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Control of Parabolic Systems with Free Boundaries — Application of Inverse Formulations

by

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In the paper some two-phase parabolic free boundary value problems are considered. For systems governed by such problems some control problems are formulated whose aim is achievement of a desired movement of free boundary or obtainment of a given terminal state. A method of solving the formulated control problems is proposed. The method is based on employing inverse formulations of parabolic problems.

1. Introduction

In the paper we propose an approach to solving some optimal control problems for parabolic processes with free boundary.

We consider a one-dimensional parabolic two-phase free boundary value problem of the Stefan type. Problems of such a type or close to them arise in mathematical modelling of numerous real processes, recall for example

— solidification or melting, in particular crystal growth [3, 6, 9, 17],
— combustion [16],
— some biochemical diffusion processes [16].

For the nonlinear model formulated in the paper we state control problem with a purpose consisting in obtaining desired dynamics of the free boundary.

The proposed method of solving such a control problem is based on exploiting integral representations of the considered model. The control problem is transformed into an inverse problem, having form of a system of linear integral equations of the first kind.

In the introduced inverse formulation one assumes motion of the free boundary to be given and determines corresponding boundary function.

Because of the non-correctness in the Hadamard sense of this inverse formulation one should apply a regularization algorithm to solve such a problem. There is a variety of works where such algorithms are proposed and their implementations are constructed [2, 8, 11, 20].

As an application we discuss an optimization problem for crystallization process.

There are several papers devoted to optimization of processes with free boundaries. In more than one dimension there are works of Saguez [13, 18], exploiting techniques of variational inequalities. In one-dimensional case there are papers making use of various inverse formulations to the free boundary problems [4, 5, 7, 19]. Except [13] all the works cited above are dealing with one-phase free boundary problems.

The method proposed in the present paper can be extended to a broader class of one-dimensional problems with free boundary, involving variable coefficients and variable values of state functions at the free boundary as well as discontinuity of state function there. The process equations and boundary conditions can be also nonlinear [15].

2. Formulation of a Control Problem for Process with Free Boundary

Let the mathematical model of the process we will consider be in form of one-dimensional two-phase parabolic free boundary value problem of the Stefan type.

Problem (S)

— process equations:

$$\frac{\partial y_i}{\partial t} - a_i^2 \frac{\partial^2 y_i}{\partial x^2} = 0 \qquad \text{in } Q_i, \tag{1}$$

— initial conditions:

$$y_i(x, t_0) = y_{i0}(x) \qquad \text{in } \Omega_{i0}, \tag{2}$$

$$s(t_0) = s_0 \tag{3}$$

— boundary conditions:

$$y_1(0, t) = u_1(t), \qquad t \in (t_0, t_f] \tag{4}$$

$$\frac{\partial y_2}{\partial x}(L, t) = 0, \qquad t \in (t_0, t_f] \tag{5}$$

— free boundary conditions:

$$y_1\big(s(t)-, t\big) = y_2\big(s(t)+, t\big) = 0, \qquad t \in (t_0, t_f] \tag{6}$$

$$\frac{ds}{dt}(t) = \lambda_1 \frac{\partial y_1}{\partial x}\big(s(t)-, t\big) - \lambda_2 \frac{\partial y_2}{\partial x}\big(s(t)+, t\big), \qquad t \in (t_0, t_f]. \tag{7}$$

In the above formulation we use the notations listed below:

t_f — a given upper bound for the time interval,

$s(t)$, $t \in [t_0, t_f]$ — position of free boundary, a priori unknown except $s(t_0) = s_0$.

$$Q_1 = \{(x, t) | 0 < x < s(t), \quad t_0 < t < t_f\},$$
$$Q_2 = \{(x, t) | s(t) < x < L, \quad t_0 < t < t_f\},$$
$$\Omega_{10} = (0, s_0), \quad \Omega_{20} = (s_0, L),$$

y_i — state function of the process, defined in Q_i,
a_i, λ_i — given positive constants: $a_1 \neq a_2$, $\lambda_1 \neq \lambda_2$.
We assume that
(A.1) $0 < s_0 < L$,
(A.2) The initial data $y_{i0} \in C^1(\bar{\Omega}_{i0})$ and satisfy the following conditions

$$\begin{aligned} d_1 < y_{10}(x) < 0, \quad & x \in \bar{\Omega}_{10} \setminus \{s_0\}, \\ 0 < y_{20}(x) < d_2, \quad & x \in \bar{\Omega}_{20} \setminus \{s_0\}, \end{aligned} \tag{8}$$

$$y_{10}(s_0) = y_{20}(s_0) = 0. \tag{9}$$

(A.3) For the initial and boundary data compatibility conditions hold

$$\begin{aligned} \frac{dy_{20}}{dx}(L) = 0, \\ y_{10}(0) = u_1(t_0). \end{aligned} \tag{10}$$

