \square \xrightarrow{B} Y$$

we obtain an observation $\varphi_k \in Y^*$, based on one measurement, for $f_k \in X_k^*$ induced by $f \in X^*$ by the formula $f_k(x_k) = f(x_k)$. Now we can trivially prove that functional $\varphi = \sum_{k=1}^{n} \varphi_k$ is a desired observation for f. Q. E. D.

Now we deal with universality of the optimal observation problem with output space $C_E[0, T]$. Consider a linear system $\dot{x} = K(t)\, x$, $y_1 = G_1(t)\, x$, or

$$X \xrightarrow{A} \square \xrightarrow{B} Y \tag{I}$$

where Y is a separable Banach space. There exists, by the universality of $C_E[0, T]$, an isometric embedding D of Y into $C_E[0, T]$:

$$X \xrightarrow{A} \square \xrightarrow{B} Y \xrightarrow{D} C_E[0, T]. \tag{II}$$

By the Hahn-Banach theorem we get the

Proposition 3.7. If φ_2^o is an optimal observation in system II (II-observation) for f, then $\varphi_1^o = D^* \varphi_2^o$ is an optimal I-observation. Conversely, if φ_1^o is an optimal I-observation, then there is an optimal II-observation φ_2^o satisfying $\|\varphi_2^o\| = \inf\{\|\varphi_2\| \colon \varphi_1^o = D^* \varphi_2\}$.

Proposition 3.8. If Y^* in problem I is strictly convex, then in problem II we have $\tilde{U}_1 = U_1 = P$.

Proof. If $f \in X^*$ is observable, then its optimal observation φ_1^o is unique by the strict convexity of Y^* and φ_1^o is an extremal point of the ball $K_{a1} \subset Y^*$ of radius $a_1 = \inf\{\|\varphi_1\| \colon f = A^*B^*\varphi_1\}$. Taking into account that for all $\varphi_2 \in (C_E[0, T])^*$ we have $\|D^*\varphi_2\| \leqslant \|\varphi_2\|$ and $a_2 = \inf\{\|\varphi_2\| \colon f = A^*B^*D^*\varphi_2\} = a_1$, we see that φ_1^o is an extremal point of $D^* K_{a_2}$, where K_{a_2} is the ball of radius a_2 in $(C_E[0, T])^*$. Therefore we can find $\varphi_2^o \in K_{a_2} \cap D^{*-1} \varphi_1^o$ such that φ_2^o is an extremal point of K_{a_2} and then φ_2^o is a required optimal observation. Q. E. D.

Remark. Proposition 3.8 has the following noteworthy consequence: There exists an output for the given process $\dot{x} = K(t)\, x$ such that $\tilde{U}_1 = X^*$. Indeed, first we take space Y such that Y^* is strictly convex and a matrix $G_1(t)$ such that $\dim BAX = n$. Next, we find an isometric embedding D of Y into $C_E[0, T]$. Then operator DB is a desired output (but this output is not given by a matrix of the form (1.1)).

4. An Explicit Solution of the Optimal Observation Problem

Another proof of Rolewicz's therem. Let y^o be a minimal element, i.e. $\|y^o\| = \inf\{\|y\| : y = BAx,\ f(x) = 1\} \stackrel{\mathrm{df}}{=} \rho^o$. Then every optimal observation φ^o satisfies the maximum property [5]

$$\varphi^o(y^o) \max_{\|\varphi\| = 1/\rho^o} (y^o) = \|\varphi^o\|\, \|y^o\| .$$

This means that φ^o is collinear with y^o (in the sense of Luenberger [8].) By analogy with equation (2.1), in virtue of the collinearity we see that φ^o is an optimal observation for f if and only if

$$f = \int_{D_{y^o}} \varphi^*(t, t_0)\, G^*(t)\, \xi^o(t)\, b(t)\, dt,$$

where $D_{y^o} = \{t \in [0, T] : \|y^o(t)\| = \|y^o\|\}$, $\xi^o(t)$ is a unit vector collinear with $y^o(t)$ and $b(t) \geqslant 0$.

