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Final observability of time-lag systems

by

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Final observability on $[0, T]$ for constant time-lag systems is defined and studied. Observability conditions depending on the length of an observation interval are formulated and proved. Two types of observability are considered: one with initial function in the space L^p, second with continuous initial function.

1. Introduction

For constant time-lag systems described by linear differential-difference equations equations observability problem can be defined in several ways.

In [1] is considered the system given by equation.

$$\begin{aligned} &\dot{x}(t)=A_0\,x(t)+A_1\,x(t-1),\ t\geqslant 0,\ x(t)\in R^n,\\ &y(t)=Cx(t),\ y(t)\in R^p, \\ &x(t)=\varphi(t),\ t\in[-1,0]. \end{aligned} \tag{1.1}$$

The observability problem in classical sense is to determine the initial state $\{\varphi(0),\ A_1\,\varphi(t),\ t\in[-1,0)\}$ where $\varphi\colon[-1,0]\mapsto R^n$ belongs to a preassumed class $\mathscr{F}\subset L^1([-1,0],R^n)$, e.g. $\varphi(\cdot)\in C([-1,0],R^n)$ or $\varphi(\cdot)\in L^p([-1,0],R^n)$, $1\leqslant p\leqslant\infty$ knowing the output $y(t)$, $t\in[0,T]$. In [1] necessery and sufficient conditions are given for observability of (1.1). Similar problem is considered in [9], where for system (1.1) general criteria are proved for determination of the initial function (not the state) $\varphi(t)$, $t\in[-1,0]$, with $\varphi(\cdot)$ belonging to the set of piecewise continuous functions provided the output $y(t)$, $t\in[0,T]$, is given. Spectral observability is discussed in [4], [8]. System (1,1) is spectral observable if and only if corresponding outputs do not vanish for all its eigensolutions.

Measurement and control problems assume quite often the knowledge of the vector $x(T)\in R^n$ at $T>0$, or the state of the system at $T>0$ given as a function $x(T+\Theta)$, $\Theta\in[-1,0]$, while only output information is available. In general case knowledge of the state at $T>0$ is not an equivalent of information about the initial state, because behaviour of the state trajectory of (1.1) is described by a semigroup (not group) property. Hence it seems to be suitable to introduce the definition of observability as the determination from output $y(t)$, $t \in [0, T]$, of vector $x(T)$ or state $x(T+\Theta)$, $T>0$, $\Theta \in [-1, 0]$.

Notation: In this paper the following notation will be used: $A \in R^{m \times p}$ means that A is a real m by p matrix. The kernel and image of a matrix A will be denoted respectively by Ker A and Im A. A^{T} is the transpose of A.

The orthogonal complement of a subspace $\mathscr{K} \subset R^n$ will be denoted by $\mathscr{K}^{\perp}$.

For given matrices $A \in R^{n \times n}$, $B \in R^{n \times m}$, $\{A|B\} = \mathrm{Im}\,[B \,\vdots\, AB \,\vdots\, \dots \,\vdots\, A^{n-1} B]$ denotes controllability subspace. If $\mathscr{B} = \mathrm{Im}\, B$ the controllability subspace will be also written $\{A|\mathscr{B}\}$. Character $\triangleq$ means equal by definition. Adjoint matrix of A will be denoted adj A. Identity p by p matrix is denoted by I_p.

2. Problem Statement and Preliminary Results

Consider the linear, time-invariant system given by the equations

$$\begin{aligned} \dot{x}(t) &= A_0\, x(t) + A_1\, x(t-1), \quad t \geqslant 0, \\ y(t) &= Cx(t), \end{aligned} \tag{2.1}$$

where $A_0, A_1 \in R^{n \times n}$, $C \in R^{p \times n}$, with initial condition: $x(t) = \varphi(t)$, $t \in [-1, 0]$, where $\varphi(\cdot) \in \mathscr{F}$. Two cases will be especially considered: $\mathscr{F}$ denotes the space $C([-1, 0], R^n)$ or $L^p([-1, 0], R^n)$, $1 \leqslant p \leqslant \infty$. A function $x_t\colon [-1, 0] \mapsto R^n$, $x_t(\Theta) \triangleq x(t+\Theta)$, $\Theta \in [-1, 0]$, can be taken as a state for (2.1) at time t. In the case when $\det|A_1 = 0$ a state of (2.1) defined above possesses overbundance of information. Therefore we define the true state which contains necessary and sufficient information to solve system equation (2.1) as follows [1]:

Definition 2.1. A pair $(x(t), A_1\, x_t) \triangleq (x(t), A_1\, x(t+\Theta), \Theta \in [-1, 0))$ is said to be a state of system (2.1) at instant t.
The fundamental definition in this paper is the following.