Following [14, 15] we are in a position to consider the boundary data u_1 either being elements of the space $C^1[t_0, t_f]$ or of the Sobolev space $H^1[t_0, t_f]$. According to the above choice we will understand boundary condition (4) either in the strong or in the following weak sense [14]:

$$\lim_{x \to 0+} \int_{t_0}^{t_f} [y_1(x, t) - u_1(t)] \eta(t) \, dt = 0, \quad \text{for every } \eta \in \mathscr{D}[t_0, t_f]. \tag{11}$$

We assume that for H^1 — boundary data

(A.4) $$c_1 \leqslant u_1(t) \leqslant c_0 < 0, \quad \text{a.e.} \quad t \in [t_0, t_f] \tag{12}$$

$$|v_1(t)| \leqslant c_2, \quad \text{almost everywhere } t \in [t_0, t_f] \tag{13}$$

where v_1 denotes the generalized derivative of u_1.

For C^1 — boundary data both the inequalities (12), (13) hold everywhere and the derivative in (13) is understood in the classical sense.

As we have proved in [14] the classical solution $\{y_1, y_2, s\}$ of Problem (S), corresponding to a function $u_1 \in H^1[t_0, t_f]$ may be obtained on the basis of solutions $\{y_{1n}, y_{2n}, s_n\}$ corresponding to smoothed boundary data $u_{1n} \in C^1[t_0, t_f]$, as the following convergences take palce

$$\|y_{in} - y_i\|_{C^{2,1}(Q_i)} \xrightarrow[n]{} 0,$$
$$\|s_n - s\|_{C^1[t_0, t_f]} \xrightarrow[n]{} 0$$

when $u_{1n} \to u_1$ in $H^1[t_0, t_f]$, at least after choosing some subsequences.

Due to this property we can restrict ourselves in the sequel to discussing Problem (S) only for smoothed controls $u_1 \in C^1[t_0, t_f]$.

Existence of a unique classical solution $\{y_1, y_2, s\}$ to this problem is guaranteed by results of [14, 15]. More precisely the following theorem holds:

THEOREM 1. Let:
— the initial data satisfy (A.1) and

$$\lambda_1 \frac{dy_{10}}{dx}(s_0) - \lambda_2 \frac{dy_{20}}{dx}(s_0) > 0,$$

— a function $u_1 \in C^1[t_0, t_f]$ satisfying (A.4) is given,
— compatibility conditions (A.3) are fulfilled.

Then there exists a classical solution $\{y_1, y_2, s\}$ to free boundary problem (S) in some time interval $[t_0, t_k]$ and this solution is unique. Value of t_k is defined by

$$t_k = \min\{t \in (t_0, t_f] | s(t) = l \quad \text{or} \quad t_k = t_f\}$$

with some given $l \in (s_0, L)$.

The above theorem can proved in two stages.

First, introducing equivalent integral representation to Problem (S) and exploiting results given in [14] one immediately obtains local existence of the solution $\{y_1, y_2, s\}$ in a nontrivial time interval $[t_0, t^*]$ as well as uniqueness of this solution.

Next one makes use of some a priori estimates [14, 15] for functions $y_i(x, t)$, $\frac{\partial y_i}{\partial x}(s(t), t)$, $s(t)$. On the basis of them it is possible to extend the local solution in a finite number of steps onto the whole time interval $[t_0, t_k]$. We recall here the integral representation of Problem (S) and the a priori estimates.

