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Finite-difference Approximations to Parabolic Free Boundary Value Problems Arising in Modelling of Underground Gas Reservoir. Part II. Convergence Proof. Numerical Results

by

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The paper presents convergence proof of the finite-difference approximations to parabolic free boundary value problems, introduced in part I. Numerical results are given.

Introduction

In part I of the paper [8] we have presented a finite-difference method for solving one-dimensional parabolic free boundary value problems suggested by the equations modelling flow of gas and water in an underground gas reservoir. The method is based on some preparatory transformation of the problem into another nonlinear parabolic problem in a domain with fixed boundary. Finite-difference scheme is of conservative type, expressing on the grid law of continuity of flow.

In Section 6 of the present paper we prove convergence of the finite-difference scheme described in [8].

We propose also a direct finite-difference method for solving free boundary value problems (Section 7). In this method preparatory transformation of the problem is not used.

In Section 8 numerical results and comparison of efficiency of both methods are presented.

In the paper we use the notations introduced in part I [8].

6. Convergence of the Finite-difference Scheme Based on Preparatory Transformation of Free Boundary Value Problem

In this section we are going to prove convergence of the finite-difference scheme (1) described in [8]. In the proof we will make use of some results of Kamynin [4, 5] relating to the continuous dependence of the solution of linear parabolic equation with discontinuous coefficients upon the boundary. In Appendix B we recall the main result of [5]. Some a priori-estimates for solution of two-layer parabolic problem associated with a given boundary Γ (Problems (b_k), $k=1, 2$, see Sec. 3) will be also useful.

Due to the maximum principle for two-layer parabolic free boundary value problems (see Theorem 6.2 [7]) the solution of Problem (b_1) corresponding to $y \in C^2 [0, T]$ can be a priori estimated as follows

$$\bar{M} \leqslant u_i (x, t) \leqslant \bar{\bar{M}}, \quad (x, t) \in \text{cl}\, D_i, \quad i=1, 2 \tag{6.1}$$

where

$$\begin{aligned} \bar{M} &= \min \{ \min_{x \in [0, y_0]} u_{10} (x), \min_{x \in [y_0, l]} u_{20} (x), \min_{t \in [0, T]} F_1 (t), \min_{t \in [0, T]} F_2 (t) \}, \\ \bar{\bar{M}} &= \max \{ \max_{x \in [0, y_0]} u_{10} (x), \max_{x \in [y_0, l]} u_{20} (x), \max_{t \in [0, T]} F_1 (t), \max_{t \in [0, T]} F_2 (t) \}. \end{aligned} \tag{6.2}$$

The following result has been proved in [1, 2] by employing the equivalent integral representation of Problem (b_k), $k=1, 2$.

Lemma 6.1. Assume that there is given a family of such curves $\{x=y(t) | 0<y(t)<l, t \in [0, T]\}$ that

$$y \in C^2 [0, T] \text{ and } |y'(t)| \leqslant c \text{ for } t \in [0, T]$$

where constant $c>0$ is the same for all y.

Let us consider the family $\{u_1, u_2\}$ of solutions to Problem (b_k) corresponding to the given family of curves y. Denote $v(t) \triangleq u_1 (y(t), t) = u_2 (y(t), t)$, $t \in [0, T]$. Then for function v' the following estimate holds

$$|v'(t)| \leqslant c' \text{ for } t \in [0, T]$$

with constant c' dependent on bounds of y, y', v and on given data of Problem (b_k).

Apart from the above facts the following interpolation lemma will be applied in the proof of convergence of the finite-difference scheme.

Lemma 6.2. Assume that there are given real numbers $a^0, a^1, \dots, a^L$; $b^0, b^1, \dots, b^L$ such that $|b^j| \leqslant B$, $j=0, 1, \dots, L$ where B is a given positive constant and

$$a^{j+1} = a^j + \tau b^j, \quad j=0, 1, \dots, L-1, \quad \tau>0.$$

Denote $t_0=0$, $t_j=j\tau$, $j=1, \dots, L$, $t_L=T$. The there exists a function $y \in C^2 [0, T]$ satisfying the following conditions:

(i) $y(t_j) = a^j$,

(ii) $y'(t_j) = b^j$, $j=0, 1, \dots, L$,

(iii) $|y'(t)| \leqslant B_1$, $t \in [0, T]$

where B_1 is a positive constant dependent only on B, i.e. $B_1 = B_1 (B)$.

Proof. To prove this lemma it is sufficient to show that there exists a function $y_j \in C^2[t_{j-1}, t_j]$ such that

(a) $y_j(t_{j-1})=a^{j-1}$, $y_j(t_j)=a^j$,

(b) $y_j'(t_{j-1})=b^{j-1}$, $y_j'(t_j)=b^j$,

(c) $y_j''(t_{j-1})=y_j''(t_j)=0$,

(d) $|y_j'(t)|\leqslant B_1$, $t\in[t_{j-1}, t_j]$.

It can be verified that the polynomial

$$y_j(t)=c_5\left(\frac{t-t_{j-1}}{\tau}\right)^5+c_4\left(\frac{t-t_{j-1}}{\tau}\right)^4+c_3\left(\frac{t-t_{j-1}}{\tau}\right)^3+$$
$$+c_2\left(\frac{t-t_{j-1}}{\tau}\right)^2+c_1\left(\frac{t-t_{j-1}}{\tau}\right)+c_0,\quad t\in[t_{j-1}, t_j] \tag{6.3}$$

where $c_5=3\tau(b^{j-1}-b^j)$, $c_4=-7\tau(b^{j-1}-b^j)$, $c_3=4\tau(b^{j-1}-b^j)$, $c_2=0$, $c_1=\tau b^{j-1}$, $c_0=a^{j-1}$, satisfies the conditions (a)–(d) with $B_1=111B$. Function y such that $y(t)=y_j(t)$ for $t\in[t_{j-1}, t_j]$, $j=1, \ldots, L$ satisfies conditions (i)–(iii). Q.E.D.

Now we are ready to formulate and to prove the main result of the paper.

THEOREM 6.1. Assume that for the solution of Problem (b_1) associated with a given function $y\in C^2[0, T]$, $0<y_m\leqslant y(t)\leqslant y_M<l$, $t\in[0, T]$ the following regularity conditions are fulfilled:

$$u_i\in C^{2,1}(\mathrm{cl}\, D_i),\quad i=1, 2;$$

$\frac{\partial u_i}{\partial x}$ satisfies Lipschitz continuity condition in $\mathrm{cl}\, D_i$ with respect to t; (6.4)

$\frac{\partial u_i}{\partial t}$ satisfies Lipschitz continuity condition in $\mathrm{cl}\, D_i$ with respect to x.

Then the finite-difference scheme (1) without iterations (see Section 5) is convergent on the grid

$$\Omega_{h\tau}^Q \triangleq \Omega_h\times\omega_\tau^Q,\quad \omega_\tau^Q\triangleq\{t_r|t_r=rQ\tau,\ r=0, 1, \ldots, K\} \tag{6.5}$$

to the solution of Problem (B_1^S), i.e.

$$\max_{(x_i, t_j)\in\Omega_{h\tau}^Q}|U_i^j-u(x_i, t_j)|\underset{h,\tau\to 0}{\to}0,$$
$$\max_{t_j\in\omega_\tau^Q}|Y^j-y(t_j)|\underset{h,\tau\to 0}{\to}0. \tag{6.6}$$

Proof. Following the finite-difference scheme (1) without iterations we obtain the approximate values

$$U_i^j,\ V^j,\ (Y')^j,\ Y^j\in(y_m, y_M),\quad i=0, 1, \ldots, N;\ j=0, 1, \ldots, L_1 \tag{6.7}$$

of functions u, v, y', y at the grid points $(x_i, t_j)\in\Omega_{h\tau}$, $t_j\in\omega_\tau$.

We can assume without loss of generality that $L_1=L$. If $L_1<L$ then we prove convergence of the difference scheme in an appropriately smaller time interval, where $y(t) \in [y_m, y_M]$.