Dfeinition 2.2. System (2.1) is $\mathscr{F}$-finally observable on $[0, T]$ iff for all $\varphi(\cdot) \in \mathscr{F}$ such that $y(t) \equiv 0$, $t \in [0, T]$, the final trajectory value $x(T) = 0$. Definition 2.2 makes sense for $T \geqslant 1$ and means that if the system (2.1) is $\mathscr{F}$-finally observable on $[0, T]$ then a unique map $\{y(t), t \in [0, T]\} \mapsto x(T)$ exists, where $y(\cdot)$ is generated by the initial function $\varphi(\cdot) \in \mathscr{F}$. Hence it is possible to determine $x(T)$ knowing the output $y(t)$ on $[0, T]$. The importance of the Definition 2.2 lies in the fact that for $\mathscr{F}$-finally observable systems on $[0, T]$ one is able to determine the final state $(x(T+1), A_1\, x_{T+1})$. Indead, for any function space $\mathscr{F}$ such that $x_t(\cdot) \in \mathscr{F}$, $t>0$, provided $\varphi \in \mathscr{F}$ we may shift the time variable and start with $x_\tau(\cdot)$ for some $\tau>0$, as an initial function. Utilizing the stationarity of system (2.1) we get the conclusion that $\mathscr{F}$-final observability on $[0, T]$ implies $\mathcal{F}$-final observability on $[0, T_1]$, $T_1 > T$. Hence one can determine $x(T+\tau)$, $\tau \in [0, 1]$, from the data $y(t)$, $t \in [0, T+1]$ and, consequently, to get the function $x_{T+1}(\cdot)$. Summarizing these considerations we have:

Proposition 2.1.

(i) If system (2.1) is $\mathcal{F}$-finally observable on $[0, T]$, $T \geq 1$, then it is $\mathcal{F}$-finally observable on $[0, T_1]$ for each $T_1 > T$ provided $\mathcal{F}$ is such that $\varphi \in \mathcal{F}$ implies $x_t(\cdot) \in \mathcal{F}$, $t > 0$.

(ii) $\mathcal{F}$-final observability of system (2.1) on $[0, T]$ implies that the function $x_{T+1}(\cdot)$ and the final state $(x(T+1), A_1 x_{T+1}(\cdot))$ can be determined from the data $y(t)$, $t \in [0, T+1]$, for any $\varphi \in \mathcal{F}$.

The statement (ii) is especially important when a linear-quadratic problem for controlled time delay systems is considered. In such a case the optimal control has the feedback form $u(t) = \mathcal{K} x_t(\cdot)$ [7]. Therefore $x_t(\cdot)$ has to be computed on-line if only output measurements are available in the system.

Proposition 2.2.

(i) If system (2.1) is $\mathcal{F}$-finally observable on $[0, T]$ and $y(t) = 0$ for all $t \in [0, T_1]$, $T_1 > T$, then $x(t) = 0$ for all $t \in [T, T_1]$.

(ii) $y(t) = 0$ for all $t \in [0, n]$ implies $y(t) = 0$ for all $t \geq 0$

(iii) $\mathcal{F}$-final observability on $[0, T]$, $T > n$, implies $\mathcal{F}$-final observability on $[0, n]$.

Proof:

(i) Follows directly from Proposition 2.1 (i).

(ii) Let λ denote differential operator: $(\lambda x)(t) = \dfrac{dx}{dt}(t)$ and z denote the right shift operator: $(zx)(t) = x(t-1)$. State equations for system (2.1) can be written as follows: $[(\lambda I - A_0 - A_1 z) x](t) = 0$, $t \geq 0$ or, denoting $\Delta(\lambda, z) \triangleq [\lambda I - A_0 - A_1 z]$, $[\Delta(\lambda, z) x](t) = 0$.

The polynomial in λ, z matrix adj $\Delta(\lambda, z)$ is nonsingular and adj $\Delta(\lambda, z) \cdot \Delta(\lambda, z) = \det \Delta(\lambda, z)$. Hence the latter implies the equation

$$[\det \Delta(\lambda, z) x](t) = 0, \ \forall t > t_1 \tag{2.2}$$

where t_1 is such that the left-hand side of (2.2) is well defined. We shall show that $t_1 = n-1$. In fact,

$$\det \Delta(\lambda, z) = \lambda^n + a_{n-1}(z) \lambda^{n-1} + \ldots + a_1(z) \lambda + a_0(z) \tag{2.3}$$

where $a_i(z)$, $i = 0, \ldots, n-1$, are polynomials in z and $\deg(a_i(z)) \leq n-i$.

Therefore, $[a_i(z) x](t)$ is defined for $t \geq n-i-1$. Since $(\lambda^n x)(t)$ is well defined for $t > n-1$, as it is easily seen by the method of steps, the conclusion follows that (2.2) holds with $t_1 = n-1$. Clearly, the output $y(t)$ satisfies the similar equation:

$$[\det \Delta(\lambda, z) y](t) = 0, \ t \geq n \tag{2.4}$$

which follows by the fact that the operator $C \det \Delta(\lambda, z)$ equals $\det \Delta(\lambda, z)\, C$. The fact that (2.4) does not hold for $t \in [n-1, n)$ in general is clear since $[a_0(z)\, y](t)$ may be undefined on this interval ($y(t)$ is not defined for $t \in [-1, 0)$). By (2.3) we note that (2.4) is actually a differential-difference equation with maximal delay not greater than n. Therefore fixing $y(t)$, $t \in [0, n]$, uniquely determines the values $y(t)$ for $t > n$. In particular,

$$y(t) = 0 \text{ on } [0, n] \text{ implies } y(t) = 0 \text{ on } [0, \infty). \tag{2.5}$$

(iii) Suppose system (2.1) is $\mathscr{F}$-finally observable on $[0, T]$, $T > n$. Let $y(t) = 0$ for $t \in [0, n]$. Then by (2.5) $y(t) = 0$ for $t \geqslant 0$ and by (i) we have $x(t) = 0$ for all $t \geqslant T$. Hence, by the theorem of Henry [10] on small solutions for functional-differential systems we get $x(n) = 0$ and thus the proof is complete.

