Control
and Cybernetics
VOL. 8 (1979) No. 3

Closedness of the Attainable Set of the Linear Neutral Control System

by

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The problem of closedness of the attainable set of the linear neutral control system is considered. Algebraic criteria (expressed in terms of coefficients of the system) for closedness of the attainable set in the Sobolev space W_1^p are derived. The proof of the main result is based on a general theorem about closedness of the image of the integral operator given by B. Jakubczyk.

1. Introduction

In this paper we consider a linear neutral control system described by the equation

$$\dot{x}(t)=A_0 x(t)+A_1 x(t-h)+A_{-1}\dot{x}(t-h)+B_0 u(t) \tag{1}$$

where $x(t)\in R^n$, $u(t)\in R^m$, $h>0$; A_0, A_1, A_{-1}, B_0 are constant matrices of suitable dimensions, $t\in[0,T]$. We assume that a contıol u belongs to the space $L^p([0,T]; R^m)$, $u(t)=x(t)=0$ if $t\leqslant 0$.

Define by

$$x_t(\tau):=x(t+\tau),\ \ t\in[-h,0],\ \ t\geqslant 0, \tag{2}$$

the "complete state" of the system (1) at time t. A natural state space is the Sobolev space $W_1^p([-h,0]; R^n)$. Define the attainable set for (1) at time t by

$$\mathscr{A}(t):=\{x_t\in W_1^p([-h,0]; R^n)\,|\,x_t \text{ is given by (2) and } x \text{ is a soluton of (1) for some } u\in L^p\}. \tag{3}$$

The main result presented in this paper is a necessary and sufficient condition for closedness of $\mathscr{A}(T)$ in W_1^p, where $T=sh$, $s=1,2,\dots,n-1$, or $T\geqslant nh$. The corresponding theorem is stated in Sect. 2. The condition has the form

$$K_i(\ker Z^0)\subset \operatorname{im} Z^0,\qquad i=1,2,\dots,ns-1,$$

where K_i are matrices defined in Sect. 2 expressed in terms of A_0, A_1, A_{-1}, B_0 and Z^0 has the form

$$Z^0 := [A_{-1}^s B_0, \ldots, A_{-1} B_0, B_0]. \tag{4}$$

Above "ker" means a kernel and "im" — an image. The related results for a delay system ($A_{-1}=0$) were obtained by Kurcyusz and Olbrot in [7]. In Sect. 2 we derive their criteria from our Theorem 1. We give also some sufficient and some necessary conditions for closedness of $\mathscr{A}(T)$ which have a simpler form than the main criterion. The proof of Theorem 1, given in Sect. 3, is based on the results of B. Jakubczyk in [5].

Our interest for closedness of the attainable set of (1) is motivated by optimal control problems for such systems, when a terminal state is infinite dimensional. Closedness of $\mathscr{A}(T)$ is needed in order that some kinds of maximum principle be nontrivial (see [1], [2]).

2. The Main Result

To state the main theorem let us define some classes of matrices (s is fixed).

$$\Delta_j^0 := \begin{cases} I, & j=0, \\ 0, & j=1, \ldots, s \end{cases} \quad (n \times n \text{ — matrices}),$$

$$\Delta_j^1 := \begin{cases} A_0 & , \quad j=0, \\ A_{-1}^{j-1}(A_{-1}A_0 + A_1), & j=1, \ldots, s, \end{cases} \tag{5}$$

$$\Delta_j^i := \sum_{k=0}^{j} \Delta_k^{i-1} \Delta_{j-k}^1, \qquad i=2, \ldots, n(s+1)-1, \qquad j=0, \ldots, s,$$

$$Z_j^i := \sum_{k=0}^{j} \Delta_k^i A_{-1}^{j-k} B_0, \qquad i=0, \ldots, n(s+1)-1, \qquad j=0, \ldots, s,$$

$$Z^i := [Z_s^i, Z_{s-1}^i, \ldots, Z_0^i], \tag{6}$$

$$\begin{aligned} &\Omega_j^i := Z_j^i - A_0 Z_j^{i-1}, \qquad i=1, \ldots, n(s+1)-1, \qquad j=0, \ldots, s, \\ &\Omega^i := [\Omega_s^i, \ldots, \Omega_0^i]. \end{aligned} \tag{7}$$