Let dim Lin $\{\varphi^*(t, t_0)\, G^*(t)\, \xi^o(t),\ t \in D_{y^o}\} = d$. Of course $d \leqslant n$. One can now choose d linearly independent vectors within the vectors $\varphi^*(t, t_0)\, G^*(t)\, \xi^o(t)$, $t \in D_{y^o}$, say the vectors with $t = t_1, \ldots, t_d$ such that

$$f = \sum_{k=1}^{d} \varphi^*(t_k, t_0)\, G^*(t_k)\, \xi^o(t_k)\, \bar{b}(t_k), \quad \bar{b}(t_k) \geqslant 0 . \tag{4.1}$$

The obtained formula defines an optimal observation based on $d \leqslant n$ measurements.

Q. E. D.

From this proof we deduce:

RULE 4.1 (of the determination of optimal observations).

1. Define a minimal element y^o of the problem.
2. Find D_{y^o}. Calculate m-dimensional unit vectors $\xi^o(t)$ collinear with $y^o(t)$ for $t \in D_{y^o}$. Functional $\varphi \in Y^*$ defined by the formula

$$\varphi(y(\cdot)) = \sum_{k=1}^{s} \left(b_k\, \xi^o(t_k), y(t_k)\right), \quad b_k \geqslant 0, \quad t_k \in D_{y^o},$$

is an optimal observation for f if and only if

$$f = \sum_{k=1}^{s} \varphi^*(t_k, t_0)\, G^*(t_k)\, \xi^o(t_k)\, b_k .$$

If the linear system is stationary and the unit ball in the space E is a polyhedron in a dual general position with respect to the system, then D_{y^o} contains only a finite number of points [12]. Thus Rule 4.1 becomes more pratical.

The following example yields an extension of the result in Example 3.2.

Example 4.1. Consider a four-dimensional system with one dimensional output: $\dot{x} = K_4 x$, $y = gx$, $0 \leqslant t \leqslant T$, $t_0 = 0$. Suppose that K_4 has single pure imaginary eigenvalues $\pm\beta i$, $\pm\gamma i$ such that $\frac{\beta}{k+1}=\frac{\gamma}{k}$ for some integer k and β is near to γ so that the phenomenon of beats happens ([3] pp. 5–7) and assume that $T\geqslant 2\pi K/\gamma$. Then every $f\in X^*$ can be optimally observable by two measurements. In fact, for all $x\in X$ the output takes the form

$$y(t)=a_1 \sin(\beta t+b_1)+a_2 \sin(\gamma t+b_2).$$

$y(t)$ has amplitude $a(t)$ which varies periodically between a_1+a_2 and $|a_1-a_2|$ with period $\frac{2\pi}{\beta-\gamma}=\frac{2\pi k}{\gamma}$ equal to the period of $y(t)$ ([3] *pp.* 5–7). Then $|y(t)|$ takes the maximum being the maximal value in the whole time axis once or twice in $\left[0, \frac{2\pi k}{\gamma}\right)$ for each $x\in X$. Now using Rule 4.1 concludes the proof.

5. Influence of the Time Interval of the Observation on the Quality and the Number of Measurements of Observations

In this section we shall examine different intervals of time assuming that the conditions on $K(t)$ and $G(t)$ in Section 1 are always satisfied. It is obvious that observability in $[T_1, T_2]$ implies observability in $[T_1, T_2]\supset[T_1, T_2]$. The following proposition whose proof is simple precises the situation in which a converse statement is true.

Proposition 5.1. Every functional observable in $[T_1, T_2]$ is also observable in a subinterval $[\bar{T}_1, \bar{T}_2]$ if and only if the number of linearly independent functions among n columns of the matrix $G(\cdot)\,\varphi(\cdot, t_0)$ for the subinterval is the same as for $[T_1, T_2]$. In particular, for stationary systems from the observability (for f) in a certain interval it follows the observability in every interval.

In [14] it was introduced a concept of differential observability and there was proved a fact similar to Proposition 5.1 for complete observability.

Proposition 5.2. Let φ and $\bar{\varphi}$ be optimal observations for f in $[T_1, T_2]$ and $[\bar{T}_1, \bar{T}_2]$ respectively. If $[\bar{T}_1, \bar{T}_2]\subset[T_1, T_2]$, then $\|\bar{\varphi}\|\geqslant\|\varphi\|$ (i.e. shortening the time interval decreases the accuracy of optimal observations).