According to the algorithm

$$\begin{aligned} &(Y')^j=(Y')^{(r-1)Q} \text{ for } (r-1)Q+1 \leqslant j \leqslant rQ-1, \\ &(Y')^{rQ}=\beta V^{rQ-1}, \ r=1, \ldots, K, \end{aligned} \tag{6.8}$$

$$\begin{aligned} &Y^j=Y^{j-1}+\tau (Y')^{j-1}, \ j=1, \ldots, L, \\ &Y^{rQ}=Y^{(r-1)Q}+Q\tau (Y')^{(r-1)Q}, \ r=1, \ldots, K. \end{aligned} \tag{6.9}$$

We start with proving the following lemmas.

LEMMA 6.3. For h, τ sufficiently small ($h \leqslant h^*$, $\tau \leqslant \bar{\tau}$)

$$|V^j| \leqslant B, \ j=0, 1, \ldots, L \tag{6.10}$$

where $B>0$ is a constant independent of h, τ.

Proof. Let us interpolate the points $(Y^0, 0), (Y^1, \tau), \ldots, (Y^L, L\tau)$ by a curve y_L satisfying the conditions:

$$y_L \in C^2 [0, T], \tag{6.11}$$

$$y_L(t_j)=Y^j, \ y_L'(t_j)=(Y')^j, \ j=0, 1, \ldots, L. \tag{6.11'}$$

In particular we can choose the curve of type (6.3), with $a^j=Y^j$, $b^j=(Y')^j$. One may assume that for sufficiently small τ ($\tau \leqslant \bar{\tau} \leqslant \tau^*$)

$$y_L(t) \in [y_m, y_M] \text{ for } t \in [0, T]. \tag{6.12}$$

Let us denote

$$\begin{aligned} &D_{1L} \triangleq \{(x, t) | x \in (0, y_L(t)), \ t \in (0, T)\}, \\ &D_{2L} \triangleq \{(x, t) | x \in (y_L(t), l), \ t \in (0, T)\}, \\ &v_L(t) \triangleq u_{1L}(y_L(t), t)=u_{2L}(y_L(t), t), \ t \in [0, T], \\ &u_L(x, t) \triangleq u_{iL}(x, t) \text{ for } (x, t) \in \operatorname{cl} D_{iL}, \ i=1, 2 \end{aligned}$$

where $\{u_{1L}, u_{2L}\}$ is a solution of Problem (b_1) corresponding to the function y_L. Note that according to the transformation (4.2)

$$u_{iL}(x, t)=\tilde{u}_{iL}(\xi, t), \ (x, t) \in \operatorname{cl} D_{iL}, \ (\xi, t) \in \operatorname{cl} \tilde{D}_i, \ i=1, 2 \tag{6.13}$$

where $\{\tilde{u}_{1L}, \tilde{u}_{2L}\}$ is a solution of Problem (4.3)–(4.6) associated with the function y_L. By $\tilde{u}_L$ and $\tilde{v}_L$ we denote

$$\begin{aligned} &\tilde{u}_L(\xi, t) \triangleq \tilde{u}_{iL}(\xi, t) \text{ for } (\xi, t) \in \operatorname{cl} \tilde{D}_i, \ i=1, 2, \\ &\tilde{v}_L(t) \triangleq \tilde{u}_{1L}\left(\frac{l}{2}, t\right)=\tilde{u}_{2L}\left(\frac{l}{2}, t\right), \ t \in [0, T]. \end{aligned} \tag{6.14}$$

Note also that in view of (4.2)

$$\tilde{v}_L(t) = v_L(t) \text{ for } t \in [0, T]. \tag{6.15}$$

By assumption (6.4) and conditions (6.11), (6.12) function $\tilde{u}_L$ satisfies (H10) (see Sec. 5). Consequently, for problem (4.3)–(4.6), associated with the function y_L, assumptions of Theorem 5.1 [8] are satisfied. It follows from this theorem that for sufficiently small h and τ ($h \leqslant h^*$, $\tau \leqslant \tau^*$)

$$\max_{(\xi_i, t_j) \in \omega_{h\tau}} |U_i^j - \tilde{u}_L(\xi_i, t_j)| \leqslant M \left(h \ln^\delta \frac{1}{h} + \tau \ln^\delta \frac{1}{\tau} \right), \tag{6.16}$$

where $M > 0$ is a constant independent of h and τ; $\delta \triangleq 1 + \varepsilon$, ε is any given positive constant. Since $V^j = U^j_{N/2}$, $j = 0, 1, \ldots, L$, by (6.16) we get the estimate

$$\max_{j \in \{0,1,\ldots,L\}} |V^j - \tilde{v}_L(t_j)| \equiv \max_{j \in \{0,1,\ldots,L\}} |V^j - v_L(t_j)| \leqslant M \left(h \ln^\delta \frac{1}{h} + \tau \ln^\delta \frac{1}{\tau} \right). \tag{6.17}$$

Now observe that by (6.1) we have a priori-estimates for the functions u_{iL}

$$|u_{iL}(x, t)| \leqslant M_1, \ (x, t) \in \operatorname{cl} D_{iL}, \ i = 1, 2 \tag{6.18}$$

where $M_1 = \max\{|\bar{M}|, |\overline{M}|\}$; $\bar{M}$, $\overline{M}$ are defined by (6.2). Here M_1 is independent of y_L, i.e. independent of h and τ. By (6.13) and (6.18) we get

$$|\tilde{u}_{iL}(\xi, t)| \leqslant M_1 \text{ for } (\xi, t) \in \operatorname{cl} \tilde{D}_i. \tag{6.19}$$

Hence, in particular

$$|\tilde{v}_L(t)| \leqslant M_1 \text{ for } t \in [0, T]. \tag{6.20}$$

Combining the inequalities (6.17) and (6.20) we get

$$|V^j| \leqslant M_1 + M \left(h \ln^\delta \frac{1}{h} + \tau \ln^\delta \frac{1}{\tau} \right), \ j = 0, 1, \ldots, L.$$

Therefore, if $h \leqslant h^* \leqslant e^{-\delta}$, $\tau \leqslant \bar{\tau} \leqslant e^{-\delta}$, then the estimate (6.10) holds with the constant $B = M_1 + M(h^* \ln^\delta 1/h^* + \bar{\tau} \ln^\delta 1/\bar{\tau})$. Thus we have shown that the constant B is independent of h and τ. Q.E.D.

Note that by (6.8), (6.10)

$$|(Y')^j| \leqslant \beta B \text{ for } j = 0, 1, \ldots, L. \tag{6.21}$$

It follows from Lemma 6.2 that

$$|y_L'(t)| \leqslant B_1 \text{ for } t \in [0, T] \tag{6.21'}$$

where B_1 is a positive constant dependent only on βB.

LEMMA 6.4. For h, τ sufficiently small ($h \leqslant h^*$, $\tau \leqslant \bar{\tau}$)

$$|V^j - V^{j-1}| \leqslant M_2 \left(h \ln^\delta \frac{1}{h} + \tau \ln^\delta \frac{1}{\tau} + \tau \right) \tag{6.22}$$

where $M_2 > 0$ is a constant independent of h, τ.

Proof. Observe that

$$|V^j - V^{j-1}| \leqslant |V^j - v_L(t_j)| + |v_L(t_{j-1}) - V^{j-1}| + |v_L(t_j) - v_L(t_{j-1})|. \qquad (6.23)$$

For the first and the second term on the right-hand side of (6.23) the estimate (6.17) holds. The third term can be estimated according to Lemma 6.1 as follows

$$|v_L(t_j) - v_L(t_{j-1})| \leqslant c'\tau \qquad (6.24)$$

where $c'>0$ depends on bounds of y_L, y_L', v_L and on given data of Problem (b_1). Thus, in view of (6.12), (6.21') and (6.20) c' is independent of h and τ. Combining (6.23), (6.17) and (6.24) we obtain (6.22). Q.E.D.

LEMMA 6.5. For h, τ sufficiently small ($h \leqslant h^*$, $\tau \leqslant \bar{\tau}$)

$$\left| \frac{(Y')^{rQ} - (Y')^{(r-1)Q}}{Q\tau} \right| \leqslant C, \quad r = 1, \ldots, K \qquad (6.25)$$

where $C>0$ is a constant independent of h, τ.

