Control
and Cybernetics
VOL. 8 (1979) No. 3

Control of Multistage Processes Under Maximization or Balanced Growth of Utility Function

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The control of multistage processes is considered. Various situations may occur at different stages of the process. For every situation another subset of utility functions is given.

Two cases are considered. In the first case assignement of a utility function to a particular stage is random although at each stage ther is only one utility function. The control results from the maximization of the expected value of the global utility function.

In the second case at each stage an appropriate subset of utility functions has to be taken into account. In this case the control results from the application of the balanced growth principle.

1. Introduction

Many technological batch processes and most of economic processes may be treated as multistage processes.

Theory of control of this type of processes, with its main result — dynamic programming [1], was developed under a general assumption that the performance of the whole process is measured by a single, strictly given performance index.

In mathematical economics, where models of economic growth are devoted a lot of space, problems of control of individual stages are not considered, while concept of technological sets is used [6], [7]. In these models performance indices are assumed in the form of functionals expressing weighted sums of investment goods produced by the whole process or consumption aggregated to one component [6], [7].

This formulation of the problem is closely connected with turnpike theorems [2], [9], [10], [11] according to which optimal trajectories have the property that significant portions of their total lengths are close to the states of balanced growth.

Papers [4] and [5] state the problem of control of dynamic multistage processes with different performance indices at individual stages for technological processes. The problem was reduced to local optimization at individual stages and to a choice

of switching points in the augmented state space (state+time), where a measure of discontinuity of control at the switching points was chosen as a utility function. Such a utility function, however is meaningful only in certain cases of technological processes.

In the present paper, the following problem is considered. A multistage process described by state equations is given. Continuity of state space trajectories is assumed. Control of the process is evaluated by means of a set of performance indices characterizing production output, its quality, energy consumption or other technological characteristics. There are given linear utility functions, determined on these indices, which may be different at different stages and there may be many such functions at each stage.

Two cases are considered. In the first case assignment of a utility function to a particular stage is random although at each stage there is only one utility function.

Such a situation arises when prices or priorities may change from one stage to another due to causes which are random from the point of view of the process control.

In the second case at each stage an appropriate set of utility functions is given. In this case application of a principle of balanced growth is postulated together with using the theory of von Neumann models [2], [6], [8] in order to determin some relations connected with the growth rate.

II. Problem Formulation

Consider a multistage, dynamic control process whose stages are described by state equations.

Stage i, $i=1, 2, ..., N$, for times $\tau \in (T_{i-1}, T_i]$ is described by the following state trajectory

$$x_i(\tau)=g_i(u_i, x_{i0}, T_{i-1}, \tau) \tag{1}$$

where $x_i \in R^n$ — state variables at stage i,

$x_{i0}=x_i(T_{i-1})$ — initial condition for stage i,
$u_i \in V_i$ — control at stage i,
$g_i: V_i \times R^n \times R^1 \times R^1 \to R^n$.

Initial condition $\bar{x}_0=(x_{10}, T_0)$ for the first stage is assumed to be given.

Continuity of the state trajectory for the whole process is assumed

$$x_{i+1,\,0}=x_i(T_i), \qquad i=1, 2, ..., N-1. \tag{2}$$

For each stage a target set is given (Fig. 1).

$$[x(T_i), T_i]=\bar{x}_i \in S_i \subset R^{n+1}, \qquad i=1, 2, ..., N. \tag{3}$$

It is assumed that the sets S_i are connected and closed.