From the definition of Z_j^i it is clear that it has the form

$$Z_j^i := \Sigma A_{-1}^{k_1}(A_{-1}A_0 + A_1)^{p_1} A_0^{r_1} A_{-1}^{k_2}(A_{-1}A_0 + A_1)^{p_2} A_0^{r_2} \ldots$$
$$\ldots A_{-1}^{k_s}(A_{-1}A_0 + A_1)^{p_s} A_0^{r_s} A_{-1}^b B_0 \tag{8}$$

where the sum is taken for all $k_1, \ldots, k_s, p_1, \ldots, p_s, r_1, \ldots, r_s, b \in N \cup \{0\}$, such that $\sum_{l=1}^{s} (p_l + r_l) = i$, $b + \sum_{l=1}^{s} (k_l + p_l) = j$. If $k_l = p_l = 0$ or $p_l = r_l = 0$ or $r_{l-1} = k_l = 0$ then

$k_m = p_m = r_m = 0$ for $m \geq l$. It means that in the above sum there are the compositions of the matrices A_{-1}, A_0, A_1, B_0 such that the number of A_0 in the composition is i and the number of A_{-1} is j. In Ω_j^i there are no components which have A_0 at the begining (from the definition of Ω_j^i) so we may express Ω_j^i also by (8) putting the additional condition $p_1 \neq 0$.

To formulate the theorem we need also matrices K_i

$$K_1 := \Omega^1,$$

$$K_i := \Omega^i - \sum_{j=1}^{i-1} \Omega^{i-j} (Z^0)^+ K_j, \qquad i = 1, \ldots, n(s+1) - 1, \tag{9}$$

where $(Z^0)^+$ is a fixed right inverse of Z^0, i.e. $Z^0 (Z^0)^+|_{im\, Z^0} = I$.

THEOREM 1. The set $\mathscr{A}(T)$, $T = (s+1)h$, is closed in W_1^p iff

$$K_i (\ker Z^0) \subset \operatorname{im} Z^0, \qquad i = 1, \ldots, n(s+1) - 1. \tag{10}$$

If $T > nh$ then we may put $s = n$ in (10).

COROLLARY 1. If $\mathscr{A}(T)$, $T = (s+1)h$, is closed in W_1^p then

$$\Omega^i (\ker Z^0) \subset \operatorname{im} Z^0 + (+_{j=1}^{i-1} \operatorname{im} \Omega^j), \qquad i = 1, \ldots, n(s+1) - 1. \tag{11}$$

Proof. The proof follows from Theorem 1 and the definition of K_i ■

COROLLARY 2. Each of the conditions given below is sufficient for closedness of $\mathscr{A}(T)$ in W_1^p

$$\operatorname{im} \Omega^i \subset \operatorname{im} Z^0, \qquad i = 1, \ldots, n(s+1) - 1, \tag{12}$$

$$\operatorname{rank} Z^0 = n, \tag{13}$$

$$\operatorname{rank} B_0 = n. \tag{14}$$

Proof. The condition (12) implies that $\operatorname{im} K_i \subset \operatorname{im} Z^0$ which implies (10). The condition (13) implies (12) and (14) implies (13). Hence Corollary follows from Theorem 1. ■

REMARK 1. The condition (13) implies a stronger fact. Namely, it is equivalent to closedness and finite codimensionality of $\mathscr{A}(T)$ in W_1^p (see [8] and [4]). Thus, the condition rank $[A_{-1}^s B_0, \ldots, B_0] = n$ is necessary for function space controllability on $[0, T]$ (i.e. $\mathscr{A}(T) = W_1^p$) for the system (1).