PROPOSITION 1 [15]. Problem (S) is equivalent to the following system of integral equations:

$$y_1(x, t) = \int_0^{s_0} y_{10}(\xi)\, G_{10}(x, \xi, t-t_0)\, d\xi + a_1^2 \int_{t_0}^{t} u_1(\tau) \frac{\partial G_{10}}{\partial \xi}(x, 0, t-\tau)\, d\tau + \\ + a_1^2 \int_{t_0}^{t} v_1(\tau)\, G_{10}(x, s(\tau), t-\tau)\, d\tau, \tag{14}$$

$$y_2(x, t) = \int_{s_0}^{L} y_{20}(\xi)\, G_{2L}(x, \xi, t-t_0)\, d\xi - a_2^2 \int_{t_0}^{t} v_2(\tau)\, G_{2L}(x, s(\tau), t-\tau)\, d\tau, \tag{15}$$

$$v_1(t) = 2 \int_0^{s_0} \frac{dy_{10}}{d\xi}(\xi)\, G_{20}(s(t), \xi, t-t_0)\, d\xi - 2 \int_{t_0}^{t} v_1(\tau)\, G_{20}(s(t), 0, t-\tau)\, d\tau - \\ - 2a_1^2 \int_{t_0}^{t} v_1(\tau) \frac{\partial G_{20}}{\partial \xi}(s(t), s(\tau), t-\tau)\, d\tau, \tag{16}$$

$$v_2(t) = 2 \int_{s_0}^{L} \frac{dy_{20}}{d\xi}(\xi)\, G_{1L}(s(t), \xi, t - t_0)\, d\xi +$$

$$+ 2a_2^2 \int_{t_0}^{t} v_2(\tau) \frac{\partial G_{1L}}{\partial \xi}(s(t), s(\tau), t - \tau)\, d\tau, \tag{17}$$

$$s(t) = s_0 + \int_{t_0}^{t} [\lambda_1 v_1(\tau) - \lambda_2 v_2(\tau)]\, d\tau \tag{18}$$

where

$$v_1(t) \triangleq \frac{du_1}{dt}(t), \qquad v_i(t) \triangleq \frac{\partial y_i}{\partial x}(s(t), t)$$

and

$$G_{i0}(x, \xi, t) = E(x - \xi, a_1^2 t) + (-1)^i E(x + \xi, a_1^2 t),$$

$$G_{iL}(x, \xi, t) = E(x - \xi, a_2^2 t) + (-1)^i E(x + \xi - 2L, a_2^2 t),$$

with

$$E(x, t) \triangleq \begin{cases} (4\pi t)^{-1/2} \exp(-x^2/4t), & t > 0, \quad x \in R \\ 0 \hfill, & t \leqslant 0, \quad x \in R \end{cases}.$$

PROPOSITION 2 [15]. Let $\{y_1, y_2, s\}$ be a solution to Problem (S), defined for $t \in [t_0, t_k]$. Then:

(P.1) There exists a positive constant m_1, dependent only upon the bounds of data, such that

$$s(t) \geqslant m_1 \qquad t \in [t_0, t_k]. \tag{19}$$

(P.2) If the initial data satisfy conditions

$$\left| \frac{dy_{i0}}{dx}(x) \right| \leqslant d_{i+2}, \qquad x \in \bar{\Omega}_{i0} \tag{20}$$

with given positive constants d_3, d_4, then there exist positive constants m_2, m_3, m_4 dependent only upon the bounds of data, such that

$$\begin{aligned} &\left| \frac{\partial y_i}{\partial x}(x, t_0) \right| \leqslant d_{i+2}, \qquad x \in \bar{\Omega}_{i0}, \\ &\left| \frac{\partial y_1}{\partial x}(0, t) \right| \leqslant m_2, \qquad t \in [t_0, t_k], \\ &\left| \frac{\partial y_1}{\partial x}(s(t), t) \right| \leqslant m_{i+2}, \qquad t \in [t_0, t_k]. \end{aligned} \tag{21}$$

(P.3) Under assumptions as in (P.2) there exist a priori known positive constants m_5, m_6 such that

$$\left|\frac{\partial y_1}{\partial x}(x,t)\right| \leqslant m_5, \qquad (x,t)\in \bar{Q}_1^k,$$
$$\left|\frac{\partial y_2}{\partial x}(x,t)\right| \leqslant m_6, \qquad (x,t)\in \bar{Q}_2^k, \tag{22}$$

where

$$Q_i^k \stackrel{\Delta}{=} Q_i \cap [R\times(t_0, t_k)].$$

Moreover by the above a priori estimates it is easy to obtain lower and upper bounds for attainable positions of the free boundary.