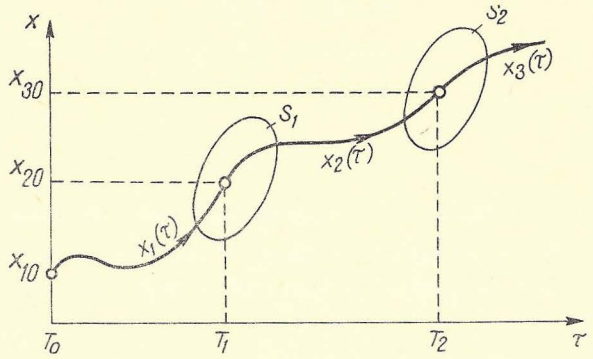

Figure 1

It follows from (1), using the notation of (3), that for $\tau = T_i$

$$x_i(T_i) = \bar{g}_i[u_i, \bar{x}_{i-1}], \quad i = 1, 2, ..., N, \tag{4}$$
$$\bar{g}_i : V_i \times R^{n+1} \to R^n.$$

Relation (4) expresses the fact that the final state at each stage is a function of control and initial conditions at this stage.

Let Q be a given set of nonnegative functionals — performance indices and let it be composed of subsets $Q_j, j = 1, 2, ..., M$,

$$\begin{aligned} &Q = \{Q_1, ..., Q_M\}, \\ &Q_j = \{q_j^1, ..., q_j^N\}, \\ &q_j^i = \bar{q}_j^i(u_i, \bar{x}_{i-1}, \bar{x}_i), \qquad j = 1, ..., M, \\ &\bar{q}_j^i : V_i \times R^{n+1} \times R^{n+1} \to R^1. \qquad i = 1, ..., N. \end{aligned} \tag{5}$$

Substitution (4) to (5) gives

$$\begin{aligned} &q_j^i = \hat{q}_j^i(u_i, \bar{x}_{i-1}), \qquad j = 1, ..., M \\ &\hat{q}_j^i : V_i \times R^{n+1} \to R^1, \qquad i = 1, ..., N. \end{aligned} \tag{6}$$

III. Determination of Switching Points by Maximizing of the Global Utility Function

Let us assume that, for each stage, there is given a vector of utility functions being linear combinations of performance functionals

$$w^i = C^i q^i \tag{7}$$

where

$$\begin{aligned} &C^i = \{c_{rj}^i\}_{R \times M} \\ &\qquad\qquad i = 1, ..., N, \\ &c_{rj}^i \geqslant 0, \qquad r = 1, ..., R, \\ &\qquad\qquad j = 1, ..., M, \end{aligned} \tag{8}$$

$$q^i=(q_1^i, ..., q_M^i)^T,$$
$$w^i=(w_1^i, ..., w_R^i)^T.$$

Substitution (6) to (7) gives

$$w^i=C^i\,\hat{q}^j(u_i, \bar{x}_{i-1}). \tag{9}$$

Assignment of a particular utility function to a particular stage is random.

It is assumed that the probability of assigning one of R utility functions to stage i is a function of such assignments at proceeding stages. It is assumed that these probabilities are known.

Formally this problem may be stated as follows.

The assignment is treated as a discrete stochastic process $\Omega(i)$, $i=1, 2, ..., N$. The process is a sequence of random variables whose realizations are numbers of utility functions.

$$\Omega(i)=\{r_i \in \{1, 2, ..., R\}\}. \tag{10}$$

Let $p_i(r_i|J_s^{i-1})$ denote a conditional probability of the event that a utility function number r_i is assigned to stage i, provided that the assignments at the proceding stages formed the sequence

$$J_s^{i-1}=(r_1, r_2, ..., r_{i-1})_s, \quad i=2, 3, ..., N. \tag{11}$$

It is assumed that the utility function assigned to the first stage has been ascribed index 1. Thus J_s^{i-1} is a $(i-1)$-element sequence that is constructed of elements of R-element set $\{1, 2, ..., R\}$ in such a way that each sequence has 1 as the first element.