3. Reformulation of the $\mathscr{F}$-Final Observability Problem

System (2.1) will be rewritten as an equivalent system with greater dimension but without delay.

LEMMA 3.1 [3], [2]. Let k be an integer, $k \geqslant 1$. To each trajectory $x(t)$, $t \in [0, k]$ of the system (2.1) there corresponds uniquely a trajectory $z_k(s) \in R^{nk}$, $s \in [0, 1]$ of system without delay defined below:

$$\dot{z}_k(s) = A^k z_k(s) + B^k f(s), \quad s \in [0, 1] \tag{3.1}$$

and satisfying additional constraints

$$z_k(0) = z_k^o + J^k z_k(1) \tag{3.2}$$

where:

$$R^{nk} \ni z_k(s) \triangleq \begin{bmatrix} x_1(s) \\ \vdots \\ x_k(s) \end{bmatrix} \quad \text{and} \quad x_i(s) \triangleq x(i-1+s), \; i = 1, \ldots, k, \; s \in [0, 1], \tag{3.3}$$

$$R^{nk} \ni z_k^0 \triangleq \begin{bmatrix} x(0) \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{3.4}$$

The matrices $A^k \in R^{nk \times nk}$, $B^k \in R^{nk \times n}$, $J^k \in R^{nk \times nk}$ have the following form:

$$A^k \triangleq \begin{bmatrix} A_0 & & & 0 \\ A_1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & A_1 & A_0 \end{bmatrix}, \quad B^k \triangleq \begin{bmatrix} A_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad J^k \triangleq \begin{bmatrix} 0 & & & 0 \\ I_n & \ddots & & \\ & \ddots & \ddots & \\ 0 & & I_n & 0 \end{bmatrix} \tag{3.5}$$

and

$$f(s) \triangleq \varphi(s-1), \quad s \in [0, 1] \tag{3.6}$$

where $\varphi(\cdot) \in \mathscr{F}$ is an initial state for (2.1). The space of functions $f(\cdot)$ satisfying condition (3.6) will be denoted $\mathscr{F}_1$. Conversely, if $z_k(s)$ is a solution to (3.1) and (3.2) then using formula (3.3) a continuous trajectory $x(t)$, $t \in [0, k]$ satisfying equation (2.1) can be obtained.

Using Lemma 3.1 any system (2.1) with delay may be transformed into system without delay equivalent of the latter on time interval $[0, k]$. Now, Definition 2.2 of $\mathscr{F}$-final observability can be expressed in terms of system (3.1), (3.2).

LEMMA 3.2. Let $T>1$, $\mathscr{N} \triangleq \operatorname{Ker} C$, $\mathscr{N}_k \triangleq \mathscr{N} \times \ldots \times \mathscr{N}$ (k-times), k be an integer satisfying $k \geqslant T > k-1$ and $z_k(s)$ be a solution of (3.1), (3.2).

System (2.1) is $\mathscr{F}$-finally observable on $[0, T]$ iff for all $f \in \mathscr{F}_1$ such that for all $s \in [0, 1]$ $z_k(s) \in \mathscr{N}_k$ implies $z_k(T-k+1) \in \mathscr{N}_{k-1} \times \{0\}$.

Proof. $z_k(s) \in \mathscr{N}_k$ if and only if $y(t) \equiv 0$ for $t \in [0, T]$. Condition $x(T)=0$ is equivalent of $z_k(T-k+1) \in \mathscr{N}_{k-1}\ x\ \{0\}$.

Lemma 3.2 shows that $\mathscr{F}$-final observability problem can be regarded as a kind of controllability problem of existing a nonzero trajectory of system (3.1) satisfying (3.2) for which $z_k(s) \in \mathscr{N}_k$, $s \in [0, 1]$, and $z_k(T-k+1) \notin \mathscr{N}_{k-1} \times \{0\}$. The latter means that corresponding solution $x(t)$ of system (2.1) is not equal to zero at $t=T$. So system (2.1) is not observable on $[0, T]$. To answer the question of the existence of such trajectory properties of some controllability subspaces will be considered.

DEFINITION 3.1 [5], [6]. Let $A \in R^{p \times p}$, $B \in R^{p \times l}$ and $\mathscr{R}$ be a subspace in R^p. The greatest (as of inclusion order) subspace $\mathscr{S}$ contained in $\mathscr{R}$ satisfying

$$A\mathscr{S} \subset \mathscr{S} + \operatorname{Im} B \tag{3.7}$$

is called maximal invariant controlled subspace and denoted Mic $(A, B, \mathscr{R})$

LEMMA 3.3 [5], [6]. Condition (3.7) is satisfied iff a matrix $D \in R^{l \times p}$ exists such that

$$(A+BD)\,\mathscr{S} \subset \mathscr{S} \tag{3.8}$$

DEFINITION 3.2 [5], [6]. Let $A \in R^{p \times p}$, $B \in R^{p \times l}$, $\mathscr{R}$ be a subspace in R^p and $\bar{A} \triangleq A+$ $+BD$ for arbitrary D satisfying (3.8), $D \in R^{l \times p}$. The subspace $\{\bar{A} | \operatorname{Mic}(A, B, \mathscr{R}) \cap$ $\cap \operatorname{Im} B\}$ is called maximal controllability subspace and denoted Mcs $(A, B, \mathscr{R})$.