Example. Let us consider the scalar n-th order control system

$$x^{(n)}(t) = b_n x^{(n)}(t-h) + a_{n-1} x^{(n-1)}(t) + b_{n-1} x^{(n-1)}(t-h) +$$
$$+ \ldots + a_0 x(t) + b_0 x(t-h) + cu(t), \qquad b_n \neq 0.$$

It can be transformed in a standard way into the neutral system (1), where

$$A_{-1}:=\begin{bmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & b_n \end{bmatrix}, \qquad A_1:=\begin{bmatrix} 0 & & \dots & 0 \\ \vdots & & & \vdots \\ 0 & & & 0 \\ b_0 & b_1 & \dots & b_{n-1} \end{bmatrix},$$

$$A_0:=\begin{bmatrix} 0 & 1 & & \dots & 0 & 0 \\ 0 & 0 & & & & \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & & 0 & 1 \\ a_0 & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \end{bmatrix}, \qquad B_0:=\begin{bmatrix} 0 \\ \vdots \\ 0 \\ c \end{bmatrix}.$$

It is easy to verify that rank $Z^0=\operatorname{rank}[A_{-1}^s B_0, \dots, B_0]=1$, so the condition (13) is not satisfied. On the other hand, im $Z^0=\operatorname{im}[0, \dots, 1]^T$ and from the form of Ω_j^i given by (8) it is clear that $\operatorname{im}\Omega_j^i \subset \operatorname{im}[0, \dots, 1]^T$ .Hence, the condition (12) holds. Thus, the attainable set for this system is closed in W_1^p but its codimension is infinite.

Corollary 3 (Kurcyusz and Olbrot [7]). If $A_{-1}=0$ then $\mathscr{A}(T)$ is closed in W_1^p iff

$$\operatorname{im} A_1 A_0^i B_0 \subset \operatorname{im} B_0, \qquad i=0, \dots, n-1. \tag{15}$$

Proof. If $A_{-1}=0$ then $Z^0=[0, \dots, B_0]$. Notice that $\Omega_0^i=0$ (for every neutral system). This implies that $\Omega^i(\ker Z^0)=\operatorname{im}\Omega^i$, so the condition (11) is equivalent to (12) and they both are equivalent to closedness of $\mathscr{A}(T)$. We will prove that (12) is equivalent to the condition given by Olbrot and Kurcyusz. Indeed, notice that $\Omega_1^i= =A_1 A_0^{i-1} B_0$ and $\operatorname{im} Z^0=\operatorname{im} B_0$ which proves that (15) follows from (12). On the other hand, (15) implies that $\operatorname{im} A_1 B_0 \subset \operatorname{im} B_0$ so $\operatorname{im} A_1^k B_0 \subset \operatorname{im} B_0$ and $\operatorname{im} A_1^k A_0^i B_0 \subset \subset \operatorname{im} B_0$ which gives the proof of second implication (see the form of Ω_j^i). ■

Remark 2. Above we considered the attainable set $\mathscr{A}(T)$ only for $T=(s+1)h$, $s=0, 1, \dots$. Jacobs and Langenhop [3] proved that $\mathscr{A}(T)$ is constant for $T>nh$. Thus our conditions for closedness are valid for all $T>nh$ if we put $s=n$ in the main theorem.

3. Proof of Theorem 1

Let us consider the control system with an output

$$\begin{aligned} &\dot{x}=Ax+Bu, \quad t\in[-h, 0], \quad h>0, \quad x(-h)=0, \\ &\dot{y}=Cx+Du, \quad x\in R^k, \quad u\in R^q, \quad y\in R^r, \end{aligned} \tag{16}$$

A, B, C, D — constant matrices.

The input-output operator corresponding to this system

$$L: L^p([-h, 0]; R^q) \to W_1^p([-h, 0]; R^r)$$

has the form

$$y(t)=(Lu)(t)=Du(t)+\int_{-h}^{t} Ce^{A(t-\tau)}Bu(\tau)\,d\tau. \tag{17}$$

The following theorem is basic for the proof.

THEOREM 2 [5]. If $D=0$ then
im L is closed in V_1^p iff rank $L_k=k\cdot$ rank CB where

$$L_k:=\begin{bmatrix} CB & 0 & \dots & 0 \\ CAB & CB & & 0 \\ \vdots & & & \vdots \\ CA^{k-1}B & CA^{k-2}B & \dots & CB \end{bmatrix} \tag{18}$$

and $V_1^p=\{f\in W_1^p \mid f(-h)=0\}$. ∎

Now we shall transform our neutral system (1) into the system (16). Let $T=(s+1)h$. We put