The number of possible such sequences is equal to the number of variations with repetitions of $i-1$ elements selected from R elements with the first term fixed.

$$\begin{aligned} &s=1, 2, ..., S_{i-1}, \\ &S_{i-1}=(R)^{i-2}, \quad i\geqslant 2. \end{aligned} \tag{12}$$

Moreover

$$\begin{aligned} &p_i(r_i|J_s^{i-1})\geqslant 0, \quad \forall i, r_i, s, \\ &\sum_{r_i=1}^{R} p_i(r_i|J_s^{i-1})=1, \quad \forall i, s. \end{aligned}$$

Probability of generating the sequence

$$(J_s^{i-1}, r_i)=J^i, \quad i=1, 2, ..., N, \quad s=1, 2, ..., S_i, \tag{13}$$

is given by a product of conditional probabilities

$$p_i(J^j)=p_i(r_i|J_s^{i-1}) \prod_{k=1}^{i-1} p_k(r_{ks}|J_s^{k-1}) \tag{14}$$

where

$$p_1(r_1=1)=1.$$

The probability of the event that the utility function r will be assigned to stage i (no matter what the preceding assignments J_s^{i-1}, $s=1, 2, ..., S_{i-1}$) is given by the following expression:

$$p_i(r_i) = \sum_{s=1}^{S_{i-1}} p_i(J_s^{i-1}, r_i). \tag{15}$$

Substituting (14) to (15) gives

$$p_i(r_i) = \sum_{s=1}^{S_{i-1}} p_i(r_i|J_s^{i-1}) \prod_{k=1}^{i-1} p_k(r_{ks}|J_s^{k-1}). \tag{16}$$

Let us consider the optimization problem now.

At each stage i local optimization can be carried on with respect to $u_i \in V_i$. The local optimization problem consists in determining u_{ir}^o, for all i and r, that satisfies the following relation:

For fixed $\bar{x}_{i-1} \in S_{i-1}$, $\bar{x}_i \in S_i$

$$\arg \max_{u_i \in V_i} w_r^i = u_{ir}^o(\bar{x}_{i-1}, \bar{x}_i). \tag{17}$$

Using (17) in (9) one can get

$$\tilde{w}_r^i = \sum_{j=1}^{M} c_{rj}^i \tilde{q}_{rj}^i(\bar{x}_{i-1}, \bar{x}_i) \tag{18}$$

where

$$\tilde{q}_{rj}^i(\bar{x}_{i-1}, \bar{x}_i) = \hat{q}_j^i[u_{ir}^o(\bar{x}_{i-1}, \bar{x}_i), \bar{x}_{i-1}].$$

The mean value of the utility function at each etage i is given by the expresiou

$$\tilde{w}_i = \sum_{r_i=1}^{R} p_i(r_i) \tilde{w}_r^i. \tag{19}$$

The global utility function W for all stages is equal to

$$W = \sum_{i=1}^{N} \tilde{w}_i$$

or after substitution of (19) and (18)

$$W = \sum_{i=1}^{N} \sum_{r_i=1}^{R} p_i(r_i) \sum_{j=1}^{M} c_{rj}^i \tilde{q}_{rj}^i(\bar{x}_{i-1}, \bar{x}_i). \tag{20}$$

Maximization of the global utility function with respect to the initial conditions $\bar{x}_{i-1} \in S_{i-1}$ and $\bar{x}_i \in S_i$ gives the switching points $\bar{x}_i^* \in S_i$, $i=1, 2, ..., N-1$, and the final point $\bar{x}_N^* \in S_N$

$$\arg \max_{\substack{\bar{x}_i \in S_i \\ i=1,...N}} \sum_{i=1}^{N} \sum_{r_i=1}^{R} p_i(r_i) \sum_{j=1}^{M} c_{rj}^i \tilde{q}_j^i(\bar{x}_{i-1}, \bar{x}_i) = (\bar{x}_1^*, ..., \bar{x}_N^*). \tag{21}$$

Having assigned an appropriate utility function to the first stage, one can determin, by performing operation (21), all switching points.

Sliding optimization can also be applied which makes possible using at the k-th stage information about the current (k-th) and preceding assignments.