Proof. Let $y^o(t)$ and $\bar{y}^o(t)$ be minimal elements in $[T_1, T_2]$ and $[\bar{T}_1, \bar{T}_2]$ resp. Then

$$\bar{p}^o=\max_{[\bar{T}_1, \bar{T}_2]}\|\bar{y}^o(t)\|\leqslant\max_{[\bar{T}_1, \bar{T}_2]}\|y^o(t)\|\leqslant\max_{[T_1, T_2]}\|y^o(t)\|=p^o.$$

Hence

$$\|\bar{\varphi}\|=\frac{1}{\bar{p}o}\geqslant\frac{1}{po}=\|\varphi\|. \qquad \text{Q. E. D.}$$

We say that observation φ_0 defined in $[\bar{T}_1, \bar{T}_2]$ is a globally optimal observation if we have $\|\varphi_0\| \leqslant \|\varphi\|$ for any observation φ in any interval. One easily proves, by an argument similar to that in the above proof, that there exists a globally optimal observation in $[\bar{T}_1, \bar{T}_2]$ if and only if there is a globally minimal element $y^o(t)$ (i.e. a function $y^o(t)$ which is a minimal element for the problem in any interval $[T_1, T_2] \supset \supset [\bar{T}_1, \bar{T}_2]$) and $\max_{[T_1, T_2]} \|y^o(t)\| = \max_{[\bar{T}_1, \bar{T}_2]} \|y^o(t)\|$ for any $[T_1, T_2] \supset [\bar{T}_1, \bar{T}_2]$. Furthermore, if all $y(t)$ are periodical with the same period, then every observable functional has a globally optimal observation in any interval with the length equal to the period. As a trivial example of such systems we can take a stationary system in which matrix K has single pure imaginary eigenvalues $\pm i\beta_k$ such that β_k are commensurable.

Now we deal with a dependency of the number of measurements on the length of time interval. One easily sees that lengthening the interval does not increase the number of measurements in observations. The situation with optimal observations is more complicated. Two examples below show that we may reduce the number of measurements in certain cases by lengthening the interval and in other cases by shortening it.

Example 5.1. Consider functional $f = (\sqrt{3}, 1)$ in the system

$$\begin{cases} \dot{x}_1 = -x_2, \\ \dot{x}_2 = x_1, \end{cases}$$
$$y = x_1.$$

There is the unique observation consisting of one measurement (at $t = 5\pi/6$) and it is globally optimal. On the contrary, if we permit to observe the output in $[0, \pi/2]$, then to optimally observe f we have to make two measurements.

Example 5.2. Given the functional $f = (1/^{10}, 10)$ and the system

$$\begin{cases} \dot{x}_1 = -10x_1 + x_2, \\ \dot{x}_2 = \qquad -10x_2, \end{cases}$$
$$y = 10x_1, \qquad 0 = t \leqslant \leqslant T, \qquad T > 100.$$

It is not hard to show that the only observation based on one point, namely $t_1 = 100$, possessing the norm $\|\varphi_1\| = \dfrac{e^{1000}}{100}$ is not optimal. If we are able to observe the output only in $[100, T]$, then the observation φ_1 is optimal. To verify this compare $\|\varphi_1\|$ with $\|\varphi_2\|$ for all φ_2 based on two measurements. (It should be pointed out that the optimality in $[100, T]$ is worse than that in $[0, T]$.)

6. Universal Points for the Observation

A set of points $\{t_1, \ldots, t_s\} = \tau_s$, $t_k \in [0, T]$, is called a universal set (of points of the measurement) for the linear system, provided that every observable functional has observations based only on measurements at τ_s.

THEOREM 6.1. If there is $t_1 \in [0, T]$ such that rank $G(t_1)=m$ and if dim $P=r$, then there exists a universal set consisting of $r-m+1$ points. Without the assumption about $G(t)$ we shall have a universal set consisting of r points.

Proof. Since rank $\varphi^*(t_1, t_0) G^*(t_1)=m$ among n rows there are m linearly independent rows, say m first ones. Then $n-m$ remained rows are linear combinations of them:

$$\begin{cases} \text{row } (m+1)=(\alpha_1^{m+1}, \ldots, \alpha_m^{m+1}) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \text{row } n \qquad =(\alpha_1^n, \ldots, \alpha_m^n) \end{cases} \tag{6.1}$$