Proof. Note that

$$\left| \frac{(Y')^{rQ} - (Y')^{(r-1)Q}}{Q\tau} \right| = \beta \left| \frac{V^{rQ-1} - V^{(r-1)Q-1}}{Q\tau} \right| \leqslant \beta \left| \frac{V^{rQ-1} - v_L(t_{rQ-1})}{Q\tau} \right| + \beta \left| \frac{v_L(t_{(r-1)Q-1}) - V^{(r-1)Q-1}}{Q\tau} \right| + \beta \left| \frac{v_L(t_{rQ-1}) - v_L(t_{(r-1)Q-1})}{Q\tau} \right| \triangleq W_1 + W_2 + W_3. \qquad (6.26)$$

Applying (6.17) and taking into account that

$$Q\tau \geqslant \max\{h^{1-\varepsilon_1}, \tau^{1-\varepsilon_2}\}$$

we can estimate the first and the second term on the right-hand side of (6.26) in the following way

$$W_\nu \leqslant \beta M \left[\frac{h \ln^\delta \frac{1}{h}}{h^{1-\varepsilon_1}} + \frac{\tau \ln^\delta \frac{1}{\tau}}{\tau^{1-\varepsilon_2}} \right] = \beta M \left(h^{\varepsilon_1} \ln^\delta \frac{1}{h} + \tau^{\varepsilon_2} \ln^\delta \frac{1}{\tau} \right), \quad \nu = 1, 2.$$

Therefore, if $h \leqslant h^* \leqslant e^{-\delta/\varepsilon_1}$, $\tau \leqslant \bar{\tau} \leqslant e^{-\delta/\varepsilon_2}$ then

$$W_\nu \leqslant \beta M \left[(h^*)^{\varepsilon_1} \ln^\delta \frac{1}{h^*} + (\bar{\tau})^{\varepsilon_2} \ln^\delta \frac{1}{\bar{\tau}} \right], \quad \nu = 1,2. \qquad (6.27)$$

By Lemma 6.1 we get

$$W_3 \leqslant \beta c' \qquad (6.28)$$

where $c'>0$ is a constant independent of h, τ. From (6.26), (6.27) and (6.28) it follows (6.25). Q.E.D.

Now we are going to show that there may be constructed a sequence of functions y_K and corresponding to it sequence of solutions of Problem (b_1) converging to the solution of Problem (B_1^s).

Making use of the construction described in the proof of Lemma 6.2 let us interpolate the points $(Y^0, 0), (Y^Q, Q\tau), \ldots, (Y^{KQ}, KQ\tau)$ by a curve y_K satisfying the conditions

$$y_K \in C' [0, T],\ y_K(t_{rQ}) = Y^{rQ},\ Y_K'(t_{rQ}) = (Y')^{rQ},\ r = 0, 1, \ldots, K. \tag{6.29}$$

One can verify that if y_K is the curve of type (6.3), then in view of (6.21), (6.25) there exist constants $B_1 = 111\beta B$, $B_2 = 168C$, independent of h and τ, such that

$$|y_K'(t)| \leqslant B_1,\ |y_K''(t)| \leqslant B_2 \text{ for } t \in [0, T]. \tag{6.30}$$

Observe that for sufficiently small τ

$$y_K(t) \in [y_m, y_M],\ t \in [0, T]. \tag{6.31}$$

Now let us denote by $\mathscr{S}_1$ and $\mathscr{S}_2$ the following sets of functions:

$$\mathscr{S}_1 \triangleq \{y_K \mid K = 1, 2, \ldots\} \subset C[0, T],$$
$$\mathscr{S}_2 \triangleq \{y_K' \mid K = 1, 2, \ldots\} \subset C[0, T].$$

From (6.30), (6.31) it follows immediately that functions being elements of the sets $\mathscr{S}_1$ and $\mathscr{S}_2$ are equi-bounded and equi-continuous. By the Ascoli-Arzelà Theorem we conclude that the sets $\mathscr{S}_1$, $\mathscr{S}_2$ are compact in the space of continuous functions $C[0, T]$ with the norm $\|f\|_0 \triangleq \sup\limits_{t \in [0, T]} |f(t)|$. Consequently there exist subsequences $\{y_{K_\nu}\}_{K_\nu = 1}^{\infty}$, $\{y_{K_\nu}'\}_{K_\nu = 1}^{\infty}$ uniformly convergent in the interval $[0, T]$ to some functions $y \in C[0, T]$ and $g \in C[0, T]$, respectively. Since

$$y_{K_\nu}(t) = y_0 + \int_0^t y_{K_\nu}'(\zeta)\, d\zeta,$$

we get

$$y(t) = y_0 + \int_0^t g(\zeta)\, d\zeta.$$

Therefore $g(t) \equiv y'(t)$ for $t \in [0, T]$.

Now we will make use of Theorem B (see Appendix B). It follows from this theorem that

$$\lim_{\|y_{K_\nu} - y\|_{1, T} \to 0} \ \sup_{(x, t) \in S^i(y_{K_\nu}, y; T)} |u_{iK_\nu}(x, t) - u_i(x, t)| = 0, \quad i = 1, 2 \tag{6.32}$$

where $\{u_1, u_2\}$ is the solution of Problem (b_1) corresponding to the function y. Taking into account definition of the norm $\|\cdot\|_{1, T}$ (see Appendix B) we get

$$\|y_{K_\nu} - y\|_{1, T} \leqslant \sup_{t \in [0, T]} |y_{K_\nu}(t) - y(t)| + \sup_{t \in [0, T]} |y_{K_\nu}'(t) - y'(t)| \xrightarrow[K_\nu \to \infty]{} 0. \tag{6.33}$$

Next we shall show that

$$\lim_{\|y_{K_\nu}-y\|_{1,T}\to 0}\ \sup_{t\in[0,T]} |u_{iK_\nu}(y_{K_\nu}(t),t)-u_i(y(t),t)|=0,\ i=1,2, \tag{6.34}$$

i.e.

$$\lim_{\|y_{K_\nu}-y\|_{1,T}\to 0}\ \sup_{t\in[0,T]} |v_{K_\nu}(t)-v(t)|=0. \tag{6.35}$$

Indeed, observe that

$$|v_{K_\nu}(t)-v(t)|\leqslant |u_{K_\nu}(y_{K_\nu}(t),t)-u(y_{K_\nu}(t),t)|+|u(y_{K_\nu}(t),t)-u(y(t),t)|. \tag{6.36}$$

It follows from (6.32) that the first term on the right-hand side of (6.36) tends to zero when $\|y_{K_\nu}-y\|_{1,T}\to 0$. The second term tends also to zero since u is continuous in $\mathrm{cl}\,D$. Therefore (6.35) is actually satisfied.
It remains to show that

$$y(t)=y_0+\beta\int_0^t v(\zeta)\,d\zeta \ \text{ for } t\in[0,T]. \tag{6.37}$$

To this end first we are going to show that

$$|V^{rQ}-v_{K_\nu}(t_{rQ})|\leqslant W_1(h,\tau),\ r=0,1,\ldots,K_\nu \tag{6.38}$$

where $W_1(h,\tau)=M\left(h\ln^\delta\frac{1}{h}+\tau\ln^\delta\frac{1}{\tau}\right)+2M\left(h\ln^\delta\frac{1}{h}+Q\tau\ln^\delta\frac{1}{Qt}\right)$. Indeed, observe that

$$|U_i^{rQ}-\tilde{u}_{K_\nu}(\xi_i,t_{rQ})|\leqslant|U_i^{rQ}-\tilde{u}_L(\xi_i,t_{rQ})|+|\bar{U}_i^{rQ}-\tilde{u}_L(\xi_i,t_{rQ})|+$$
$$+|\bar{U}_i^{rQ}-\tilde{u}_{K_\nu}(\xi_i,t_{rQ})| \tag{6.39}$$

where $\bar{U}_i^{Qr}$, $i=0,1,\ldots,N$; $r=0,1,\ldots,K_\nu$ is a solution of the following finite-difference scheme