The idea of the method is that at time T_{k-1}, that is at the beginning of k-th stage, when the sequence J^k is known, switching points $\bar{x}_k^*, \bar{x}_{k+1}^*, \ldots, \bar{x}_{N-1}^*, \bar{x}_N^*$ are determined by means of maximizing the utility function at stages $k, k+1, \ldots, N$

$$\arg \max_{\substack{\bar{x}_i \in S_i \\ i=k,\ldots,N}} \sum_{i=k}^{N} \sum_{r_i=1}^{R} p_i(r_i|J^{k-1}) \sum_{j=1}^{M} c_{rj}^i \tilde{q}_{rj}^i(\bar{x}_{i-1}, \bar{x}_i) = (\bar{x}_k^*, \ldots, \bar{x}_N^*). \tag{22}$$

The result of operation (22) is $\bar{x}_i^*$ for $i=k, \ldots, N$, but only $\bar{x}_k^*$ is used. When the process gets to the point $\bar{x}_k^*$, a proper utility function is assigned to the stage $k+1$ and optimization is performed again in order to determin $\bar{x}_{k+1}^*, \ldots, \bar{x}_N^*$.

Formal description of such a procedure may be the following: The sequence J_s^{i-1} is broken into two sequences

$$J_s^{i-1} = (J_s^{k-1}, J_s^{i-k})$$

where

$$\begin{aligned} J_s^{k-1} &= (1, r_2, \ldots, r_{k-1})_s, \\ J_s^{i-k} &= (r_k, \ldots, r_{i-1})_s. \end{aligned} \tag{23}$$

After $k-1$ stages the sequence J_s^{k-1} is a realization whose terms are known

$$J_s^{k-1} = J_{\hat{s}}^{k-1}.$$

Thus the probability of occurrence of sequence J^i after $k-1$ stages is given by the following formula

$$p_i^k(J^i) = p_i(J_{\hat{s}}^{k-1})\, p_i(r_i|J_s^{i-1}) \prod_{l=k}^{i-1} p_l(r_{ls}|J_s^{l-1}). \tag{24}$$

Substitution of (24) to (15) yields a probability of assigning r-th utility function to stage i for the realized sequence $J_{\hat{s}}^{k-1}$

$$p_i^k(r_i) = p_i(J_{\hat{s}}^{k-1}) \sum_{s=1}^{S_i-1} p_i(r_i|J_s^{i-1}) \prod_{l=k}^{i-1} p_l(r_{ls}|J_s^{l-1}). \tag{25}$$

Global value of the overall utility function, which is maximized at stage k in order to determin $\bar{x}_k^*, \ldots, \bar{x}_N^*$, is the following

$$\begin{aligned} W^o(k) = \sum_{i=1}^{k-1} p_i^i(r_i) \sum_{j=1}^{M} c_{rj}^i \tilde{q}_{rj}^i(\bar{x}_{i-1}^*, \bar{x}_i^*) + \\ + \max_{\substack{\bar{x}_i \in S_i \\ i=k,\ldots,N}} \sum_{i=k}^{N} \sum_{r_i=1}^{R} p_i(r_i|J^{k-1}) \sum_{j=1}^{M} c_{rj}^i \tilde{q}_{rj}^i(\bar{x}_{i-1}, \bar{x}_i). \end{aligned} \tag{26}$$

Formula (26) expresses the optimality principle for the case considered: the final stages of the optimal trajectory form the optimal trajectory, no matter what the preceding switching points are.

Numerical algorithmization of the methods proposed requires specification of assumptions on performance indices (linear or quadratic), of target sets and of a character of the stochastic process (Markov processes).

IV. Determination of Switching Points Applying Balanced Growth Principle of Utility Function

Let us consider a situation in which every stage is characterized by a set of utility functions being linear combinations of performance functionals

$$w^i = C^i q \tag{27}$$

where:

$$q=(q_1, ..., q_M)^T \in R_+^M,$$
$$q_j \in Q_j \subset R_+^1,$$
$$C^i = \{c_{rj}^i\}_{R \times M},$$
$$c_{rj}^i \geqslant 0, \qquad \text{for } i=1, 2, ..., N, \quad r=1, 2, ..., R, \quad j=1, 2, ..., M.$$

Having R utility functions simultaneously at each stage, it seams reasonable to require that each of these functions grows from one stage to the next and that their growth should be possibly uniform.

Let us make a formal statement of the above problem.