$$x_i(t):=x(t+hi),\quad u_i(t):=u(t+hi),\quad t\in[-h,0],\quad i=1,\dots,s+1$$

$$\underline{x}:=\begin{bmatrix} x_1 \\ \vdots \\ x_{s+1}\end{bmatrix},\qquad \underline{u}:=\begin{bmatrix} u_1 \\ \vdots \\ u_{s+1}\end{bmatrix},$$

$$\underline{A}:=\begin{bmatrix} A_0 & 0 & \dots & 0 & 0 \\ A_1 & A_0 & & & \vdots \\ 0 & A_1 & & & \\ \vdots & & & A_0 & 0 \\ 0 & 0 & \dots & A_1 & A_0 \end{bmatrix}$$

$$\bar{A}:=\begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ A_{-1} & 0 & & & \\ 0 & A_{-1} & & & \vdots \\ \vdots & & & & \\ 0 & 0 & \dots & A_{-1} & 0 \end{bmatrix} \tag{19}$$

$$J:=\begin{bmatrix} 0 & \dots & 0 & 0 \\ I & 0 & & \\ 0 & I & & \vdots \\ \vdots & & & \\ 0 & \dots & I & 0 \end{bmatrix}$$

$n(s+1)\times n(s+1)$ — matrices

$$\bar{B}:=\begin{bmatrix} B_0 & 0 & \dots & 0 \\ 0 & B_0 & & \\ \vdots & & & \vdots \\ 0 & 0 & \dots & B_0 \end{bmatrix}$$

$n(s+1)\times m(s+1)$ — matrix.

With this notation we can write (1) as

$$\underline{\dot{x}}(t)=\underline{Ax}(t)+\bar{A}\underline{\dot{x}}(t)+\bar{B}\underline{u}(t), \qquad t\in[-h,0].$$

Since the matrix $I-\bar{A}$ is invertible we can define

$$\begin{aligned} &A:=(I-\bar{A})^{-1}\underline{A}, \qquad B:=(I-\bar{A})^{-1}\bar{B}, \\ &C:=[0,\ldots,0,I] \ -n\times n(s+1) \text{ matrix}, \qquad D=0. \end{aligned} \tag{20}$$

We must write also the continuity condition for the function x. It has the form

$$J\underline{x}(0)=\underline{x}(-h). \tag{21}$$

If we define

$$M_0(\underline{u}):=\int_{-h}^{0} e^{-A\tau}Bu(\tau)\,d\tau$$

then one can compute $x(-h)$

$$\underline{x}(-h)=(I-Je^{Ah})^{-1}JM_0(\underline{u}).$$

Now we have

$$x_{s+1}(t)=y(t)=(L\underline{u})(t)+(L_f\underline{u})(t)$$

where L is defined by (17) and L_f is finite dimensional operator $(L_f\underline{u})(t)=$ $=Ce^{A(t+h)}(I-Je^{Ah})^{-1}JM_0(\underline{u})$ (we must put $k=n(s+1)$, $q=m(s+1)$, $r=n$).

We may omit the operator L_f in our considerations because of the following theorem.

THEOREM 3. Let $A,B\colon X\to Y$ be linear bounded operators; X, Y — Banach spaces, B is finite dimensional operator. Then

$$\operatorname{im}(A+B) \text{ is closed in } Y \text{ iff } \operatorname{im} A \text{ is closed in } Y.$$

This is a trivial consequence from the known theorem about operators which have the closed image with finite codimension.

THEOREM 4. [6]. Let im A be closed in Y and codim im $A<\infty$. Assume that B is finite dimensional operator. Then

$$\operatorname{im}(A+B) \text{ is closed and } \operatorname{codim}\operatorname{im}(A+B)<\infty. \qquad ■$$

In order to use Theorem 2 to solve the problem of closedness of the attainable set we need some lemmas.