$$B_i^{rQ}\frac{\bar{U}_i^{rQ}-\bar{U}_i^{(r-1)Q}}{Q\tau}-\frac{1}{h}\left[A_{i+1}^{rQ}\frac{\bar{U}_{i+1}^{rQ}-\bar{U}_i^{rQ}}{h}-A_i^{rQ}\frac{\bar{U}_i^{rQ}-\bar{U}_{i-1}^{rQ}}{h}\right]$$
$$-C_i^{rQ}\left[\kappa A_{i+1}^{rQ}\frac{\bar{U}_{i+1}^{rQ}-\bar{U}_i^{rQ}}{h}-(1-\kappa)A_i^{rQ}\frac{\bar{U}_i^{rQ}-\bar{U}_{i-1}^{rQ}}{h}\right]=0,$$
$$i=1,\ldots,N-1;\ r=1,\ldots,K_\nu, \tag{6.40}$$

$$\bar{U}_0^{rQ}=F_1(t_{rQ}),\ \bar{U}_N^{rQ}=F_2(t_{rQ}),\ r=1,\ldots,K_\nu, \tag{6.41}$$

$$\bar{U}_i^0=\tilde{u}_0(\xi_i),\ i=0,1,\ldots,N; \tag{6.42}$$

A_i^{rQ}, B_i^{rQ}, C_i^{rQ} are defined by (5.11), (5.28).

For the first term on the right-hand side of (6.39) the estimate (6.16) holds. According to Theorem 5.1 [8]

$$|\bar{U}_i^{rQ}-\tilde{u}_{K_\nu}(\xi_i,t_{rQ})|,\ |\bar{U}_i^{rQ}-\tilde{u}_L(\xi_i,t_{rQ})|\leqslant M\left(h\ln^\delta\frac{1}{h}+Q\tau\ln^\delta\frac{1}{Q\tau}\right),$$
$$i=0,1,\ldots,N;\ r=0,1,\ldots,K_\nu. \tag{6.43}$$

From (6.39), (6.16) and (6.43) it follows that

$$|U_i^{rQ}-\tilde{u}_{K_\nu}(\xi_i, t_{rQ})|\leqslant W_1(h,\tau), \quad i=0,1,...,N; \quad r=0,1,...,K_\nu. \tag{6.44}$$

Since $V^{rQ}=U^{rQ}_{N/2}$ and $v_{K_\nu}(t)=\tilde{v}_{K_\nu}(t)$, we get by (6.44) the estimate (6.38). Now observe that

$$y_{K_\nu}(t_{rQ})=y_{K_\nu}(t_{(r-1)Q})+Q\tau(Y')^{(r-1)Q}=y_0+Q\tau\sum_{s=0}^{r-1}(Y_0')^{sQ}=$$

$$=y_0+\beta Q\tau\Big[V^0+\sum_{s=1}^{r-1}V^{sQ-1}\Big], \quad r=1,...,K_\nu. \tag{6.45}$$

Noting that

$$|V^{sQ-1}-v_{K_\nu}(t_{sQ})|\leqslant|V^{sQ-1}-V^{sQ}|+|V^{sQ}-v_{K_\nu}(t_{sQ})|, \quad s=1,...,K_\nu \tag{6.46}$$

and taking into consideration (6.22) as well as (6.38) we get

$$|V^{sQ-1}-v_{K_\nu}(t_{sQ})|\leqslant W(h,\tau), \quad s=1,...,K_\nu \tag{6.47}$$

where $W(h,\tau)=W_1(h,\tau)+M_2\left(h\ln^\delta\frac{1}{h}+\tau\ln\delta\frac{1}{\tau}+\tau\right)$.

Moreover, let us note that

$$V^0=v_{K_\nu}(t_0). \tag{6.48}$$

From (6.45), (6.47) and (6.48) we obtain the following estimates

$$y_0+\beta Q\tau\sum_{s=0}^{r-1}v_{K_\nu}(t_{sQ})-\beta Q\tau(r-1)W(h,\tau)\leqslant y_{K_\nu}(t_{rQ})\leqslant$$

$$\leqslant y_0+\beta Q\tau\sum_{s=0}^{r-1}v_{K_\nu}(t_{sQ})+\beta Q\tau(r-1)W(h,\tau), \quad r=1,...,K_\nu. \tag{6.49}$$

Now let $h,\tau\to 0$ (consequently $K_\nu\to\infty$, $Q\to\infty$). Passing to the limit in (6.49) and taking into consideration that

$$\sup_{t\in[0,T]}|y_{K_\nu}(t)-y(t)|\to 0, \qquad \sup_{t\in[0,T]}|v_{K_\nu}(t)-v(t)|\to 0$$

as well as that $Q\tau\to 0$, $W(h,\tau)\to 0$ when $h,\tau\to 0$, we get (6.37).

It follows from the above considerations that there exist the limit functions u_1, u_2, y satisfying all the conditions of Problem (B_1^S). Due to uniqueness of the solution to this problem [7] we can claim that the entire sequences $\{y_K\}_{K=1}^\infty$, $\{y_K'\}_{K=1}^\infty$, $\{v_K\}_{K=1}^\infty$, $\{u_{iK}\}_{K=1}^\infty$, $i=1,2$ are convergent.

So we have constructed the sequence $\{u_{1K}, u_{2K}\}_{K=1}^\infty$ of solutions to Problem (b_1) associated with the sequence of functions $\{y_K\}_{K=1}^\infty$ and we have proved that the limit functions

$$y=\lim_{K\to\infty}y_K, \quad u_i=\lim_{K\to\infty}u_{iK}, \; i=1,2$$

are the solution of Problem (B_1^S). To complete the proof let us note that

$$|U_i^{rQ}-u(x_i,t_{rQ})|\leqslant|U_i^{rQ}-u_K(x_i,t_{rQ})|+|u_K(x_i,t_{rQ})-u(x_i,t_{rQ})|,$$

$$|Y^{rQ}-y(t_{rQ})|\leqslant|Y^{rQ}-y_K(t_{rQ})|+|y_K(t_{rQ})-y(t_{rQ})|,\qquad i=0,1,\dots,N,\qquad r=0,1,\dots,K.$$

Hence by (6.44), (6.32), (6.29) and (6.33) we get the assertion of Theorem 6.1.
Q.E.D.

7. The Direct Finite-difference Scheme

Apart from the method employing the preparatory transformation of the problem we propose a direct method. The direct method is applicable to solving free boundary value problems without any preparatory stages. In case of the Problem (B_1^S) (see Section 3) one can prove convergence of this method in much the same manner as in Section 6 for the method with preparatory transformation.

In [7] apart from Problems (B_k^S) we have introduced free boundary value problems, denoted by (A_k), which differ from (B_k^S) in form of the ordinary differential equation describing dynamics of the free boundary. In the case of Problems (A_k) we have not got proof of convergence neither of the method with preparatory transformation nor the direct method, but we have obtained computational results, suggesting efficiency of both algorithms. In the next section we will present these results and discuss them.

Now we recall Problems (A_k). Find functions $\{p_1, p_2, y\}$ satisfying the following conditions:

$$\frac{\partial p_i}{\partial t}-\alpha_i\frac{\partial^2 p_i}{\partial x^2}=0 \text{ in } D_i,\qquad i=1,2,\tag{7.1}$$

$$y(0)=y_0,\qquad p_i(x,0)=p_{i0}(x) \text{ in } Z_i,\tag{7.2}$$

$$p_1(y(t),t)=p_2(y(t),t),\tag{7.3}$$

$$a_1\frac{\partial p_1}{\partial x}(y(t),t)=a_2\frac{\partial p_2}{\partial x}(y(t),t),\qquad t\in(0,T],\tag{7.4}$$

$$\text{for Problem } (A_1)\text{: } p_i(l_i,t)=f_i(t),\qquad t\in(0,T],\tag{7.5}$$

$$\text{for Problem } (A_2)\text{: } a_i\frac{\partial p_i}{\partial x}(l_i,t)=F_i(t),\qquad t\in(0,T],\tag{7.5'}$$

$$\frac{dy}{dt}(t)=-\beta a_1\frac{\partial p_1}{\partial x}(y(t),t),\qquad t\in(0,T].\tag{7.6}$$

Here α_i, a_i, β, $y_0\in(0,l)$ are given positive constants and p_{i0}, f_i, F_i are given functions. The above problems form mathematical model of a controlled underground gas reservoir, expressed in terms of pressure distribution [3, 6]. The boundary conditions depend on the type of control.