Requirement of growth of each of the utility functions from one stage to another has a form of a set of inequalities

$$C^i q \leqslant C^{i+1} q, \quad i=1, 2, ..., N-1, \quad q \geqslant 0 \tag{28}$$

Let us define the growth rate $\alpha_i(q)$ at stage i as the smallest of the quotients of r-th utility function at stage $i+1$ to r-th utility function at stage i

$$\alpha_i(q) = \min_{1 \leqslant r \leqslant R} \frac{\sum_{j=1}^{M} c_{rj}^{i+1} q_j}{\sum_{j=1}^{M} c_{rj}^i q_j}, \qquad i=1, 2, ..., N-1. \tag{29}$$

The growth rate of the whole process $\alpha(q)$ is the minimum value of $\alpha_i(q)$ over $i=1, 2, ..., N-1$.

$$\alpha(q) = \min_{1 \leqslant i \leqslant N-1} \alpha_i(q) = \min_{1 \leqslant i \leqslant N-1} \min_{1 \leqslant r \leqslant R} \frac{\sum_{j=1}^{M} c_{rj}^{i+1} q_j}{\sum_{j=1}^{M} c_{rj}^i q_j}. \tag{30}$$

Maximum growth rate of an N-stage process is called a quantity

$$\alpha^* = \max_{q \geqslant 0} \alpha(q) = \alpha(\lambda q^*) \quad \forall\, \lambda > 0 \tag{31}$$

where λ is a coefficient that appears because of uniformness of $\alpha_i(q)$, $\alpha(q)$, α^* with respect to q^*

$$q^* = (q_1^*, \ldots, q_M^*)^T$$

where $q_j^* \geqslant 0$ is a direction in R_+^M in which the growth rate is maximum.

For $q = \lambda q^*$, for $\forall \lambda > 0$ and for $\forall i \in [1, \ldots, N-1]$ the following inequality holds

$$\alpha^*\, C^i\, q^* \leqq C^{i+1}\, q^*. \tag{32}$$

For the whole process

$$\alpha^* P^0\, q^* \leqq P^1\, q^* \tag{33}$$

where

$$P^0 = \begin{bmatrix} C^1 \\ \cdots\cdots \\ \cdots\cdots \\ C^{N-1} \end{bmatrix}_{S \times M}, \qquad P^1 = \begin{bmatrix} C^2 \\ \cdots \\ \cdots \\ C_N \end{bmatrix}_{S \times M}, \qquad S = (N-1)\, R.$$

Symbol $\leqq$ means that at least one of the inequalities (33) is satisfied as equality.

Matrices P^0 and P^1 transform subspace R_+^M into a convex plyhedral cone (pyramid) spanned on columns of these matrices. Direction λq^* lies in this cone (Fig. 2.)

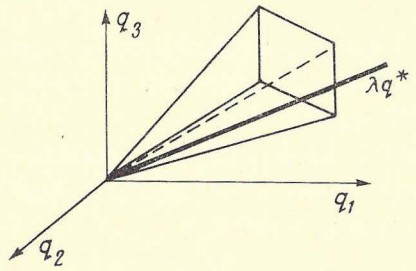

Figure 2

To summarize, the problem of determining a pair (α^*, q^*) is a problem of solving the following operation

$$\alpha^* = \max \{\gamma \geqslant 0 : \gamma\, P^0\, q \leqslant P^1\, q; \quad q \geqslant 0\} \tag{34}$$

where α^* is attained along the direction q^* for vectors λq^*, $\lambda > 0$.

The above problem is well known in mathematical economics [2], [6], [7]. In the problems of mathematical economics, the model described by a system of linear inequalities of type (33) is known as the von Neumann model. Let us write the von Neumann model in the form

$$\begin{aligned} &Ax \leqslant Bx \\ &A = \{a_{ij}\}_{S \times M}, \quad B = \{b_{ij}\}_{S \times M}, \\ &a_{ij} \geqslant 0, \quad b_{ij} \geqslant 0, \quad x \geqslant 0. \end{aligned} \tag{35}$$

Economic interpretation of this model is the following. The economy is composed of M technological processes which make S goods. Each good produced is used in the production process.