PROPOSITION 3 [15]. Let the initial data satisfy (20). Then there exists a positive constant $m_0 = m_0(m_3, m_4)$ such that

$$-\lambda_2 m_0(t-t_0)+s_0 \leqslant s(t) \leqslant \lambda_1 m_0(t-t_0)+s_0, \qquad t\in[t_0, t_k]. \tag{23}$$

From (23) it follows immediately the lower bound for t_k:

$$t_k \geqslant \frac{1}{\lambda_1 m_0}(l-s_0)+t_0. \tag{24}$$

On the basis of (24) in the case $t_k \geqslant t_f$ one can indicate necessary modifications of the set of admissible controls or of the initial data.

For the introduced model with free boundary we are going to formulate a control problem. As it is often suggested by technological reasons we would like to secure a desired motion of the free boundary (cf. Section 4).

One of the possibilities of formulating such a control problem is to introduce a functional dependent on the difference

$$s(t; u_1) - \sigma(t)$$

where function σ corresponds to the desired evolution of the free boundary.

We propose a different approach, based on employing an inverse formulation of the free boundary problem.

3. Inverse Formulation of Problem (S)

To introduce the proposed inverse formulation of Problem (S) we make use of the equivalent integral representation (14)–(18) to the model.

We will consider the control problem whose aim is to achieve the desired motion of the free boundary as an inverse two-phase parabolic problem in non-cylindrical domains.

The inverse formulation consists here in finding functions y_1, y_2, u_1 under assumption that the function s describing position of the free boundary is known and

$$s(t)=\sigma(t), \qquad t\in[t_0, t_k] \tag{26}$$

where t_k is defined by

$$t_k=\min\{t|\sigma(t)=l\} \tag{27}$$

with some $l\in(s_0, L)$.

Exploiting the equivalence of Problem (S) and the system (14)–(18) we can formulate the discussed control problem as a problem of solving a system of integral equations.

More precisely, observe that to determine boundary control u_1 it is sufficient to know only the function v_2.

In this connection we ought to solve the following problem, obtained from the system (14)–(18) by simple calculations.

Control Problem (IS)

Determine an admissible boundary control u_1 such that the corresponding system of Volterra integral equations

$$v_2(t)-\int_{t_0}^{t} k_2(t,\tau;\sigma)\,v_2(\tau)\,d\tau=r_2(t;\sigma), \tag{28}$$

$$\lambda_1\int_{t_0}^{t} k_1(t,\tau;\sigma)\,v_1(\tau)\,d\tau=r_1(t;\sigma,v_2) \tag{29}$$

is satisfied for $t\in[t_0, t_k]$.

In the above system we use the following notations

$$k_2(t,\tau;\sigma)=\begin{cases}(2a_2\sqrt{\pi})^{-1}(t-\tau)^{-3/2}\{[\sigma(t)-\sigma(\tau)]\cdot\\ \cdot\exp[-(\sigma(t)-\sigma(\tau))^2/4a_2^2(t-\tau)]+[\sigma(t)+\sigma(\tau)-2L]\cdot\\ \cdot\exp[-(\sigma(t)+\sigma(\tau)-2L)^2/4a_2^2(t-\tau)]\}, & \text{if} \quad t_0\leqslant\tau<t\leqslant t_k\\ 0 & \text{for other } t,\tau\end{cases}$$

$$r_2(t;\sigma)=2\int_{s_0}^{L}\frac{dy_{20}}{d\xi}(\xi)\,G_{1L}(\sigma(t),\xi,t-t_0)\,d\xi,$$

$$k_1(t,\tau;\sigma)=\begin{cases}(\lambda_1/a_1\sqrt{\pi})^{-1}(t-\tau)^{-1/2}\exp[-\sigma^2(t)/4a_1^2(t-\tau)],\\ \qquad\qquad \text{if} \quad t_0\leqslant\tau<t\leqslant t_k,\\ 0 \qquad\qquad \text{for other } t,\ \tau\end{cases}$$

$$r_1(t;\sigma, v_2) = -\frac{1}{2}\frac{d\sigma}{dt}(t) + \lambda_1 \int\limits_0^{s_0} \frac{dy_{10}}{d\xi}(\xi)\, G_{20}(\sigma(t), \xi, t-t_0)\, d\xi -$$

$$-\lambda_2 \int\limits_{s_0}^{L} \frac{dy_{20}}{d\xi}(\xi)\, G_{1L}(\sigma(t), \xi, t-t_0)\, d\xi -$$