where, for simplicity, in the right sides we write only the coefficients of the combinations. Assume that $r>m$ (if $r=m$, then the following procedure of the proof will be finished after the first step). Since dim $P=r$ the number of linearly independent functions among n columns of matrix $G(\cdot)\,\varphi(\cdot, t_0)$ is equal to r and one can choose $t_2 \in [0, T]$ such that for the matrix $\varphi^*(t_2, t_0)\,\mathbf{G}^*(t_2)$ there is at least one equality in (6.1), which becomes an inequality, say row $(m+1) \neq (\alpha_1^{m+1}, \ldots, \alpha_m^{m+1})$. Clearly in the matrix $[\varphi^*(t_1, t_0) G^*(t_1), \varphi^*(t_2, t_0) G^*(t_2)]$ $m+1$ first rows are linearly independent. Continuing the procedure we finally obtain

$$\text{rank } [\varphi^*(t_1, t_0) G^*(t_1), \ldots, \varphi^*(t_{r-m+1}, t_0) G^*(t_{r-m+1})]=r.$$

As dim $P=r$ a glance at the equation (2.1) assures us that the set $\{t_1, \ldots, t_{r-m+1}\}$ is universal. Q. E. D.

The following proposition has elementary proof. which is omitted.

PROPOSITION 6.2:

(a) The set of all universal sets consisting of s points is open in $[0, T]^s=[0, T]\times\ldots$ $\ldots\times[0, T]$.

(b) The mentioned set is everywhere dense in $[0, T]^s$ if the number of linearly independent functions among n columns of matrix $G(\cdot)\,\varphi(\cdot, t_0)$ is equal to s in each subinterval of $[0, T]$. If, further, rank $G(t)=m$ for all $t \in [0, T]$, then we have the same fact but for universal sets consisting of $s-m+1$ points and for $[0, T]^{s-m+1}$. If, moreover, dim $P=n$, then setting $s=n$ in the above statements we obtain a necessary and sufficient condition for the density.

Let us note that for any observable functional f there is a universal set such that f is optimally observable only by points of this set. Indeed, by (4.1) f is optimally observable by $t_1, \ldots, t_d$. Of course rank $[\varphi^*(t_1, t_0) G^*(t_1), \ldots, \varphi^*(t_d, t_0)\cdot$ $\cdot G^*(t_d)] \leqslant \dim P$. If we have the strict inequality, then we can choose points $t_{d+1}, \ldots, t_s$ so that the inequality for $t_1, \ldots, t_s$ becomes an equality and we get a required universal set.

Conversely, for given universal set τ there is the "worst" functional f^* in the sense that

$$\inf_{\varphi} \|\varphi_\tau^{f^*}\| = \sup_{\|f\|\leqslant 1} \inf_{\varphi} \|\varphi_\tau^f\|$$

where φ_τ^f denotes an observation for f, based on τ and the infimum is taken over all such observations. Precisely, for $(t_1, \ldots, t_s)=\tau$ the formula $f=\sum_{k=1}^{s} \varphi^*(t_k, t_0)\, G^*(t_k)\, \xi_k$ defines a linear operator Q from $E^*\times\ldots\times E^*$ onto P. Since the unit ball in $E^*\times\ldots\times E^*$ is compact, by a method of Rolewicz ([12], Theorem V.6.2), we see that the $\inf_{\varphi} \|\varphi_\tau^f\| = \inf_{Q\varphi=f} \|\varphi\|$ is attained at an element $\bar{\varphi}_\tau^f$. By the mentioned Banach theorem the linear operator Q^{-1} from P onto $E^*\times\ldots\times E^*/\mathrm{Ker}\, Q$ is bounded. Therefore $\|\bar{\varphi}_\tau^f\|=\|Q^{-1}(f)\|$ is a continuous function of f. Thus by the compactness of the unit ball in P for a certain $f^*\in P$ we have: $\sup_{\|f\|\leq 1} \|\bar{\varphi}_\tau^f\|=\|\bar{\varphi}_\tau^{f*}\|$.

We say that universal set τ^1 is better than τ^2 if $\|\bar{\varphi}_{\tau_1}^{f_1^*}\|<\|\bar{\varphi}_{\tau_2}^{f_2^*}\|$. Unfortunitely, in general, the best (optimal!) universal sets do not exist since the set of all universal sets is open.

7. An Extension to the Observation for Systems of Functionals

Let the linear system (1.2) be given. Consider a system of functionals $F=(f_1, \ldots, f_q)$. F can be regarded as an operator mapping X into a q-dimensional Banach space H. System F is said to be observable if there is an operator ψ such that the diagram

$$\begin{tikzcd} X \arrow[r, "A"] \arrow[dr, "F"'] & \square \arrow[r, "B"] & Y \arrow[dl, "\psi"] \\ & H & \end{tikzcd}$$

is commutative and to be optimally observable if there is such a ψ with minimal norm [12].