As in [8] we will perform the process of constructing the direct finite-difference scheme in two stages. Note that in Problems (A_k) there occurs an additional difficulty associated with approximation of $\frac{\partial p_1}{\partial x}$ at the free boundary. On account of this we restrict ourselves to the version of the direct finite-difference scheme related to Problems (A_k).

Stage I. First we approximate the auxiliary problems (7.1)–(7.5′) associated with a given function $y \in C^2[0, T]$. We will assume boundary conditions in the form more general than (7.5) and (7.5′):

$$a_i \frac{\partial p_i}{\partial x}(l_i, t) + (-1)^i \sigma_i(t) p_i(l_i, t) = g_i(t), \quad t \in (0, T], \qquad i = 1, 2 \tag{7.7}$$

where σ_i satisfy condition (H9) (see Section 5). We introduce the regular grid $\omega_{h\tau}$ (see Section 2). Using the integral-interpolation method [9] we obtain the following system of implicit finite-difference equations for P_i^j, $(x_i, t_j) \in \omega_{h\tau}$ (P_i^j denote approximate values of a solution to problem (7.1)–(7.4), (7.7)):

$$\tilde{\mathscr{F}}_1[P_i^j] \triangleq B_i^j \frac{P_i^j - P_i^{j-1}}{\tau} - \frac{1}{h}\left[A_{i+1}^j \frac{P_{i+1}^j - P_i^j}{h} - A_i^j \frac{P_i^j - P_{i-1}^j}{h}\right] = 0$$

$$i = 1, \ldots, N-1, \quad j = 1, \ldots, L, \tag{7.8}$$

$$\tilde{\mathscr{F}}_2[P_i^j] \triangleq A_i^j \frac{P_1^j - P_0^j}{h} - \sigma_1(t_j) P_0^j - \frac{a_1 h}{2\alpha_1} \frac{P_0^j - P_0^{j-1}}{\tau} = g_1(t_j),$$

$$\tilde{\mathscr{F}}_3[P_i^j] \triangleq A_N^j \frac{P_N^j - P_{N-1}^j}{h} + \sigma_2(t_j) P_N^j + \frac{a_2 h}{2\alpha_2} \frac{P_N^j - P_N^{j-1}}{\tau} = g_2(t_j), \tag{7.9}$$

$$j = 1, \ldots, L,$$

$$\tilde{\mathscr{F}}_4[P_i^j] \triangleq P_i^0 = p_0(x_i), \quad i = 0, 1, \ldots, N \tag{7.10}$$

where

$$A_i^j = \left[\frac{1}{h} \int_{x_{i-1}}^{x_i} \frac{dx}{a(x, t_j)}\right]^{-1}, \quad B_i^j = \frac{1}{h} \int_{x_{i-1/2}}^{x_{i+1/2}} b(x, t_j)\, dx, \tag{7.11}$$

$$a(x, t) \triangleq \begin{cases} a_1 & \text{for } (x, t) \in \operatorname{cl} D_1 \setminus \operatorname{cl} \Gamma, \\ a_2 & \text{for } (x, t) \in \operatorname{cl} D_2, \end{cases} \tag{7.12}$$

$$b(x, t) \triangleq \begin{cases} \dfrac{a_1}{\alpha_1} & \text{for } (x, t) \in \operatorname{cl} D_1 \setminus \operatorname{cl} \Gamma, \\ \dfrac{a_2}{\alpha_2} & \text{for } (x, t) \in \operatorname{cl} D_2, \end{cases} \tag{7.12}$$

$$p_0(x) \triangleq \begin{cases} p_{10}(x) & \text{for } x \in [0, y_0) \\ p_{20}(x) & \text{for } x \in [y_0, l]. \end{cases}$$

From the maximum principle for implicit schemes [9] it follows that there exists a unique solution of (7.8)–(7.10). The following result is valid (see [6] for the proof).

LEMMA 7.1. Assume that:

(i) there exists a unique solution of problem (7.1)–(7.4), (7.7) corresponding to a given function $y \in C^2 [0, T]$ and this solution satisfies the regularity condition (6.4);

(ii) $\tau \leqslant h$;

(iii) $\alpha_1 a_2 = \alpha_2 a_1$;

(iv) σ_i, $i=1, 2$ satisfy condition (H9) (see Sec. 5).

Then for sufficiently small h $(h \leqslant h^{**})$

$$\max_{(x_i, t_j) \in \omega_{h\tau}} |P_i^j - p(x_i, t_j)| \leqslant M h^{1/2} \tag{7.14}$$

where M is a positive constant independent of h, τ; $p(x, t) \triangleq p_i(x, t)$ for $(x, t) \in \text{cl}\, D_i$, $i=1, 2$.

Before we pass on to the description of finite-difference schemes corresponding to Problems (A_k) we will show how one can approximate $\dfrac{\partial p}{\partial x}$ at a node $(x_i, t_j) \in \omega_{h\tau}$ on the basis of values P_i^j. Observe that in view of (7.14) neither the forward difference $(P_{i+1}^j - P_i^j)/h$ nor the backward one $(P_i^j - P_{i-1}^j)/h$ do approximate $\dfrac{\partial p}{\partial x}(x_i, t_j)$. But there may be chosen an integer μ, dependent on h and τ, such that

$$\lim_{h, \tau \to 0} \left| \frac{P_{i+\mu}^j - P_i^j}{\mu h} - \frac{\partial p}{\partial x}(x_i, t_j) \right| = 0. \tag{7.15}$$

To show this note that

$$\left| \frac{P_{i+\mu}^j - P_i^j}{\mu h} - \frac{\partial p}{\partial x}(x_i, t_j) \right| \leqslant \left| \frac{P_{i+\mu}^j - p(x_{i+\mu}, t_j)}{\mu h} \right| + \left| \frac{p(x_i, t_j) - P_i^j}{\mu h} \right| +$$

$$+ \left| \frac{p(x_{i+\mu}, t_j) - p(x_i, t_j)}{\mu h} - \frac{\partial p}{\partial x}(x_i, t_j) \right| \triangleq W_1 + W_2 + W_3 .$$

If $\mu h \geqslant h^{1/2 - \varepsilon_1}$ where

$$\varepsilon_1 \in \left(0, \frac{1}{2}\right) \tag{7.16}$$

then by (7.14) $W_1 \leqslant M h^{\varepsilon_1} \xrightarrow[h \to 0]{} 0$. By the same arguments $W_2 \xrightarrow[h \to 0]{} 0$. To satisfy condition (7.16) we choose

$$\mu = E(h^{-1/2 - \varepsilon_1}) + 1 . \tag{7.17}$$

For such μ, $\mu h \leqslant h^{1/2 - \varepsilon_1} + h \xrightarrow[h \to 0]{} 0$ and $W_3 \xrightarrow[h \to 0]{} 0$.

Thus (7.15) is actually satisfied.

Stage II. Now we will present a direct finite-difference scheme for solving Problems (A_K) on the grid $\omega_{h\tau}$ satisfying condition (7.13). For given h, τ we define

$$\tilde{Q}=E(h^{\delta_1-\varepsilon_2}\tau^{-1})+1 \tag{7.18}$$

where $\delta_1 \triangleq \min\{\varepsilon_1, 1/2-\varepsilon_1\}$, $\varepsilon_2 \in (0, \delta_1)$; ε_1 is defined by (7.16). Let us introduce

$$\tilde{K} \triangleq L/\tilde{Q}. \tag{7.19}$$

Observe that $\tilde{Q}\to\infty$, $\tilde{K}\to\infty$, $\tilde{Q}\tau\to 0$ when $h, \tau\to 0$.

Finite-difference scheme (2). *Algorithm without iterations*

Given:

$$h, \tau, \tilde{Q}; \quad Y^0=y_0, \quad P_i^0=p_0(x_i), \quad i=0, 1, \ldots, N; \quad \text{set } j=1, \quad r=1.$$

Step 1.