$$-a_1^2 \int\limits_{t_0}^{t} \frac{d\sigma}{d\tau}(\tau) \frac{\partial G_{20}}{\partial \xi}(\sigma(t), \sigma(\tau), t-\tau)\, d\tau -$$

$$-\lambda_2 \int\limits_{t_0}^{t} \left[v_2(\tau;\sigma)\, a_1^2 \frac{\partial G_{20}}{\partial \xi}(\sigma(t), \sigma(\tau), t-\tau) + a_2^2 \frac{\partial G_{1L}}{\partial \xi}(\sigma(t), \sigma(\tau), t-\tau) \right] d\tau,$$

$$v_1(t) = \frac{du_1}{dt}(t).$$

As the set of admissible controls we take

$$U_{ad}^1 = \{u \in C^1[t_0, t_f] \,|\, u(t_0) = y_{10}(0), \quad c_1 \leqslant u(t) \leqslant c_0 < 0, \quad t \in [t_0, t_f],$$
$$|v(t)| \leqslant c_2, \quad t \in [t_0, t_f]\}$$

where $v \triangleq \frac{du}{dt}$.

We define above the functions u on the whole time interval $[t_0, t_f]$ only for simplicity. Actually, for any given σ we can restrict ourselves to controls defined on corresponding interval $[t_0, t_k]$, with t_k given by (27).

(28) is Volterra equation of the second kind with continuous kernel and right-hand side, so its unique solution v_2 exists in $[t_0, t_k]$ and is Lipschitz continuous [12]. Moreover function v_2 may be a priori estimated by a constant independent of t_k and σ (see Proposition 2).

(29) is Volterra equation of the first kind with respect to v_1. In view of properties of the Green function G_{20} this equation is non-transformable to any equation of the second kind [2, 11]. Kernel $k_1(t, \tau; \sigma)$ of this equation is continuous, so the corresponding Volterra operator is well defined in the space $C[t_0, t_k]$.

As it is well known, problem of solving such an equation is non-correct in the Hadamard sense [2, 11]. To solve this problem it is necessary to apply a regularization method. Because there are many works concerning both theory and algorithmic implementations of various regularization methods, we only refer to some of them [2, 8, 11, 14, 20].

By means of regularization algorithms one can solve non-correctly posed problems of type (29), even knowing the kernel and the right-hand side only approximatively, what is particularly important in the case when some parameters are determined from measurements [11, 14]. There are regularization algorithms, stable with respect to perturbations of the data [8, 11] and with respect to approximation of the model [11]. As a result of using such algorithms one obtains sequence of elements convergent in some, generally set — theoretical sense to so called normal solution of the exact problem or to a quasi-solution in the case when such a solution does not exist [11, 20].

4. Application to Control of Crystallization Process

In some cases it is possible to describe dynamics of crystal-growth processes by model with free boundaries for coupled physical fields of temperature and mass concentration [3, 6, 9, 14].

For the process of crystal-growth from a two-component substance containing one of the components as a vestigial admixture the model will have form of a parabolic free boundary system.

We introduce the notations:

T_i — temperature,

C_i — mass concentration of the admixture,

b_i — diffusion coefficients ($b_1 \neq b_2$).

Lower index $i=1$ corresponds everywhere to the solid phase, $i=2$ — to the liquid. Other notations are as in the previous sections.

Then the model has the following form:

Process equations:

$$\frac{\partial T_i}{\partial x} - a_i^2 \frac{\partial^2 T_i}{\partial x^2} = 0 \tag{30}$$

in Q_i.

$$\frac{\partial C_i}{\partial t} - b_i \frac{\partial^2 C_i}{\partial x^2} = 0 \tag{31}$$

Initial conditions:

$$s(t_0) = s_0 \quad \text{where} \quad 0 < s_0 < L, \tag{32}$$

$$T_i(x, t_0) = T_{i0}(x), \qquad x \in \Omega_{i0}, \tag{33}$$

$$C_i(x, t_0) = C_{i0}(x), \qquad x \in \Omega_{i0}. \tag{34}$$

Free boundary conditions:

$$T_1(s(t)-, t) = T_2(s(t)+, t) = T_K, \qquad t \in (t_0, t_k], \tag{35}$$

$$\begin{aligned} C_1(s(t)-, t) &= \psi_1(T_K), \qquad t \in (t_0, t_k], \\ C_2(s(t)+, t) &= \psi_2(T_K), \qquad t \in (t_0, t_k], \end{aligned} \tag{36}$$