Using Krein's method of moments [1] and noting that the finite dimensional space H has the separable extension property [7] if and only of it is (isometric and isomorphic to) a space with the norm

$$\|(h_1, \ldots, h_q)\| = \max_{1\leq i\leq q} |h_i| \tag{7.1}$$

(see [7] [9]) we get the following extension of a realtionship obtained by Krasovskii [5]: For each operator $F\colon Y\to H$ the following relation is valid

$$\inf\{\|\psi\|: F(x)=\psi(BAx)\}=\frac{1}{\rho^0},$$

where $\rho^o=\inf\{\|y\|: y=BAx,\ \|F(x)\|_H=1\}$, if H has the norm (7.1). Conversely, if for given space H we have the mentioned relation for each linear system (1.2) and each operator F, then (by the universality of $C_E[0, T]$) H has the norm (7.1).

We easily prove the following

PROPOSITION 7.1. Let H have the norm (7.1) and let y^o be a minimal element of the problem for F (i.e. $y^o=BAx$, $\|F(x)\|_H=1$, $\|y^o\|=\rho^o$). The observation $\psi: \psi(y(\cdot))=$

$= \sum_{k=1}^{s} V_k y(t_k)$, $V_k \in B(E \to H)$, $t_k \in [0, T]$, is optimal if and only if for $k=1, \ldots, s$ we have:

(a) $t_k \in D_{y^o} = \{t \in [0, T]: \|y^o(t)\| = \|y^o\|\}$;

(b) V_k and $y^o(t_k)$ are collinear, i.e. $\|V_k y^o(t_k)\|_H = \|V_k\| \, \|y^o(t_k)\|$;

(c) $\|\sum_{k=1}^{s} V_k y^o(t_k)\|_H = \sum_{k=1}^{s} \|V_k y^o(t_k)\|_H$.

Rule 7.2 (of the determination of optimal observations when H has the norm (7.1))

(1) Find a minimal element y^o given by definition

$$\|y^o\| = \inf\{\|y\| : y = BAx, \|F(x)\|_H = 1\} = \rho^o.$$

There exist optimal observations ψ if and only if $\rho^o > 0$ and then $\|\psi\| = \dfrac{1}{\rho^o}$.

(2) Define D_{y^o} and q by m matrices V_k^o with $\|V_k^o\| = 1$ collinear with $y^o(t_k)$ for $t_k \in D_{y^o}$. Operator $\psi: Y \to H$ defined by

$$\psi(y(\cdot)) = \sum_{k=1}^{s} b_k V_k^o y(t_k), \qquad b_k \geqslant 0, \qquad t_k \in D_{y^o},$$

is an optimal observation for F if and only if

$$F = \sum_{k=1}^{s} b_k V_k^o G(t_k) \varphi(t_k, t_0),$$

$$\left\| \sum_{k=1}^{s} b_k V_k^o y^o(t_k) \right\|_H = \sum_{k=1}^{s} \|b_k V_k^o y^o(t_k)\|_H.$$

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O pomiarach wykonywanych na podstawie obserwacji lub optymalnych obserwacji układów liniowych o parametrach skupionych

Podano wyniki dotyczące obserwacji i optymalnych obserwacji, których podstawę stanowi skończona liczba pomiarów. Podano własności zbiorów U_s i $\tilde{U}_s$ wszastkich funkcji obserwowalnych odnoszących się do s pomiarów. Przedstawiono modyfikację minimaksowej zasady Krasowskiego. Przeanalizowano wpływ długości przedziału obserwacji na jakość i liczbę pomiarów. Określono punkty układu dogodne dla dokonywania obserwacji.

Об измерениях выполняемых на основе наблюдений и оптимальных наблюдений линейных систем с сосредоточенными параметрами

Работа содержит результаты касающиеся наблюдений и оптимальных наблюдений, основой которых является конечное число наблюдений. Даны свойства множеств U_s и $\tilde{U}_s$ всех наблюдаемых функций и оптимально наблюдаемых функций относящихся к s измерениям. Представлена модификация минимаксного принципа Красовского. Анализируется влияние величины интервала наблюдений на качество и число измерений. Определены точки системы, удобные для проведения наблюдений.