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Choice Function and its Representation

by

JUHANI NIEMINEN

Department of Mathematics Faculty of Technology
University of Oulu
90100 Oulu, Finland

The problems of representing a choice function posed by Fishburn are solved. A new way of representing a choice function is given.

The purpose of this paper is to consider conditions for representing a choice function C by means of a weak order. The ideas and observations here are based on Szpilrajn's order extension [3] and Fishburn's work [1]. As noted by Fishburn [1], all possible representations of C by a weak order are of importance in economics.

Let $\mathscr{X}$ be a non-empty set of elements and $\mathscr{X}$ a non-empty family of non-empty subsets of $\mathscr{X}$. A choice function C determined on $\mathscr{X}$ is a function with the domain $\mathscr{X}$ such that C maps each $Y \in \mathscr{X}$ into a non-empty subset $C(Y)$ of Y. Note that $\mathscr{X}$ can contain an infinite number of elements as well as $Y \in \mathscr{X}$ but *every set $C(Y)$ is here assumed to be finite.*

Let P be an asymmetric binary relation on X. P is a strict partial order on X if and only if it is transitive, and a weak order if and only if it is negatively transitive, i.e. xPy and $z \in X$ imply xPz or zPy. With indifference relation $\sim_P$ defined from P by $x \sim_P y$ if and only if neither xPy nor yPx, it can be shown that P is a weak order if and only if P is a strict partial order and the relation $\sim_P$ is transitive. P is a linear order if an only if it is a complete strict partial order, i.e. $x \neq y$ implies xPy or yPx.

Following Fishburn, we define the set $C(Y, P)$ as follows:

$$C(Y, P) = \{x | x \in Y \text{ and } yPx \text{ for no } y \in Y\}.$$

A choice function C from $\mathscr{X}$ into the non-empty subsets of X is called widely representable (representable) by a binary relation of a specified type if and only if there is a binary relation P on X of this type such that $C(Y, P) \cap C(Y) \neq \emptyset$ ($\emptyset \neq \neq C(Y, P) \subset C(Y)$) for every Y in $\mathscr{X}$. Further, C is exactly representable by a weak order P if and only if $C(Y, P) = C(Y)$ for every $Y \in \mathscr{X}$. The representability is introduced in [1] by Fishburn and the exact representability by Richter in [2].

The handling of the problem is more concrete when the relation P needed in the representations can be derived from a relation determined by C on X. As noted by Richter, such a relation exists. We shall use a modified version of Richter's relation V and denote it by U:

$$xUy \Leftrightarrow \text{there is a subset } Y \in \mathscr{X} \text{ such that } x \in C(Y) \text{ and } y \in Y \setminus C(Y).$$

If there is a sequence $u_1, ..., u_m$ of elements of X such that $xUu_1, u_1 Uu_2, ...,$ $..., u_{m-1} Uu_m, u_m Uy$, we write xWy. It can be easily shown that W is the smallest transitive relation on X including U. In fact, W is the transitive closure U^t of U on X.

THEOREM 1. A choice function C from X into the non-empty subsets of X is representable by a linear order P on X if and only if U includes a relation R satisfying the conditions (i) and (ii):

(i) For every $Y \in \mathscr{X}$ there exists at least one element $x \in C(Y)$ such that xRy for each $y \in Y \setminus C(Y)$.

(ii) R^t is a strict partial order on X.

Proof. According to the defintions of U and R, U includes R. Assume that C can be represented by a linear order P. According to the definition of the representation of C, every Y in $\mathscr{X}$ has a greatest element x and $C(Y, P)=\{x\}$. Then xPy for every $y \in Y \setminus C(Y)$. We define the relation R on X as follows: $xRy \Leftrightarrow$ there is a subset $Y \in \mathscr{X}$ where $x, y \in Y$, xPy, x is the greatest element with respect to P in Y and $Y \in Y \setminus C(Y)$. Because P is transitive and asymmetric, R^t is too, and thus R satisfies (i) and (ii). According to the construction U includes R.

Conversely, let R be the relation with the properties given in the theorem. According to Szpilrajn's order extension construction, R^t can be extended into a linear order P including R^t. Let Y be an arbitrary element from $\mathscr{X}$. As P is a linear order, there is a greatest element, say w, in $C(Y)$ with respect to P. On the other hand, let $x \in Y$ have the property (i). Because P includes R^t (and also R) and because wPx, wPy for every $y \in Y \setminus \{w\}$. Thus $\emptyset \neq C(Y, P)=\{w\} \subset C(Y)$, and C is representable by P.

In [1, Lemma 1] Fishburn proved that C is representable by a linear order if and only if it is representable by a weak order. Hence we can write

COROLLARY. A choice function C can be represented by a weak order if and only if there exists the relation R satisfying the demands of Theorem 1.