with some given functions ψ_1, ψ_2 [9, 14], as well as

$$\lambda_1 \frac{\partial T_1}{\partial x}(s(t)-, t) - \lambda_2 \frac{\partial T_2}{\partial x}(s(t)+, t) = \frac{ds}{dt}(t), \qquad t \in (t_0, t_k], \tag{37}$$

$$\begin{aligned} &[C_2(s(t)+, t) - C_1(s(t)-, t)] \frac{ds}{dt}(t) = \\ &= b_1 \frac{\partial C_1}{\partial x}(s(t)-, t) - b_2 \frac{\partial C_2}{\partial x}(s(t)+, t), \qquad t \in (t_0, t_k]. \end{aligned} \tag{38}$$

Boundary conditions:

$$\frac{\partial T_2}{\partial x}(L, t) = 0, \qquad t \in (t_0, t_k], \tag{39}$$

$$\frac{\partial C_1}{\partial x}(0, t) = 0, \qquad t \in (t_0, t_k], \tag{40}$$

and

$$T_1(0, t) = u_1(t), \qquad t \in (t_0, t_k], \tag{41}$$

$$\frac{\partial C_2}{\partial x}(L, t) = u_2(t), \qquad t \in (t_0, t_k] \tag{42}$$

with some $t_k \leqslant t_f$.

Assuming that the admixture is contained in the crystallizing substance only in vestigial quantities we can simplify the free boundary conditions (36), (38) respectively to the form [9, 14]:

$$C_1(s(t)-, t) = C_2(s(t)+, t) = C_K, \qquad t \in (t_0, t_k], \tag{43}$$

$$b_1 \frac{\partial C_1}{\partial x}(s(t)-, t) - b_2 \frac{\partial C_2}{\partial x}(s(t)+, t) = 0, \qquad t \in (t_0, t_k]. \tag{44}$$

Further we will consider problem of crystal-growth in the above particular case.

Having as a purpose obtainment of a crystal with some desired physical properties [3, 6, 9, 14] we can formulate the following requirements.

(R.1) The free boundary $s(t)$, $t \geqslant t_0$ should move after a given pattern $\sigma(t)$, $t \geqslant t_0$ (cf. Section 3). The terminal time moment t_k is then defined as in Section 3.

(R.2) The final distribution of the admixture $C_1(x, t_k)$, $x \in [0, s(t_K)]$ should be equal to a given pattern $\chi(x)$, $x \in [0, s(t_k)]$.

As controls we take functions u_1, u_2 enclosed in boundary conditions (41), (42). Such boundary controls have clear physical interpretation. The control u_1 enclosed in the Dirichlet type condition (41) corresponds to assignment of the heating power [9, 14], whereas the control u_2 enclosed in the Neumann type condition (42) — to defining flux of the admixture.

The sets of admissible controls we define as in Section 3 in case of u_1 and in the following way in case of u_2:

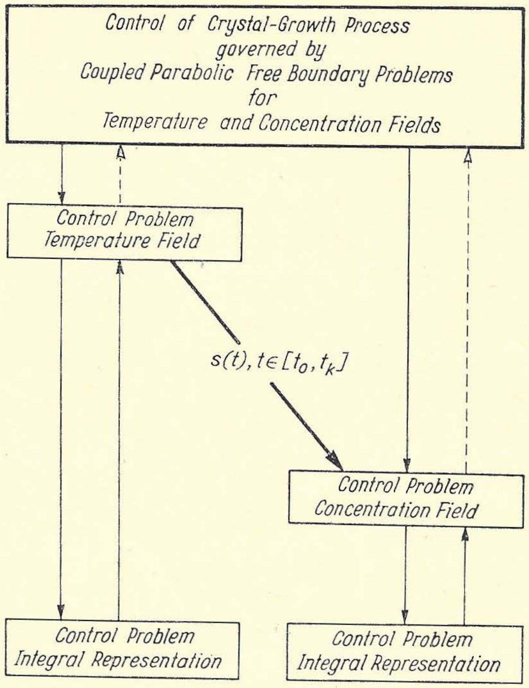
$$U_{ad}^2 = \{u_2 \in C[t_0, t_k] | u_2(t_0) = \frac{dC_{20}}{dx}(L),$$

$$c_3 \leqslant u_2(t) \leqslant c_4, \qquad t \in [t_0, t_k]\}.$$

We make also an additional simplifying assumption.