(a) Set

$$V^{j-1}=-\kappa a_1 \frac{P_{\theta-1}^{j-1}-P_{\theta-1-\bar{\mu}}^{j-1}}{\bar{\mu}h}-(1-\kappa)\,a_2\frac{P_{\theta+\bar{\mu}}^{j-1}-P_{\theta}^{j-1}}{\bar{\mu}h}$$

where

$$\Theta \triangleq \min\{i\in\{0, 1, \ldots, N\} \mid Y^{j-1}\leqslant ih\}, \tag{7.20}$$

$$\bar{\mu}=\begin{cases}\mu & \text{if } h\leqslant h^{**}\\ 1 & \text{if } h>h^{**},\end{cases}$$

μ is defined by (7.17), $\kappa\in[0, 1]$.

(b) If $j=1$ then set $(Y')^{j-1}=\beta V^{j-1}$ and go to 1 (d), otherwise go to 1 (c).

(c) If $j-1<r\tilde{Q}$ then set $(Y')^{j-1}=(Y')^{(r-1)\tilde{Q}}$, otherwise set $(Y')^{j-1}=\beta V^{j-1}$ and $r\leftarrow r+1$.

(d) Set $Y^j=Y^{j-1}+\tau(Y')^{j-1}$.

(e) If $0<Y^j<l$ then go to Step 2, otherwise STOP.

Step 2.

(a) Compute A_i^j, B_i^j, $i=1, \ldots, N-1$ on the basis of expressions (7.11), (7.12) with $y(t_j)$ replaced by Y^j.

(b) Compute P_i^j, $i=0, 1, \ldots, N$ by solving the system of difference equations

$$\tilde{\mathscr{F}}_1[P_i^j]=0, \quad i=1, \ldots, N-1$$

with conditions:

— for Problem (A_1): $P_0^j=f_1(t_j)$, $P_N^j=f_2(t_j)$,

— for Problem (A_2): conditions (7.9) where

$$\sigma_k(t_j)=0, \quad g_k(t_j)=F_k(t_j), \quad k=1, 2.$$

(c) If $j<L$ then go to Step 3, otherwise STOP.

Step 3. $j\leftarrow j+1$ and return to Step 1.

REMARK 7.1. The above algorithm may be used in an iterative version, similarly as the finite-difference scheme (1) (see Section 5).

REMARK 7.2. One can easily modify the finite-difference scheme (2) to make it applicable for solving Problems (B_k^S) (see [6] for details). Convergence of the finite-difference scheme (2) to the solution of Problem (B_1^S) can be proved in the way similar to that presented in Section 6.

8. Numerical Results

In this section we present the results of numerical experiments which have been performed to test the methods suggested in the previous sections. We have applied the method with preparatory transformation and the direct method for solving Problems (A_k). For Problems (A_k) there are available nontrivial analytical solutions so a comparison of efficiency of both methods is possible.

We have solved Problems (A_k) in domain

$$D(T_0, T) \triangleq \{(x, t) | x \in (0, 1),\ t \in (T_0, T)\} \text{ where } 0 < T_0 < T.$$

Functions y, p defined in the following way

$$y(t) = 2\sqrt{t}, \quad t \in [T_0, T],$$

$$p(x, t) = \begin{cases} p_1(x, t) = \bar{a}_1 \operatorname{erf}\left(\dfrac{x}{2\sqrt{\alpha_1 t}}\right) + \bar{a}_2, & (x, t) \in \operatorname{cl} D_1(T_0, T) \\ p_2(x, t) = \bar{b}_1 \operatorname{erf}\left(\dfrac{x}{2\sqrt{\alpha_2 t}}\right), & (x, t) \in \operatorname{cl} D_2(T_0, T) \end{cases} \tag{8.1}$$

where

$$\operatorname{erf}(x) \triangleq \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\, dt,$$

$$\bar{a}_1 = -\frac{\sqrt{\pi \alpha_1}}{\beta a_1} \exp\left(\frac{1}{\alpha_1}\right), \quad \bar{b}_1 = -\frac{\sqrt{\pi \alpha_2}}{\beta a_2} \exp\left(\frac{1}{\alpha_2}\right),$$
$$\bar{a}_2 = -\bar{a}_1 \operatorname{erf}\left(\frac{1}{\alpha_1}\right) + \bar{b}_1 \operatorname{erf}\left(\frac{1}{\alpha_2}\right), \tag{8.2}$$

$$D_1(T_0, T) = \{(x, t) | x \in (0, y(t)), \quad t \in (T_0, T)\},$$

$$D_2(T_0, T) = \{(x, t) | x \in (y(t), 1), \quad t \in (T_0, T)\}$$

satisfy Problem (A_1) with the initial conditions

$$y(T_0) = 2\sqrt{T_0},$$

$$p(x, T_0) = \begin{cases} \bar{a}_1 \operatorname{erf}\left(\dfrac{x}{2\sqrt{\alpha_1 T_0}}\right) + \bar{a}_2, & x \in [0, y(T_0)] \\ \bar{b}_1 \operatorname{erf}\left(\dfrac{x}{2\sqrt{\alpha_2 T_0}}\right), & x \in [y(T_0), 1] \end{cases} \tag{8.3}$$

and with the Dirichlet boundary conditions

$$p(0,t)=\bar{a}_2, \quad p(1,t)=\bar{b}_1 \operatorname{erf}\left(\frac{1}{2\sqrt{\alpha_2 t}}\right), \quad t \in [T_0, T]. \tag{8.4}$$

Functions y, p defined by (8.1) satisfy also Problem (A_2) with initial condition (8.3) and with the Neumann boundary conditions

$$\frac{\partial p}{\partial x}(0,t)=\frac{\bar{a}_1}{\sqrt{\pi\alpha_1 t}}, \quad \frac{\partial p}{\partial x}(1,t)=\frac{\bar{b}_1}{\sqrt{\pi\alpha_2 t}} \exp\left(-\frac{1}{4\alpha_2 t}\right), \quad t \in (T_0, T]. \tag{8.5}$$

The finite-difference schemes (1) and (2) in noniterative as well as in iterative versions were applied for solving Problem (A_1) with conditions (8.3), (8.4) and for solving Problem (A_2) with conditions (8.3), (8.5).

The following values of parameters were assumed:

$$T_0=8\cdot 10^{-3}, \quad T=33\cdot 10^{-3}, \quad a_1=100, \quad a_2=1, \quad \alpha_1=1, \quad \alpha_2=100, \quad \beta=5, \quad Q=1, \quad \tilde{Q}=1.$$

In the domain $D(T_0, T)$ the regular grid $\omega_{h\tau}$ with $h=1/N$, $\tau=(T-T_0)/L$ was introduced.

We present the result of computations which have been performed on computer Odra-1325.

Table 1 lists errors of approximation to Problem (A_1) by using the finite-difference scheme (1) in noniterative as well as in iterative versions (Euler-Cauchy, Milne and Hamming predictor-corrector routines were used). The discrete values of solution to Problem (A_1) were obtained on the grid $\Omega_{h\tau}$ (see Section 5). The computations were performed for $h=0.05$ and $\tau=0.001$ ($N=20$, $L=25$).

The errors listed in Table 1 are the following:

— error of approximation to $y(t)$ for $t=t_j$, $j=0, 1, \ldots, L$

$$\operatorname{err}[y(t_j)] \triangleq \frac{(y(t_j)-Y^j)}{y(t_j)} 100\%; \tag{8.6}$$

— error of approximation to $p(y(t), t)$ for $t=t_j$, $j=0, 1, \ldots, L$

$$\operatorname{err}[p(y(t_j), t_j)] \triangleq \frac{(p(y(t_j), t_j)-P^j_{N/2})}{p(y(t_j), t_j)} 100\%. \tag{8.7}$$

The maximal absolute value of the relative error of approximation to function p in $t=t_j$ was achieved at the node $(x_{N/2}, t_j)$ (i.e. at the free boundary) for both versions of the algorithm.