In [5], [6] are given algorithms for computing Mic $(A, B, \mathscr{R})$ and Mcs $(A, B, \mathscr{R})$ and properties of both subspaces are studied.

LEMMA 3.4 [2]. Each solution of (3.1) satisfying condition $z_k(s) \in \mathscr{N}_k$, $s \in [0, 1]$ has the form

$$z_k(s) = e\bar{A}^k z_k(0) + \tilde{z}_k(s), \quad s \in [0, 1] \tag{3.9}$$

where $z_k(0) \in \operatorname{Mic}(A^k, B^k, \mathscr{N}_k)$, $\tilde{z}_k(s) \in \operatorname{Mcs}(A^k, B^k, \mathscr{N}_k)$ and $\bar{A}^k \triangleq A^k + B^k D_k$ for arbitrary $D_k \in R^{n \times nk}$ satisfying $\bar{A}^k(\operatorname{Mic}(A^k, B^k, \mathscr{N}_k)) \subset \operatorname{Mic}(A^k, B^k, \mathscr{N}_k)$.

Furthermore, the set of all points $z_k(s)$ attainable from $z_k(0) \in \mathrm{Mic}(A^k, B^k, \mathcal{N}_k)$ equals

$$e^{\bar{A}^k} z_k(0) + \mathrm{Mcs}(A^k, B^k, \mathcal{N}_k).$$

COROLLARY 3.1. If $\mathrm{Mic}(A^k, B^k, \mathcal{N}_k) = \{0\}$ then system (2.1) is $\mathcal{F}$-finally observable on $[0, T]$, $T \geq k$, for any suitable $\mathcal{F} \subset L^1$.

Proof. If $\mathrm{Mic}(A^k, B^k, \mathcal{N}_k) = \{0\}$ then $\mathrm{Mcs}(A^k, B^k, \mathcal{N}_k) = \{0\}$ and only $z_k(s) = 0$, $s \in [0, 1]$, is the solution of the form (3.9). Hence $y(t) = 0$, $t \in [0, k]$, implies $x(k) = 0$, i.e., system (2.1) is $\mathcal{F}$-finally observable on $[0, k]$. By Proposition 2.1 (i) system (2.1) is $\mathcal{F}$-finally observable on $[0, T]$, $T > k$.

The following Lemma will be useful in the sequal.

LEMMA 3.5. Let $A \in R^{n \times p}$, $\mathcal{P}$ be subspace of R^p and $P^\perp$ is any matrix with independent rows spanning $\mathcal{P}^\perp$ (if $\dim \mathcal{P}^\perp = k$, then $P^\perp \in R^{k \times n}$). Define set $A^{-1}\mathcal{P} \triangleq \{x \in R^n : \exists y \in \mathcal{P} \text{ and } y = Ax\}$. Then the following equality holds:

$$A^{-1}\mathcal{P} = \mathrm{Ker}\, P^\perp A.$$

Proof. Let $x \in A^{-1}\mathcal{P}$. Then $y = Ax$ for some $y \in \mathcal{P}$. By definition of $P^\perp$, $P^\perp y = 0$ which implies $P^\perp Ax = 0$, i.e., $x \in \mathrm{Ker}\, P^\perp A$ Conversely, let $x \in \mathrm{Ker}\, P^\perp A$. Then $P^\perp Ax = 0$ implies $Ax \in \mathcal{P}$ which means that $y \in \mathcal{P}$ exists such that $y = Ax$, i.e., $x \in A^{-1}\mathcal{P}$.

4. L^p-Final Observability on $[0, k]$

Now, it will be assumed that $\mathcal{F} = L^p([-1, 0], R^n)$, $1 \leq p \leq \infty$. When $T = 1$ then condition (3.2) is always fulfilled because $J^1 z_1(1) = 0$, $z_1(0) = x_1(0) = x(0)$ and $z_1^0 = f(1) = x(0)$. The system will be L^p-finally observable on $[0, 1]$ if each trajectory of (3.1) contained in $\mathcal{N}$: $z_1(s) \in \mathcal{N}$ $\forall s \in [0, 1]$ has the property:

$$0 = z_1(1) = e^{\bar{A}^1} z_1(0) + \tilde{z}_1(1)$$

$$\forall z_1(0) \in \mathrm{Mic}(A_0, A_1, \mathcal{N}) \quad \text{and} \quad \forall \tilde{z}_1(1) \in \mathrm{Mcs}(A_0, A_1, \mathcal{N})$$

This implies, in virtue of nonsingularity of $e^{\bar{A}^1}$, that $\mathrm{Mic}(A_0, A_1, \mathcal{N}) = \{0\}$. Conversaly, if $\mathrm{Mic}(A_0, A_1, \mathcal{N}) = \{0\}$ then the property above holds since, by Definition 3.2, $\mathrm{Mcs}(A_0, A_1, \mathcal{N}) \subset \mathrm{Mic}(A_0, A_1, \mathcal{N})$.

Thus we get the following simple result.