A is called an input matrix while B is called an output matrix. Elements of vector x are interpreted as activity levels the production processes. Scalar product of the i-th row of matrix A and vector x, $a_i^T x$, expresses the quantity of good i that is used in the production processes of the economy. Scalar product of the i-th row of matrix B and vector x, $b_i^T x$, is the quantity of good i produced. Inequality (35) says that quantities of all goods should grow in the process of production.

In the references concerning mathematical economics, given above, it is proved that provided that the following assumptions are met:

(i) for each j there exists an i such that $a_{ij}>0$ (each production process uses at leas one good),

(ii) for each i there exists an j such that $b_{ij}>0$ (each good is made by at least one production process),

a solution (α^*, x^*) of the problem

$$\alpha^* = \max [\gamma > 0 : \gamma Ax \leqslant Bx, \quad x \geqslant 0] \tag{36}$$

exists. There may be more than one direction x^*. Number α^* is called maximum technological efficiency of the economy and vector λx^*, $\lambda > 0$, is called technologically optimal vector of production levels.

Moreover, there exists a vector of prices

$$p^* = (p_1^*, \ldots, p_s^*) \geqslant 0 \tag{37}$$

such that

$$\alpha^* (p^*)^T Ax \geqslant (p^*)^T Bx, \quad \forall x \geqslant 0, \tag{38}$$
$$(p^*)^T Bx > 0.$$

Economic interpretation of vector p^* is the following. When the economy is controlled by means of prices, the prices p^* stimulate the production processes to settle their levels at proportions that are technologically optimal and that provide maximum technological efficiency of the economy.

The tripple (α^*, x^*, p^*) is called an equilibrium state of the von Neumann model.

Let us note that if $x^* > 0$ then the equilibrium state is unique. In that case the direction determined by x^* is called the von Neumann ray.

The von Neumann models are also used to describe processes of economic growth. In such cases vector x is devided into groups, each of which represents activity levels in different periods of growth.

In this situation different terminology is used. α^* is called a technological rate of growth, the von Neumann ray is called a turnpike. The latter comes from the fact that when the optimization goal is maximization of linear, or even under certain assumptions — nonlinear [8], functionals of the final state, considerable portions of optimal trajectories are in a neighbourhood of the turnpike. There are many so called turnpike theorems, which under differing assumptions give proofs of the above statement (see [2], [3], [9], [10], [11]).

It follows from the above considerations that the problem of determining α^* satysfying (33) is similar to the problem of determining the turnpike in the von Neumann model, although the model considered falls into a different class of problems and interpretation of the quantities involved is different.

In the problem discussed in this paper assumption (i) says that each performance index q_j should appear at least in one of the utility functions taken into consideration at the first $N-1$ stages.

Assumption (ii), on the other hand, requires that there should be no stage, beginning from stage 2, such that any of the utility functions taken into consideration would be identically zero.

Considering the nature of the problems discussed it can be assumed that both assumptions are satisfied. This allows for stating the existence of a solution of problem (34)

$$0<\alpha^*<\infty, \quad q^*\geqslant 0. \tag{39}$$

To complete the above discussion let us describe shortly a method of determining α^*.

The problem is to find a pair (α, q), $\alpha>0$, $q\in R_+^M$ such that among all pairs satisfying the following constraints

$$\alpha P^0 q\leqslant P^1 q, \tag{40}$$

$$\sum_{j=1}^{M} q_j=1, \tag{41}$$

it contains the largest α. The equality constraint (41) determines one point of the direction required.

In order to solve this problem let us solve the following linear programming subproblem. Find a minimum value of function Z in R^1 subject to the constraints

$$\begin{aligned} &Ze+P^1 q-\gamma P^0 q\geqslant 0,\\ &\sum_{j=1}^{M} q_j=1, \quad q\geqslant 0, \end{aligned} \tag{42}$$

where $\gamma\geqslant 0$ is a given parameter, e is a unit vector, $e=[1, 1, ..., 1]^T$.