LEMMA 1. Let A, B, C be such as in (20) and Z_j^i as in (6). Then

$$CA^iB=[Z_s^i, Z_{s-1}^i, \ldots, Z_0^i].$$

Proof. It is easy to see that

$$(I-\bar{A})^{-1}=\begin{bmatrix} I & 0 & \dots & 0 \\ A_{-1} & I & & \vdots \\ \vdots & & & \\ A_{-1}^{s} & A_{-1}^{s-1} & \dots & I \end{bmatrix} \quad \text{and}$$

$$A=\begin{bmatrix} A_0 & 0 & \dots & 0 \\ A_{-1}A_0+A_1 & A_0 & & \vdots \\ A_{-1}(A_{-1}A_0+A_1) & A_{-1}A_0+A_1 & & \\ \vdots & \vdots & A_0 & 0 \\ A_{-1}^{s-1}(A_{-1}A_0+A_1) & \dots & A_{-1}A_0+A_1 & A_0 \end{bmatrix}$$

$$B=\begin{bmatrix} B_0 & 0 & \dots & 0 \\ A_{-1}B_0 & B_0 & & \vdots \\ \vdots & & & \\ A_{-1}^{s}B_0 & & \dots & B_0 \end{bmatrix}.$$

According to (5) we may write A as

$$\begin{bmatrix} \Delta_0^1 & 0 & \dots & 0 & 0 \\ \Delta_1^1 & \Delta_0^1 & & & \vdots \\ \vdots & & & \Delta_0^1 & 0 \\ \Delta_s^1 & & \dots & \Delta_1^1 & \Delta_0^1 \end{bmatrix}.$$

Now the conclusion follows from the rules of multiplication of matrices of such a form (Toeplitz's matrices) and the form of C. ■

Next lemmas are elementary facts fıom linear algebra, but we include the proofs for completness.

LEMMA 2. Let A, B, C be matrices of dimensions $k\times l$, $m\times l$, $m\times r$ respectively. Then

$$\operatorname{rank}\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}=\operatorname{rank} A+\mathit{rank}\ C \text{ iff} \tag{22}$$

$$B(\ker A)\subset \operatorname{im} C. \tag{23}$$

Proof. Let $A=[a_1, \dots, a_p, 0, \dots, 0]$ where a_i — linearly independent vectors, $B=[b_1, \dots, b_l]$, $C=[c_1, \dots, c_r]$. Then (22) is equivalent to the condition that $b_{p+1}, \dots, b_l$ are the linear combinations of $c_1, \dots, c_r$ so im $[b_{p+1}, \dots, b_l]\subset$ im C; but im $[b_{p+1}, \dots, b_l]=B(\ker A)$ so (22) is equivalent to (23).

If A has an arbitrary form we may change the coordinate system in such a way that A will have the torm as above and the equivalence will hold. ■

Let $P_0, \dots, P_{k-1}$ be the matrices of dimension $n\times m$. Let

$$D_k:=\begin{bmatrix} P_0 & 0 & \dots & 0 \\ P_1 & P_0 & & 0 \\ \vdots & & & \vdots \\ P_{k-1} & P_{k-2} & \dots & P_0 \end{bmatrix} \tag{24}$$

LEMMA 3. The following conditions are equivalent

$$\operatorname{rank} D_k = k \cdot \operatorname{rank} P_0, \tag{25}$$

$$\begin{bmatrix} P_1 \\ \vdots \\ P_{k-1} \end{bmatrix} (\ker P_0) \subset \operatorname{im} D_{k-1}, \tag{26}$$

$$\forall x \in \ker P_0 \, \exists x_1, \ldots, x_{k-1}:$$

$$(R_{k-1}) \begin{array}{l} P_1 x = P_0 x_1 \\ P_2 x = P_1 x_1 + P_0 x_2 \\ \vdots \\ P_{k-1} x = P_{k-2} x_1 + \ldots + P_0 x_{k-1} \end{array} \tag{27}$$

Proof. The equivalence of (25) and (26) follows from Lemma 2. The equivalence of (26) and (27) is evident. ■

Now we shall try to give the conditions for the solvability of the system (R_{k-1}) in (27). Let

$$\begin{aligned} K_1 &:= P_1, \\ K_i &:= P_i - \sum_{l=1}^{i-1} P_{i-l} P_0^+ K_l, \qquad i = 1, \ldots, k-1 \end{aligned} \tag{28}$$

where P_0^+ is some fixed right inverse of P_0, i.e.

$$P_0 P_0^+|_{im\, P_0} = I \text{ and let}$$

$$\tilde{x}_i := P_0^+ K_i x, \qquad x \in \ker P_0, \qquad i = 1, \ldots, k-1. \tag{29}$$