THEOREM 4.1: System (2.1) is L^p-finally observable on $[0, 1]$ iff

$$\mathrm{Mic}(A_0, A_1, \mathcal{N}) = \{0\}.$$

Now the case when $T=k$ and k is an integer $k \geqslant 2$ will be considered. In accordance with Lemma 3.4 and Lemma 3.2 the question of L^p-final observability is a question whether all trajectories having the form:

$$z_k(s)=e^{\bar{A}^k} z_k(0)+\tilde{z}_k(s)$$

for

$$z_k(0)=z_k^o+J^k z_k(1) \tag{4.1}$$

where $z_k(0) \in \text{Mic}(A^k, B^k, \mathcal{N}_k)$, $\tilde{z}_k(s) \in \text{Mcs}(A^k, B^k, \mathcal{N}_k)$, $s \in [0, 1]$, fulfil the condition:

$$z_k(1) \in \mathcal{N}_{k-1} \times \{0\}. \tag{4.2}$$

Using the equality $z_k(1)=e^{\bar{A}^k} z_k(0)+\tilde{z}_k(1)$ condition (4.1) can be written as follows:

$$[I_{kn}-J^k e^{\bar{A}^k} \vdots -J^k] \begin{bmatrix} z_k(0) \\ \tilde{z}_k(1) \end{bmatrix} = z_k^o. \tag{4.3}$$

Let M_k, P_k be matrices whose linearly independent columns span subspaces $\text{Mic}(A^k, B^k, \mathcal{N}_k)$ and $\text{Mcs}(A^k, B^k, \mathcal{N}_k)$ respectively. If $m=\dim \text{Mic}(A^k, B^k, \mathcal{N}_k)$ and $j=\dim \text{Mcs}(A^k, B^k, \mathcal{N}_k)$ then $M_k \in R^{nk \times m}$, $P_k \in R^{nk \times j}$.

From (3.4) and definition of matrices M_k and P_k condition (4.3) can have the form

$$[(I_{kn}-J^k e^{\bar{A}^k}) M_k \vdots -J^k P_k] \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = z_k^o \in R^n \times \{0\}_{k-1},\ z^1 \in R^m,\ z^2 \in R^j, \tag{4.4}$$

where $\{0\}_{k-1}$ is the zero subspace in $R^{n(k-1)}$.

Similarly condition (4.2) is equivalent of:

$$z_k(1)=[e^{\bar{A}^k} M_k \vdots P_k] \begin{bmatrix} z_1 \\ z^2 \end{bmatrix} \in \mathcal{N}_{k-1} \times \{0\},\ z^1 \in R^m,\ z^2 \in R^j. \tag{4.5}$$

The system (2.1) will be L^p-finally observable on $[0, k]$ iff for all vectors $\begin{bmatrix} z_1 \\ z^2 \end{bmatrix}$ for which (4.4) holds, condition (4.5) is simultaneously fulfilled.

The set of vectors $\begin{bmatrix} z^1 \\ z_2 \end{bmatrix} \in R^m \times R^j$ for which (4.4) holds is according to Lemma 3.5 given by

$$\left\{ \begin{bmatrix} z^1 \\ z^2 \end{bmatrix} \in R^m \times R^j \colon [(I_{kn}-J^k e^{\bar{A}^k}) M_k \vdots -J^k P_k] \begin{bmatrix} z^1 \\ z^2 \end{bmatrix} \in R^n \times \{0\}_{k-1} \right\} =$$
$$= \text{Ker}\left(\hat{I}_k ([I_{nk}-J^k e^{\bar{A}^k}] M_k \vdots -J^k P_k)\right)$$

where $\hat{I}_k \in R^{n(k-1) \times nk}$ is a matrix whose linearly independent rows span the subspace $(R^n \times \{0\}_{k-1})^\perp = \{0\} \times R^{n(k-1)}$. For example:

$$\hat{I}_k = \begin{bmatrix} 0 & I_n & 0 & \dots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ & & & & 0 \\ 0 & \dots & \dots & 0 & I_n \end{bmatrix}.$$

The L^p-final observability is equivalent of the following inclusion

$$[e^{\bar{A}k}\, M_k \,\vdots\, P_k]\,(\operatorname{Ker}\,(\hat{I}_k\,([I_{nk}-J^k\, e^{\bar{A}k}]\, M_k \,\vdots\, -J^k P_k))) \subset \mathcal{N}_{k-1} \times \{0\}. \tag{4.6}$$

Using again Lemma 3.5 the latter inclusion is fulfilled iff

$$\operatorname{Ker}\,(\hat{I}_k\,[(I_{nk}-J^k\, e^{\bar{A}k})\, M_k \,\vdots\, -J^k P_k]) \subset \operatorname{Ker}\,(N_k^{\perp}\,(e^{\bar{A}k}\, M_k \vdots i P_k)) \tag{4.7}$$

where $N_k^{\perp} \in R^{i \times nk}$ is a matrix whose linearly independent rows span the subspace $(\mathcal{N}_{k-1} \times \{0\})^{\perp} = (\operatorname{Im} C^T)_{k-1} \times R^n$ and $i \triangleq \dim\,(\mathcal{N}_{k-1} \times \{0\})^{\perp}$. If rank $C=p$ (what is usually assumed) $i=p\,(k-1)+n$ and the following matrix can be chosen as the

$$N_k^{\perp} = \begin{bmatrix} \begin{matrix} C & & \\ & \ddots & \\ 0 & & C \end{matrix} & 0 \\ & I_n \end{bmatrix}. \tag{4.8}$$

Denote: $\Phi_k = \hat{I}_k\,((I_{nk}-J^k\, e^{\bar{A}k})\, M_k \vdots -J^k P_k)$.