It follows from the results given in Table 1 that the iterative version of the finite-difference scheme (1) yields only small improvement in the approximation.

Considering the fact that time of computation for iterative version is about twice longer than for noniterative one, we conclude that if number of time steps is relatively small then the noniterative version is to be preferred. Employment of the iterative version becomes profitable when the discretization with respect

Table 1.

Finite-difference scheme (1)—Problem (A_1); h=0.05, τ=0.001

Number of the time step j	err $[y(t_j)]$				err $[p(y(t_j), t_j)]$			
	Without iterations	Iterative versions			Without iterations	Iterative versions		
		Euler	Milne	Hamming		Euler	Milne	Hamming
3	—0.159	0.277	0.277	0.277	—6.825	—6.363	—6.363	—6.363
5	—0.107	0.471	0.474	0.473	—8.333	—7.799	—7.999	—7.799
7	—0.012	0.658	0.665	0.662	—9.244	—8.676	—8.738	—8.675
9	0.110	0.842	0.850	0.848	—9.833	—9.249	—9.294	—9.307
11	0.248	1.022	1.032	1.028	—10.227	—9.636	—9.756	—9.668
13	0.394	1.197	1.210	1.205	—10.493	—9.900	—10.051	—9.962
15	0.545	1.369	1.383	1.378	—10.671	—10.078	—10.247	—10.173
17	0.698	1.537	1.554	1.547	—10.785	—10.195	—10.374	—10.307
19	0.853	1.701	1.721	1.713	—10.852	—10.266	—10.452	—10.390
21	1.007	1.862	1.885	1.876	—10.885	—10.303	—10.493	—10.434
23	1.161	2.021	2.046	2.036	—10.891	—10.313	—10.506	—10.450
25	1.315	2.177	2.204	2.194	—10.876	—10.403	—10.497	—10.443
Time of computations [s]	53	90	74	82				

Table 2.

Finite-difference scheme (2) — Problem (A_1); h=0,05, τ=0,001

Number of time step j	err $[p(y(t_j)]$				err $[p(y(t_j), t_j)]$				$\max_i$ err $[p(x_i, t_j)]$			
	Without iterations	Iterative versions			Without iterations	Iterative versions			Without iterations	Iterative versions		
		Euler	Milne	Hamming		Euler	Milne	Hamming		Euler	Milne	Hamming
3	—0.071	0.364	0.364	0.364	—5.304	—6.105	—6.105	—6.105	—5.308	—6.114	—6.114	—6.114
5	—0.137	0.496	0.478	0.475	—3.966	—4.354	—4.355	—4.355	—7.391	—7.998	—8.007	—8.006
7	0.068	0.826	0.817	0.820	—4.353	—4.296	—4.300	—4.300	—13.853	—14.662	—14.669	—14.668
9	0.281	1.061	1.050	1.060	—9.410	—9.614	—9.614	—9.616	—9.419	—9.631	—9.631	—9.634
11	0.350	1.176	1.165	1.174	—7.515	—7.633	—7.632	—7.635	—7.537	—7.858	—7.850	—7.859
13	0.470	1.373	1.369	1.372	—6.752	—6.807	—6.807	—6.809	—11.921	—12.695	—12.687	—12.696
15	0.704	1.640	1.640	1.641	—11.782	—8.269	—8.266	—8.166	—11.790	—14.614	—14.609	—14.617
17	0.844	1.820	1.817	1.821	—9.281	—10.266	—10.261	—10.267	—9.299	—10.410	—10.397	—10.405
19	0.948	1.946	1.943	1.947	—8.283	—8.834	—8.829	—8.835	—9.543	—9.744	—9.737	—9.745
21	1.112	2.119	2.119	2.119	—7.784	—8.130	—8.125	—8.131	—12.938	—13.147	—13.140	—13.149
23	1.328	2.343	2.344	2.344	—8.514	—8.038	—8.034	—8.039	—14.655	—15.431	—15.424	—15.433
25	1.514	2.544	2.547	2.548	—10.673	—11.667	—11.678	—11.672	—10.695	—12.095	—12.108	—12.102
Time of computations [s]	54	95	91	92								

to t is significantly more dense (in that case cumulation of errors following from the use of Euler-Cauchy extrapolation method of solving ordinary differential equations plays important role).

For the considered discretizations the process of iterations in iterative versions of finite-difference scheme (1) was quickly convergent. As a rule after two iterations difference between two succeeding approximate solutions turned out to be less than 10^{-6}.

Table 2 lists the following results obtained by using scheme (2) both in noniterative and iterative versions:

— error of approximation to $y(t)$ for $t=t_j$, defined by (8.6);

— error of approximation to $p(y(t), t)$ for $t=t_j$, $j=0, 1, ..., L$

$$\text{err}\,[p(y(t_j), t_j)] \triangleq \frac{(p(y(t_j), t_j)-W^j)}{p(y(t_j), t_j)}\,100\%$$

where

$$W^j=\kappa P^j_{\theta-1}+(1-\kappa)\,P^j_\theta, \quad \kappa=\frac{a_1(x_\theta-Y^j)}{a_1(x_\theta-Y^j)+a_2(Y^j-x_{\theta-1})},$$

$$x_\theta=\Theta h, \quad \Theta=\min_{i\in\{0,1,...,N\}}\{i\,|\,Y^j\leqslant ih\};$$

— maximal error of approximation to $p(x, t)$ for $t=t_j$, $j=0, 1, ..., L$

$$\max_i \text{err}\,[p(x_i, t_j)]=\frac{(p(x_m, t_j)-P^j_m)}{p(x_m, t_j)}\,100\%$$

where

$$m=\arg\max_{i\in\{0,1,...,N\}}\left|\frac{p(x_i, t_j)-P^j_i}{p(x_i, t_j)}\right|.$$

Table 3.

Finite-difference scheme (1) without iterations — Problem (A_1), dependance on the discretization step h; $\tau=0.001$

Number of time step j	err $[y(t_j)]$ $h=0.05$	err $[y(t_j)]$ $h=0.025$	err $[y(t_j)]$ $h=0.0125$	err $[p(y(t_j), t_j)]$ $h=0.05$	err $[p(y(t_j), t_j)]$ $h=0.025$	err $[p(y(t_j), t_j)]$ $h=0.0125$
3	—0.159	—0.297	—0.374	—6.825	—3.661	—1.902
5	—0.107	—0.356	—0.495	—8.333	—4.477	—2.331
7	—0.012	—0.361	—0.557	—9.244	—4.977	—2.596
9	0.110	—0.334	—0.582	—9.833	—5.307	—2.773
11	0.248	—0.287	—0.586	—10.277	—5.532	—2.895
13	0.394	—0.227	—0.575	—10.493	—5.689	—2.298
15	0.545	—0.160	—0.555	—10.671	—5.798	—3.043
17	0.698	—0.087	—0.528	—10.785	—5.872	—3.086
19	0.853	—0.012	—0.498	—10.852	—5.921	—3.116
21	1.007	0.066	—0.464	—10.885	—5.951	—3.136
23	1.161	0.145	—0.428	—10.891	—5.965	—3.148
25	1.315	0.225	—0.391	—10.876	—5.968	—3.153
Time of computations [s]	53	102	168			

Table 4.