The constraints (42) may be written in the following form

$$\begin{aligned} &Z+\sum_{j=1}^{M} (\pi_{kj}^1-\gamma\pi_{kj}^0) q_j\geqslant 0, \quad k=1, ..., S,\\ &\sum_{j=1}^{M} q_j=1, \quad q_j\geqslant 0, \quad j=1, ..., M \end{aligned} \tag{43}$$

where π_{kj}^0, π_{kj}^1 are elements of matrices P^0 and P^1 respectively.

Let $V(\gamma)$ denote a solution of the linear programming subproblem. It is given by the following expression

$$V(\gamma)=\min_{q\in\mathcal{F}}\ \min_{1\leqslant k\leqslant S}\ \sum_{j=1}^{M}\left[-\left(\pi_{kj}^{1}-\gamma\pi_{kj}^{0}\right)\right]q_j \tag{44}$$

where

$$\mathcal{F}=\left\{q=(q_1,\ldots,q_M)^T:\ \sum_{j=1}^{M}q_j=1,\quad q_j\geqslant 0\right\}.$$

As $\mathcal{F}$ is a bounded closed set and the function

$$\min_{1\leqslant k\leqslant S}\ \sum_{j=1}^{M}\left[-\left(\pi_{kj}^{1}-\gamma\pi_{kj}^{0}\right)q_j\right]=\varphi(\gamma,q) \tag{45}$$

is continuous with respect to q, the minimum in (44) is attainable and a solution of the linear programming subproblem exists.

It can be easily proved [7], that function $V(\gamma)$ has the following properties:

— it is continuous,

— it is monotonically decreasing, and

$$\begin{aligned} &V(\gamma)>0 \quad \text{for} \quad \gamma<\alpha^*,\\ &V(\gamma)\equiv 0 \quad \text{for} \quad \gamma=\alpha^*,\\ &V(\gamma)<0 \quad \text{for} \quad \gamma>\alpha^*. \end{aligned} \tag{46}$$

Figure 3 shows the graph of function $V(\gamma)$.

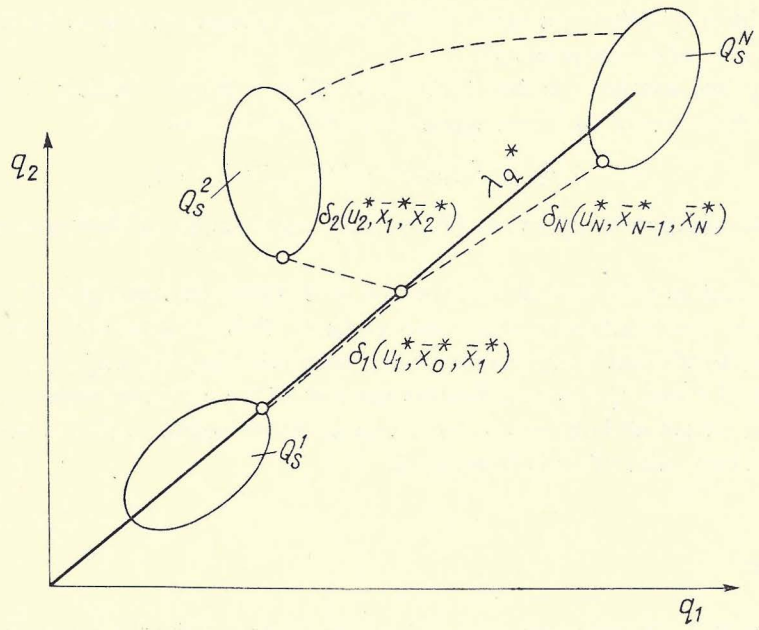

Figure 3

$Q_s^t=\{\bar{q}^t(u_i,\bar{x}_{i-1},\bar{x}_i):\ u_i\in V_i,\ \bar{x}_{i-1}\in S_{i-1},\ \bar{x}_i\in S_i\}$

Thus the problem of determining α^* is reduced to a problem of determining the largest root of function $V(\gamma)$. This can be done by means of any iterative method (e.g. Newton method) and it must be remembered that $V(\gamma)$ is to be computed in every iteration by means of solving the linear programming subproblem.