Inclusion (4.6) can be rewritten as the following rank condition:

$$\operatorname{rank}\,[\Phi_k^T] = \operatorname{rank}\,[\Phi_k^T \,\vdots\, (N_k^{\perp}\, e^{\bar{A}k}\, M_k \,\vdots\, N_k^{\perp}\, P_k)^T].$$

Hence the following theorem was proved:

THEOREM 4.2. System (2.1) is L^p-finally observable on $[0, k]$ k– an integer $k \geqslant 2$ iff

$$\operatorname{rank}\,[\Phi_k^T] = \operatorname{rank}\,[\Phi_\kappa^T \,\vdots\, (N_k^{\perp}\, e^{\bar{A}k}\, M_k \,\vdots\, N_k^{\perp}\, P_k)^T].$$

Example. As the simple example of the consideration given above the following system equation (2.1) will be examined.

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} x(t) + \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} x(t-1), \qquad t \geqslant 0. \tag{4.9}$$
$$y(t) = [0 \;\; 1]\, x(t).$$

If $k=1$ then $\operatorname{Ker} C = \mathcal{N} = \left\{ n : n = \begin{bmatrix} n_1 \\ 0 \end{bmatrix} \text{ and } n_1 \in R \right\}$. It is easily to check that $\operatorname{Mic}\,(A_0, A_1, \mathcal{N}) \neq \{0\}$ then applying Theorem 4.1 system (4.9) is not L^p-finally observable on $[0, 1]$.

Now, let $k=2$. By (3.5) the matrices A^2, B^2 are given by:

$$A^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ -1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 2 \end{bmatrix}, \quad B^2 = \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\mathcal{N}_2 = \mathcal{N} \times \mathcal{N} = \left\{ n : n = \begin{bmatrix} n_1 \\ 0 \\ n_3 \\ 0 \end{bmatrix} \text{ and } n_1, n_3 \in R \right\}$. By inclusion (3.7) we can find that $\operatorname{Mic}\,(A^2, B^2, \mathcal{N}_2) = \{0\}$.

By Corollary 3.1 the system (4.9) is L^p-finally observable on $[0, 2]$.

5. C-Final Observability on $[0, k]$

In this part of the paper the case $\mathscr{F}=C([-1,0],R^n)$ will be considered. Let k denote a positive integer and $k\geqslant 2$. (The case $k=1$ will be a simple corollary from theorem for $k\geqslant 2$).

Each continuous function $f(\cdot)$, for which the trajectory of equation (3.1) is included in $\mathscr{N}_k$ for all $s\in[0,1]$ has the form [6]:

$$f(s)=D_k z_k(s)+G_k v(s),\ s\in[0,1] \tag{5.1}$$

where $D_k\in R^{n\times nk}$ is arbitrary satisfying $(A^k+B^k D_k)(\mathrm{Mic}(A^k, B^k, \mathscr{N}_k))\subset \subset \mathrm{Mic}(A^k, B^k, \mathscr{N}_k)$ and $G_k\in R^{n\times n}$ satisfies

$$\mathrm{Im}\, B^k G_k=\mathrm{Mic}(A^k, B^k, \mathscr{N}_k)\cap \mathrm{Im}\, B^k.$$

$v(\cdot)$ is a function belonging to $C([0,1], R^n)$.

According to Lemmas 3.2 and 3.4 the system (2.1) will be C-finally observable on $[0,k]$ iff

$$z_k(1)\in \mathscr{N}_{k-1}\, x\, \{0\}$$

for each

$$z_k(1)=e^{\bar{A}^k} z_k(0)+\tilde{z}_k(1),\ z_k(0)\in \mathrm{Mic}(A^k, B^k, \mathscr{N}_k),\ \tilde{z}_k(1)\in \mathrm{Mcs}(A^k, B^k, \mathscr{N}_k) \tag{5.2}$$

where from (3.2)

$$z_k(0)=z_k^o+J^k z_k(1) \tag{5.3}$$

and from (5.1)

$$f(1)=D_k z_k(1)+G_k\cdot v(1). \tag{5.4}$$

Using the definition of z_k^0 given by (3.4) and defining $I^k\in R^{nk\times n}$ as

$$I^k \triangleq \begin{bmatrix} I_n \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

condition (5.4) can be formulated as follows:

$$z_k^o=I^k D_k z(1)+I^k G_k v(1). \tag{5.5}$$

From (5.3) and (5.5) we have

$$[I_{kn}-(I^k D_k+J^k)\, e^{\bar{A}^k}]\, z_k(0)=(I^k D_k+J^k)\,\tilde{z}_k(1)+I^k G_k v(1). \tag{5.6}$$