Finite-difference scheme (2) without iterations — Problem (A_1), dependence on the discretization step h; τ=0.001

Number of time step j	err $[y(t_j)]$			err $[p(y(t_j), t_j)]$			$\max_i$ err $[p(x_i, t_j)]$		
	h=0.05	h=0.025	h=0.0125	h=0.05	h=0.025	h=0,0125	h=0.05	h=0.025	h=0.0125
3	—0.071	—0.221	—0.364	—5.304	—2.497	—1.736	—5.308	—2.498	—3.674
5	—0.137	—0.272	—0.390	—3.966	—4.648	—3.369	—7.391	—4.646	—3.366
7	0.068	—0.327	—0.460	—4.353	—3.387	—3.027	—13.853	—5.910	—4.373
9	0.281	—0.271	—0.452	—9.410	—4.246	—3.613	—9.419	—4.247	—3.896
11	0.350	—0.218	—0.420	—7.515	—5.538	—3.979	—7.537	—5.536	—3.975
13	0.470	—0.184	—0.379	—6.752	—4.380	—4.078	—11.921	—6.007	—4.075
15	0.744	—0.062	—0.335	—11.782	—5.933	—4.108	—11.790	—5.934	—4.106
17	0.844	—0.018	—0.289	—9.281	—4.838	—4.108	—9.299	—6.407	—4.105
19	0.948	0.101	—0.239	—8.283	—6.077	—4.125	—9.543	—6.079	—4.123
21	1.112	0.157	—0.185	—7.784	—5.109	—4.175	—12.938	—6.470	—4.174
23	1.328	0.272	—0.129	—8.524	—6.250	—4.236	—14.655	—6.253	—4.235
25	1.514	0.338	—0.072	—10.673	—5.277	—4.263	—10.695	—5.340	—4.263
Time of computations [s]	54	99	158						

Computations were carried out for the same discretization as previously, i.e. $N=20$, $L=25$. If follows from the results given in Table 2 that in the case of scheme (2) its iterative versions yield the errors greater than noniterative one. The results obtained suggest that maximal absolute values of approximation errors for every t_j, $j=0, 1, \ldots, L$ are attained at one of the nodes neighbouring to $x=Y^j$.

Comparing results given in Tables 1, 2 one arrives at the conclusion that in the case of the method with preparatory transformation errors are smaller from those obtained by the direct method.

Tables 3, 4 ilustrate influence of the spatial discretization (i.e. value of N) on accuracy of approximate solutions to Problem (A_1), obtained by means of the finite-difference schemes (1) and (2) in noniterative versions. It follows from these results that the rate of convergence of the method with preparatory transformation is greater than the rate of convergence of the direct method.

In summary, we remark that the method with preparatory transformation seems to be preferred whenever possible, i.e. when it is known that in the problem considered free boundary Γ between layers does not touch fixed parts of the boundary of the domain D.

In the case when the set $\mathrm{cl}\, \Gamma \cap \{(x, t) | x=l_i,\ t \in [0, T]\}$ can be nonempty, one ought to use the direct method.

APPENDIX B

Denote by $\mathcal{L}$ the family of curves $\{x=y(t) | y(t) \in (0, l),\ t \in [0, T]\}$ satisfying Lipschitz continuity condition

$$|y(t_1)-y(t_2)| \leqslant c\, |t_1-t_2|, \qquad t_1, t_2 \in [0, T]$$

with constant c, the same for all $y \in \mathcal{L}$.

Let $Q \subset R^2$ and $(x, t) \in Q$, then $H_{x_\nu}(Q)$ where $\nu=1, 2$, $x_1=x$, $x_2=t$ will denote the class of functions Hölder continuous in Q with respect to x_ν, with the Hölder index from the interval $(0, 1]$ and $H(Q) \triangleq H_x(Q) \cap H_t(Q)$.

For a given function $y \in \mathcal{L}$ let us consider the following problem.

Find functions u_1, u_2 satisfying:

— system of parabolic equations

$$\frac{\partial^2 u_i}{\partial x^2}(x, t)=a_i(x, t)\frac{\partial u_i}{\partial t}(x, t)+b_i(x, t)\frac{\partial u_i}{\partial x}(x, t)+$$
$$+c_i(x, t)\, u_i(x, t)+f_i(x, t) \quad \text{for } (x, t) \in D_i, \quad i=1, 2; \tag{B.1}$$

— initial conditions

$$u_i(x, 0)=u_{i0}(x) \text{ in } Z_i; \tag{B.2}$$

— boundary conditions

$$\frac{\partial u_i}{\partial x}(l_i, t)+(-1)^i \sigma_i(t)\, u_i(l_i, t)=\varphi_i(t), \qquad t \in (0, T] \tag{B.3}$$

or

$$u_i(l_i, t) = \mu_i(t), \qquad t \in (0, T]; \tag{B.3'}$$

— conditions at the curve y

$$u_1(y(t), t) - u_2(y(t), t) = r(t), \tag{B.4}$$

$$\gamma_1(t) \frac{\partial u_1}{\partial x}(y(t), t) - \gamma_2(t) \frac{\partial u_2}{\partial x}(y(t), t) = s(t), \qquad t \in (0, T]. \tag{B.5}$$

Assume that the compatibility conditions are fulfilled, i.e.

$$u_{10}(y_0) - u_{20}(y_0) = r(0),$$

$$\gamma_1(0)\, u'_{10}(y_0) - \gamma_2(0)\, u'_{20}(y_0) = s(0), \tag{B.6}$$

$$u'_{i0}(l_i) + (-1)^i \sigma_i(0)\, u_{i0}(l_i) = \varphi_i(0) \quad \text{or } u_{i0}(l_i) = f_i(0), \text{ respectively.}$$

In [4, 5] the following theorem has been proved:

THEOREM B. Assume that the data of problem (B.1)–(B.6) satisfy the following conditions:

(i) $0 < a_0 \leqslant a_i(x, t) \leqslant A_0$ for $(x, t) \in \operatorname{cl} D$ where a_0, A_0 are given constants;

$\frac{\partial a_i}{\partial x}, \frac{\partial a_i}{\partial t}, b_i, c_i, f_i \in H_x(\operatorname{cl} D)$ or

$\frac{\partial a_i}{\partial x}, \frac{\partial a_i}{\partial t}, b_i, c_i, f_i \in H_t(\operatorname{cl} D)$, respectively;

$\sqrt{a_1(y(t), t)}\, \gamma_2(t) + \sqrt{a_2(y(t), t)}\, \gamma_1(t) \neq 0, \; t \in [0, T]$;

(ii) $u_{i0} \in C^2(\operatorname{cl} Z_i)$, $u''_{i0} \in H(\operatorname{cl} Z_i)$;

(iii) $\gamma_i, \sigma_i, \varphi_i, s \in H[0, T]$ $\left(\text{with Hölder indices } > \frac{1}{2}\right)$;

(iv) $r', f'_i \in H[0, T]$;

(v) $y, y^* \in \mathscr{L}$.

Then

$$\lim_{\|y - y^*\|_{1,T} \to 0} \; \sup_{(x,t) \in S^i(y, y^*; T)} |u_i(x, t) - u_i^*(x, t)| = 0$$

and

$$\lim_{\|y - y^*\|_{1,T} \to 0} \; \sup_{(x,t) \in S^i(y, y^*; T)} \left| \frac{\partial u_i}{\partial x}(x, t) - \frac{\partial u_i^*}{\partial x}(x, t) \right| = 0, \qquad i = 1, 2$$

where

$$S^1(y, y^*; t) \triangleq \{(x, \tau) | 0 \leqslant x \leqslant \min\{y(\tau), y^*(\tau)\}, \quad 0 \leqslant \tau \leqslant t\},$$

$$S^2(y, y^*; t) \triangleq \{(x, \tau) | \max\{y(\tau), y^*(\tau)\} \leqslant x \leqslant l, \quad 0 \leqslant \tau \leqslant t\},$$

$$\|y\|_{1,t} = \sup_{\tau \in [0, t]} |y(\tau)| + \sup_{\tau_1, \tau_2 \in [0, t]} \frac{|y(\tau_2) - y(\tau_1)|}{|\tau_2 - \tau_1|},$$

u_i and u_i^*, $i = 1, 2$, are solutions of problems (B.1)–(B.6) associated respectively with y and y^*.

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Aproksymacje różnicowe parabolicznych zagadnień brzegowych ze swobodną granicą opisujących dynamikę podziemnego zbiornika gazu. Część II. Dowód zbieżności. Wyniki numeryczne.

W artykule przedstawiono dowód zbieżności wprowadzonych w części I aproksymacji różnicowych dla jednowymiarowych parabolicznych zagadnień brzegowych ze swobodną granicą. Podano wyniki eksperymentów numerycznych.

Разностные аппроксимации параболических краевых задач со свободной границей возникающих при моделировании подземного газохранилища. Часть II. Доказательство сходимости. Численные результаты.

В статье доказана сходимость описанных в части I разностных схем для решения одномерных параболических краевых задач со свободной границей. Представлены численные результаты.