LEMMA 4. $\tilde{x}_i$ given by (29), $i=1, \ldots, k-1$, are the solution of (R_{k-1}) (for every $x \in \ker P_0$) iff

$$K_i (\ker P_0) \subset \operatorname{im} P_0, \qquad i = 1, \ldots, k-1. \tag{30}$$

Proof. Suppose that $\tilde{x}_i$ are given by (29); then by (30) we get

$$\begin{aligned} P_0 \tilde{x}_1 &= P_0 P_0^+ P_1 x = P_1 x, \\ P_0 \tilde{x}_2 &= P_0 P_0^+ K_2 x = K_2 x = P_2 x - P_1 \tilde{x}_1, \\ &\vdots \quad \vdots \\ P_0 \tilde{x}_{k-1} &= \ldots = K_{k-1} x = P_{k-1} x - \sum_{i=1}^{k-2} P_{k-1-i} \tilde{x}_i. \end{aligned}$$

Thus $\tilde{x}_i$ are the solution of (R_{k-1}).

Suppose that (30) is not satisfied for some i. Then there is $x \in \ker P_0$ such that the equation $P_0 x_i = K_i x$ has no solution. This contradicts the assumption. ■

The next Lemma gives more information about solutions of the system (R_{k-1}).

LEMMA 5. The following conditions are equivalent

$$\text{there are } x_1, \ldots, x_{k-1} \text{ satisfying } (R_{k-1}) \text{ (for } x \in \ker P_0), \tag{31}$$

$$\tilde{x}_i \text{ defined by (29) satisfy } (R_{k-1}). \tag{32}$$

Proof. The implication (32) $\Rightarrow$ (31) is trivial.

To see the opposite implication let x_i, $i=1, \ldots, k-1$, be the solution of (R_{k-1}). We shall prove that for every $i=1, \ldots, k-1$

$$\tilde{x}_1, \ldots, \tilde{x}_i \text{ satisfy } (R_i) \text{ and} \tag{33}$$

there are $z_1, \ldots, z_i \in \ker P_0$ such that

$$\tilde{x}_j = x_j - z_j - \sum_{p=1}^{j-1} u_p^{j+1-p}, \quad j=1, \ldots, i, \tag{34}$$

where $u_p^j = P_0^+ K_j z_p$.

Let $i=1$; (R_1) has the form $P_1 x = P_0 x_1$. From the existence of the solution x_1 it follows that $P_1 (\ker P_0) \subset \operatorname{im} P_0$ so $\tilde{x}_1$ satisfies (R_1) (from Lemma 4) and $\tilde{x}_1 = x_1 + z_1$ for some $z_1 \in \ker P_0$ (because $\tilde{x}_1 - x_1 \in \ker P_0$).

Let us assume now that (33) and (34) are satisfied for some i, $1 \leqslant i < k-1$. We will show that they are satisfied for $i+1$.

$$P_{i+1} x = P_i x_1 + \ldots + P_0 x_{i+1} = \sum_{j=1}^{i} P_{i+1-j} \tilde{x}_j + P_0 x_{i+1} -$$

$$- \sum_{j=1}^{i} P_{i+1-j} z_j + \sum_{p=2}^{i} P_{i+1-p} \sum_{j=1}^{p-1} u_p^{p+1-j}.$$

Notice that from the definition of K_j it follows that

$$\sum_{j=1}^{i} P_{i+1-j} z_j = \sum_{j=1}^{i} K_{i+1-j} z_j + \sum_{j=1}^{i-1} \sum_{p=2}^{i-j} P_{i+2-p-j} P_0^+ K_p z_j =$$

$$= \sum_{j=1}^{i} P_0 u_j^{i+1-j} + \sum_{j=1}^{i-1} \sum_{p=2}^{i-j} P_{i+2-p-j} u_j^p$$

($K_l (\ker P_0) \subset \operatorname{im} P_0$ for $l=1, \ldots, i$ from the induction assumption and Lemma 4). Now consider a sum

$$\sum_{p=2}^{i} P_{i+1-p} \sum_{j=1}^{p-1} u_p^{p+1-j}.$$

Let $s=p+1-j$. Then $p=s+j-1$, $p \leqslant i$, $s=2, \ldots, i$.