Let $\Lambda_k\in R^{nk\times nk}$, $\Omega_k\in R^{nk\times nk}$ be defined by:

$$\begin{aligned} \Lambda_k &\triangleq I_{nk}-(I^k D_k+J^k)\, e^{\bar{A}^k}, \\ \Omega_k &\triangleq I^k D_k+J^k, \end{aligned} \tag{5.7}$$

and as before M_k, P_k are matrices whose linearly independent columns span subspaces Mic $(A^k, B^k, \mathcal{N}_k)$ and Mcs $(A^k, B^k, \mathcal{N}_k)$ respectively. If $m \triangleq \dim \mathrm{Mic}(A^k, B^k, \mathcal{N}_k)$ and $j \triangleq \dim \mathrm{Mic}(A^k, B^k, \mathcal{N}_k)$ then $M_k \in R^{nk \times m}$, $P_k \in R^{nk \times j}$. Using this definitions (5.6) is equivalent of:

$$[\Lambda_k M_k \,\vdots\, -\Omega_k P_k] \cdot \begin{bmatrix} z_1 \\ z^2 \end{bmatrix} = I^k Gv(1) \text{ where } \begin{bmatrix} z^1 \\ z^2 \end{bmatrix} \in R^m \times R^j. \tag{5.8}$$

System (2.1) is C-finally observable iff condition (5.8) implies

$$[e^{\bar{A}^k} M_k \,\vdots\, P_k] \cdot \begin{bmatrix} z^1 \\ z^2 \end{bmatrix} \in \mathcal{N}_{k-1} \times \{0\}. \tag{5.9}$$

By Lemma 3.5 this implication can be transformed into the form

$$\mathrm{Ker}\,[\Sigma_k (\Lambda_k M_k \,\vdots\, -\Omega_k P_k)] \subset \mathrm{Ker}\,[N_k^{\perp} (e^{\bar{A}^k} M_k \,\vdots\, P_k)] \tag{5.10}$$

where $N_k^{\perp}$ is defined by (4.8) and Σ_k is a matrix whose linearly independent rows span subspace:

$$(\mathrm{Im}\, I^k G_k)^{\perp} = \mathrm{Ker}\,(I^k G_k)^T = \mathrm{Ker}\, G_k^T (I^k)^T.$$

Inclusion (5.10) may be transformed into some equivalent rank condition:

$$\mathrm{rank}\,[\Sigma_k \Lambda_k M_k \,\vdots\, -\Sigma_k \Omega_k P_k]^T =$$
$$= \mathrm{rank}\,[(\Sigma_k \Lambda_k M_k \,\vdots\, -\Sigma_k \Omega_k P_k)^T \,\vdots\, (N_k^{\perp} e^{\bar{A}^k} M_k \,\vdots\, N_k^{\perp} P_k)^T]$$

Hence the following theorem was proved:

THEOREM 5.1. System (2.1) is C-finally observable on $[0, k]$, k an integer $k \geqslant 2$ iff:

$$\mathrm{rank}\,[\Sigma_k \Lambda_k M_k \,\vdots\, -\Sigma_k \Omega_k P_k]^T =$$
$$= \mathrm{rank}\,[(\Sigma_k \Lambda_k M_k \,\vdots\, -\Sigma_k \Omega_k P_k)^T \,\vdots\, (N_k^{\perp} e^{\bar{A}^k} M_k \,\vdots\, N_k^{\perp} P_k)^T]. \tag{5.11}$$

COROLLARY 5.1. System (2.1) is C-finally observable on $[0, 1]$ iff

$$\mathrm{rank}\,[\Sigma_1 M_1 - \Sigma_1 D_1 e^{\bar{A}^1} M_1 \,\vdots\, -\Sigma_1 D_1 P_1]^T =$$
$$= \mathrm{rank}\,[(\Sigma_1 M_1 - \Sigma_1 D_1 e^{\bar{A}^1} M_1 \,\vdots\, -\Sigma_1 D_1 P_1)^T \,\vdots\, (e^{\bar{A}^1} M_1 \,\vdots\, P_1)^T] \tag{5.12}$$

where Σ_1 is a matrix whose linearly independent rows span Ker G_1 and M_1, P_1 are matrices whose linearly independent columns span Mic $(A_0, A_1, \mathcal{N})$ and Mcs $(A_0, A_1, \mathcal{N})$ respectively.

Proof. if $k=1$ then $\Lambda_1 = I_n - D_1 e^{\bar{A}^1}$, $\Omega_1 = D_1$, condition (5.9) has the following form:

$$[e^{\bar{A}^1} M_1 \,\vdots\, P_1] \cdot \begin{bmatrix} z^1 \\ z^2 \end{bmatrix} = 0$$

what implies $N_1^{\perp} = I_n$.

Example. We will consider system given by equations (4.9). We have:

$$A_0=\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}, \quad A_1=\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}, \quad C=[0\ 1].$$

Let $k=1$. Then $\mathcal{N}_1=\mathcal{N}=\left\{n: n=\begin{bmatrix} n_1 \\ 0 \end{bmatrix} \text{ and } n_1 \in R\right\}$. By definitions (3.5) of A^1, B^1 and J^1 we get:

$$A^1=A_0, \quad B^1=A_1, \quad J^1=0.$$

Using inclusion (3.8) we can find $\operatorname{Mic}(A_0, A_1, \mathcal{N})=\mathcal{N}=\left\{n: n=\begin{bmatrix} n_1 \\ 0 \end{bmatrix}\right.$ and $\left. n_1 \in R\right\}$.

The matrix D_1 satisfying (3.8) can be chosen as

$$D_1=\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

what implies $\bar{A}^1=A_0+A_1 D_1=\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ and $e^{\bar{A}^1}=\begin{bmatrix} e & e^2-e \\ 0 & e^2 \end{bmatrix}$.