Let us return to the considered control problem.

On way of determining the controls u_i and the switching points $\bar{x}_i$ is minimization of a sum of distances of $\bar{q}_j^1(u_i, \bar{x}_{i-1}, \bar{x}_i)$ (see (5)) from a point on half line λq^* determined by a choice of λ (Fig. 3) witch respect to $\bar{x}_i \in S_i$, $u_i \in V_i$, $i=1, ..., N$.

So u_i and $\bar{x}_i$ should be determined by means of minimization of the norm of a difference of vectors

$$\sum_{i=1}^{N} \delta_i = \sum_{i=1}^{N} \|\lambda q^* - \bar{q}^i(u_i, \bar{x}_{i-1}, \bar{x}_i)\|. \tag{49}$$

The determined values of $u_i = u_i^*$ and $\bar{x}_i = \bar{x}_i^*$ satisfies the following relation

$$(u^*, \bar{x}) = \arg \min_{\substack{u_i \in V_i \\ \bar{x}_i \in S_i \\ i=1,...,N \\ \lambda > 0}} \sum_{i=1}^{N} \delta_i, \tag{50}$$

where

$$u^* = (u_1^*, ..., u_N^*)^T,$$

$$\bar{x}^* = (\bar{x}_1^*, ..., \bar{x}_N^*)^T,$$

$$\bar{q}_1^i(u_i, \bar{x}_{i-1}, \bar{x}_i) = [\bar{q}_1^i(u_i, \bar{x}_{i-1}, \bar{x}_i), ..., \bar{q}_M^i(u_i, \bar{x}_{i-1}, \bar{x}_i)]^T.$$

A problem of defining a norm in R^M arises. It seems reasonable to relate this norm to the equilibrium prices p^*.

Taking into account the fact that prices affect q trough matrices C^i (27), the following definition of the norm appearing in (49) is proposed

$$\delta_i = p^{*T} C^i |\lambda q^* - q^i(\bar{x}_{i-1}, \bar{x}_i)|. \tag{51}$$

Algorithmization of operation (50) requires specification of functionals q_j and sets S_i.

Finally, let us note, that also in the case of applying balanced growth principle, using sliding optimization procedure is reasonable. This procedure requires repeting operation (50) after each stage in order to determin switching points $\bar{x}_k^*, \bar{x}_{k+1}^*, ..., \bar{x}_{N-1}^*$ and the final point $\bar{x}_N^*$, where k denotes number of the currently considered stage.

Having passed all bottlenecks, i.e. stages at which inequality (33) is satisfied as equality, a new value of α^* is determined.

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Received, October 1978.

Sterowanie procesami wieloetapowymi przy maksymalizacji lub zrównoważonym wzroście funkcji użyteczności

Rozpatrzono sterowanie procesami wieloetapowymi. Na każdym etapie może zaistnieć inna sytuacja. Dla każdej sytuacji obowiązuje inny zbiór funkcji użyteczności.

Rozważono dwa różne przypadki. Jeden z nich sprowadza się do maksymalizacji wartości oczekiwanej globalnej funkcji użyteczności. W drugim przypadku na każdym etapie uwzględnia się odpowiedni podzbiór funkcji użyteczności, a sterowanie wyznacza się z zasady zrównoważonego wzrostu.

Управление многоэтапными процессами при максимизации или при уравновешенном росте функции полезности

Рассматривается управление многоэтапными процессами. На отдельных этапах могут возникнуть различные ситуации. Для каждой ситуации обязывает соответствующее подмножество функций полезности.

Рассматриваются два случая. Один из них основан на максимизации ожидаемого значения глобальной функции полезности. Во втором случае на каждом этапе учитывается соответствующее подмножество функций полезности, а управление определяется исходя из принципа уравновешенного роста.