After the transformation the sum is equal to

$$\sum_{s=2}^{i} \sum_{j=1}^{i-1} P_{i+2-s-j} u_j^s$$

with the condition $s+j-1\leqslant i$ which gives

$$\sum_{j=1}^{i-1}\sum_{p=2}^{i-j} P_{i+2-p-j}u_j^p .$$

Finally we have

$$P_{i+1}x=\sum_{j=1}^{i} P_{i+1-j}\tilde{x}_j+P_0\Big(x_{i+1}-\sum_{j=1}^{i} u_j^{i+1=j}\Big)$$

so $\tilde{x}_1, \ldots, \tilde{x}_i$ and $\tilde{x}_{i+1}:=x_{i+1}+z_{i+1}-\sum_{j=1}^{i} u_j^{i+1-j}$ $(z_{i+1}\in\ker P_0)$ satisfy (R_{i+1}). For some z_{i+1} we have that $\tilde{x}_{i+1}=P_0^+K_{i+1}x$ by the latter equation and the definition of K_{i+1}. This proves the lemma. ∎

Lemma 6

$$\operatorname{rank} D_k=k\cdot\operatorname{rank} P_0 \text{ iff } K_i(\ker P_0)\subset\operatorname{im} P_0, \qquad i=1, \ldots, k-1 .$$

Proof. Follows from Lemmas 3, 4 and 5. ∎

Lemma 6 gives the proof of Theorem 1. Indeed, from the definition of Ω^k we have

$$\operatorname{rank} L_k=\operatorname{rank}\begin{bmatrix} Z^0 & 0 & \ldots & 0\\ Z^1 & Z^0 & & 0\\ \vdots & & & \vdots\\ Z^{k-1} & Z^{k-2} & \ldots & Z^0\end{bmatrix}=$$

$$=\operatorname{rank}\begin{bmatrix} Z^0 & 0 & \ldots & 0\\ Z^1-A_0Z^0 & Z^0 & & \vdots\\ \vdots & & & \\ Z^{k-1}-A_0Z^{k-2} & \ldots & & Z^0\end{bmatrix}$$

$$=\operatorname{rank}\begin{bmatrix} Z^0 & 0 & \ldots & 0\\ \Omega^1 & Z^0 & & \\ \vdots & & & \\ \Omega^{k-1} & \Omega^{k-2} & \ldots & Z^0\end{bmatrix}.$$

Assuming that $P_0=Z^0$, $P_i=\Omega^i$, $i=1, \ldots, k-1$, $k=n(s+1)$ we obtain the conclusion of Theorem 1 from Theorem 2 and Lemma 6. Notice that $\mathscr{A}(T)=$ $=\operatorname{im}(L+L_f)\subset W_1^p$ and from Theorem 3 the fact that $\mathscr{A}(T)$ is closed in W_1^p is equivalent to $\operatorname{im} L$ to be closed in V_1^p. This completes the proof of Theorem 1.

Remark 3. From the proof of Lemma 6 it is clear that in Theorem 1 we may change matrices K_i into $\bar{K}_i$ expressed in terms of Z^i

$$\bar{K}_i=Z^i-\sum_{j=1}^{i-1} Z^{i-j}(Z^0)^+K_j .$$

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Received, December 1978.

Domkniętość zbioru osiągalnego dla liniowego neutralnego układu sterowania

Rozważono problem domkniętości zbioru osiągalnego dla liniowego neutralnego układu sterowania. Wyprowadzono algebraiczne kryteria domkniętości zbioru osiągalnego w przestrzeni Sobolewa W_1^p. Kryteria są wyrażone w terminach współczynników układu. Dowód głównego wyniku sformułowano na podstawie ogólnego twierdzenia dotyczącego domkniętości obrazu operatora całkowego uzyskanego przez B. Jakubczyka.

Замкнутость достижимого множества линейной нейтральной системы управления

В работе рассматривается вопрос замкнутости достижимого множества для линейной нейтральной системы управления. Выводятся алгебраические критерии замкнутости достижимого множества в пространстве Соболева W_1^p. Критерии выражены в тержинах коэффициентов системы. Доказательство основного результата основано на общей теореме, касающейся замкнутости образа интегрального опрератора, полученной Б. Якубчиком.