To find subspace $\operatorname{Mcs}(A_0, A_1, \mathcal{N})$ we observe that

$$\operatorname{Im} A_1 \cap \operatorname{Mic}(A_0, A_1, \mathcal{N})=\operatorname{Mic}(A_0, A_1, \mathcal{N})$$

and by inclusion (3.8) which shows that $\operatorname{Mic}(A_0, A_1, \mathcal{N})$ is $\bar{A}^1$ — invariant we get:

$$\operatorname{Mcs}(A_0, A_1, \mathcal{N})=\operatorname{Mic}(A_0, A_1, \mathcal{N}).$$

The matrix G_1, which fulfils condition

$$\operatorname{Im} A_1 \cap \operatorname{Mic}(A_0, A_1, \mathcal{N})=\operatorname{Im} A_1 G_1$$

can be chosen as:

$$G_1=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$\operatorname{Ker} G_1^T=\operatorname{Ker} G_1=\left\{g: g=\begin{bmatrix} 0 \\ g^2 \end{bmatrix} \text{ and } g_2 \in R\right\}$.

Because subspace $\operatorname{Mic}(A_0, A_1, \mathcal{N})$, $\operatorname{Mcs}(A_0, A_1, \mathcal{N})$ and $\operatorname{Ker} G_1^T$ are determined we can choose matrices M_1, P_1, Σ_1 as:

$$P_1=M_1=\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \Sigma_1=[0\ 1]$$

So:

$$[\Sigma_1 M_1-\Sigma_1 D_1 e^{\bar{A}^1} M_1 \ \vdots -\Sigma_1 D_1 P_1]=[-e\ \ -1],$$

$$[e^{\bar{A}^1} M_1 \ \vdots\ P_1]=\begin{bmatrix} e & 1 \\ 0 & 0 \end{bmatrix}.$$

The rank condition (5.12) is fulfilled because:

$$\operatorname{rank}\begin{bmatrix}-e\\-1\end{bmatrix}=\operatorname{rank}\begin{bmatrix}-e & e & 0\\-1 & 1 & 0\end{bmatrix}.$$

Therefore system given by equation (4.9) is C-finally observable on [0, 1].

6. Concluding Remarks

The observability criteria given in this paper can be generalized for system (2.1) with delay $h\neq 1$ using substitution $t=\tau h$ and $\tilde{x}(\tau)=x(\tau h)$ or for many comensurable delays as well. Theorems 4.2 and 5.1 are applyed for testing $\mathscr{F}$-final observability on $[0, k]$ where k is an positiv integer. By Lemmas given above one can prove conditions of $\mathscr{F}$-final observability on $[0, T]$, $T>1$. According to Proposition 2.1 (i) the necessary condition is $\mathscr{F}$-final observability on $[0, k]$ where k fulfils inequality $k-1<T\leqslant k$, moreover the rank condition of some matrices must be fulfilled. For systems (2.1) with greater dimension n and when the observation interval is long it seems to be necessery to use computers to test observability. In this case some numerical problems with determination of subspaces and computation of exponents of matrices can apear.

The important problem which has not been solved yet is construction of observer determining $x(T)$ using output information $y(t)$, $t\in[0, T]$.

References

1. Olbrot A.W.: Observability tests for constant time-lag systems. *Control a. Cyber.* **4**, 2 (1975).
2. Olbrot A.W.: Algebraic criteria of controllability to zero function for linear constant time-lag systems. *Control a. Cyber.* **2**, 1–2 (1973).
3. Olbrot A.W.: On degeneracy and related problems for linear constant time-lag systems. *Ricerche di Automatica* **3**, 3 (1972).
4. Olbrot A.W.: Observability and observes for a class of linear systems with delays (to appear).
5. Basile G., Marro G.: Controlled and conditioned invariant subspaces in linear system theory. *J. Optimiz. Theory Appl.* **3**, 5, (1969).
6. Wonham W. M.: Linear multivariable Control. Berlin: Springer-Verlag 1974.
7. Delfour M. C., McCalla C., Mitter S. K.: Stability and the infinite-time quadratic cost problem for linear hereditary differential systems. *SIAM J. Control* **13** (1975).
8. Manitius A.: Controllability, observability, stabilizability, of retarded systems. Proceedings of the 1976 IEEE Conference on Decision and Control.
9. Metelskii A.W., Miniuk S. A.: On theory of complete observability of systems with delays, (in Russian) *Differen. Uravnenia* **15**, 4 (1978).
10. Henry D.: Small solutions of linear autonomous functional differential equations. *J. Differ. Equations* **8** (1970).

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Obserwowalność końcowa układów z opóźnieniem

Zdefiniowano i zanalizowano pojęcie obserwowalności „końcowej„ na przedziale $[0, T]$ dla układów ze stałym opóźnieniem. Sformułowano i udowodniono warunki obserwowalności zależne od długości przedziału obserwacji. Rozpatrzono dwa rodzaje obserwowalności: w pierwszym zakłada się, że funkcja początkowa jest klasy L^p, w drugim przyjmuje się, że funkcja ta jest ciągła.

Окончательная наблюдаемость систем с запаздыванием

В работе определяется и анализируется понятие окончательной наблюдаемости в интервале $[O, T]$ для систем с постоянным запаздыванием. Формулируются и доказываются условия наблюдаемости, зависящие от величины интервала наблюдений. Рассмотрены два вида наблюдаемости: в первом предоплагается, что начальная функция определена в пространстве L^p, во втором — эта функция непрерывна.