As it follows from physical experiments in the problem we consider evolution of the free boundary first of all depends on the dynamics of the thermal processes

[6]. The diffusion processes have only slender influence on this evolution. That is why we can apply the following idea of solving the considered problem (see Figure).



Idea of the proposed method of solving control problem for the crystal-growth process

First we solve the control problem for the temperature field (cf. Section 3) with model including free boundary $s(t)$, $t \geq t_0$ and with the purpose in satisfying (R.1).

Next assuming that the function $s(t)$, $t \in [t_0, t_k]$ is known, we solve the control problem for concentration field. The purpose consists now in achieving (R.2).

For the proposed method of solving the first subproblem we refer to Section 3. The second subproblem we propose to solve using similar approach.

Employing the thermal potentials' method we introduce again an equivalent integral representation of the model [9, 14, 15].

Then we formulate an appropriate inverse problem, consisting in determining boundary control u_2 and functions C_1, C_2 satisfying problem (31), (34), (40), (42), (43), (44) for the concentration field with known dynamics of the interface $s(t)$, $t \in [t_0, t_k)$ and including the terminal condition

$$C_1(x, t_k) = \chi(x), \qquad x \in [0, s(t_k)]. \tag{45}$$

This problem may be resolved into the system of integral equations [14]:

$$b_2 \int_{t_0}^{t_k} H_{20}(x, s(\tau), t_k-\tau)\, w_2(\tau)\, d\tau =$$

$$= \chi(x) - \int_0^{s_0} C_{10}(\xi)\, H_{20}(x, \xi, t_k-t_0)\, d\xi +$$

$$+ C_K \int_{t_0}^{t_k} \left[b_1 \frac{\partial H_{20}}{\partial \xi}(x, s(\tau), t_k-\tau) - \right.$$

$$\left. - \frac{ds}{d\tau}(\tau)\, H_{20}\big(x, s(\tau), t_k-\tau\big) \right] d\tau, \qquad x \in [0, l], \tag{46}$$

$$b_2 \int_{t_0}^{t} u_2(\tau) \frac{\partial H_{1L}}{\partial \xi}\big(s(t), L, t-\tau\big)\, d\tau =$$

$$= -\frac{1}{2} w_2(t) + b_2 \int_{t_0}^{t} w_2(\tau) \frac{\partial H_{1L}}{\partial \xi}\big(s(t), s(\tau), t-\tau\big)\, d\tau +$$

$$+ \int_{s_0}^{L} \frac{dC_{20}}{d\xi}(\xi)\, H_{1L}\big(s(t), \xi, t-t_0\big)\, d\xi, \qquad t \in (t_0, t_k] \tag{47}$$

where

$$w_2(t) \triangleq \frac{\partial C_2}{\partial x}\big(s(t), t\big),$$

$$l = \sigma(t_k) \qquad \text{(cf. Section 3)},$$

$$H_{20}(x, \xi, t) \triangleq = E(x-\xi, b_1 t) + E(x+\xi, b_1 t),$$

$$H_{1L}(x, \xi, t) \triangleq E(x-\xi, b_2 t) - E(x+\xi-2L, b_2 t).$$

In the above system (46) is a Fredholm equation of the first kind with respect to w_2 and (47) is a Volterra equation of the first kind with respect to u_2 (with w_2 known on the basis of (46)).

Making use of a regularization techniques (cf. Section 3) we are in a position to determine function u_2 being regularized solution or quasi-solution to the inverse problem for the concentration field.

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Sterowanie parabolicznymi zagadnieniami ze swobodną granicą – zastosowanie sformułowań odwrotnych

W pracy są rozważane dwufazowe paraboliczne zagadnienia brzegowe ze swobodną granicą. Dla takich zagadnień zostają sformułowane zadania sterowania, których celem jest zapewnienie pożądanego przemieszczania się swobodnej granicy bądź osiągnięcie wyznaczonego stanu końcowego. W pracy została zaproponowana metoda rozwiązywania takich zagadnień, wykorzystująca pewne sformułowania odwrotne problemów wyjściowych.

Управление параболическими задачами со свободной границей — применение обратных формулировок

В работе рассматривается двухфазные параболические задачи со свободной границей. Для этих задач сформулированы проблемы управления с показателями зависящими от движения свободной границы либо от финального состояния. Эти проблемы формулируется как обратные параболические задачи.