Fishburn stated in [1] that the representability by strict partial orders and by acyclic orders merit further examination. We shall consider these representations here. A binary relation P on X is acyclic, if there is no $n>1$ and $x_1, ..., x_n$ in X such that $x_1 Px_2, x_2 Px_3, ..., x_{n-1} Px_n$, and $x_n Px_1$.

THEOREM 2. A choice function C from $\mathscr{X}$ into the non-empty susbsets of X is representable by a strict partial order if and only if it is representable by a linear order.

Proof. If C is representable by a linear order, it is trivially representable by a strict patrial order, too. Hence, let P be a strict partial order on X with respect to which C is representable. But then $\emptyset \neq C(Y, P) \subset C(Y)$ for every $Y \in \mathscr{X}$ and thus xPy for each $x \in C(Y, P)$ and each $y \in Y \setminus C(Y)$. Accordingly, we can define a relation R, satisfying the conditions of Theorem 1 as follows: xRy there is a subset $Y \in \mathscr{X}$ such that $x, y \in Y$, xPy, $x \in C(Y, P)$ and $y \in Y \setminus C(Y)$. As P includes R and as P is transitive, P includes also R^t, whence R^t is a strict partial order; the other conditions of Theorem 1 are obviously valid, whence Theorem 1 implies the desired result.

Assume that P is an acyclic order on X and that C can be represented by P. Then for every $Y \in \mathscr{X}$ there exists at least one $x \in C(Y)$ such that xPy for every $y \in Y / C(Y)$ because $C(Y, P) \subset C(Y)$. The relation R of Theorem 1 can now be derived from P as follows: $xRy \Leftrightarrow$there is a subset $Y \in \mathscr{X}$ such that $x, y \in Y$, xPy, $x \in C(Y, P)$ and $y \in Y \setminus C(Y)$. As P includes R and P is acyclic, R^t is a symmetric and hence a strict partial order on X. Obviously U includes R. This shows the first part of the following theorem and as the second is trivially valid, we can write

THEOREM 3. A choice function C from X into the non-empty subsets of X is representable by an acyclic order if and only if it is representable by a linear order.

Fishburn considered in [1] an example analogous to the following: Let $X = \{x, y, z, w\}$, $\mathscr{X} = \{Y, Z\}$, where $Y = \{x, z, w\}$ and $Z = \{y, z, w\}$, $C(Y) = \{w\}$ and $C(Z) = \{z\}$. The only way to construct the relation R of Theorem 1 requires wRx, wRz, zRy and zRw. But then, because wRz and zRw, R^t is symmetric. As any of the relations cannot be removed, the demands of Theorem 1 does not hold and thus C cannot be represented by an asymmetric order. On the other hand, by using a relation P, where wPx, wPy, zPy and zPx, C can be widely represented by P.

Each choice function C from $\mathscr{X}$ into the non-empty subsets of X can always be widely represented by a weak order P by defining P as follows: $x \sim_P y$ for every two elements $x, y \in X$. Thus the problem is to find a weak order P such that it takes into account the maximum amount of the information of U and provides the wide representation of C. At first we prove a theorem illuminating the connection between the representability and the wide representability.

THEOREM 4. A choice function C from $\mathscr{X}$ into the non-empty subsets of X is widely representable by a linear order P if and only if it is representable by P.

Proof. According to the definitions, if C is representable by P then it is also widely representable by P. Conversely, let C be widely representable by a linear order P on X. Because P is a linear order, $C(Y, P)$ is a single element, say x, and because $C(Y, P) \cap C(Y) \neq \emptyset$, $x \in C(Y)$. Thus $C(Y, P) = \{x\} \subset C(Y)$ for every $Y \in \mathscr{X}$, whence C is representable by P.

Let C be a choice function from $\mathscr{X}$ into the non-empty subsets of X. We form a new relation T from U on X as follows: $xTy \Leftrightarrow xUy$, and yUx does not hold; if xUy and yUx, then $x \sim_T y$.

If T^t is asymmetric, then C can be widely represented by the strict partial order T^t. If moreover for every $Y \in \mathscr{X}$ there is an element $x \in C(Y)$ such that xTy for every $y \in Y \setminus C(Y)$, then C can be represented by a linear order according to the observations above. When X is finite, an asymmetric T^t can always be completed into a weak order with respect to which C can be widely represented.

References

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Funkcja wyboru i jej reprezentacja

Rozwiązano zagadnienie reprezentacji funkcji wyboru sformułowanej przez Fishburna. Podano pewien nowy sposób reprezentacji funkcji wyboru.

Функция выбора и ее представление

Решается вопрос представления функции выбора, сформулированной Фишбурном. Дается некоторый новый способ представления функции выбора.