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A linearly convergent approximation of quadratic cost control problems for hyperbolic systems

by

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A class of quadratic-cost optimal control problems for systems governed by second order hyperbolic equations is considered. Basing on [2], conditions for linear convergence of the finite-dimensional Galerkin approximation applied to this class are formulated. The general result is numerically verified on the one-dimensional energy minimization problem for a vibrating string.

1. Introduction

The paper delas with some distributed-control quadratic-cost optimal control problems related to vibrating systems governed by the second order hyperbolic equation. Solution of such a problem can be searched, exept a few trivial cases, only by finite-dimensional approximation. In [2] the rate of convergence estimates for a discrete-time Galerkin approximation applied to some general case has been formulated. A numerical verification of these convergence results is however a significant problem.

Our present aim is to construct a nontrivial example which could be solved analytically or numerically with arbitrarily high accuracy, making possible a practical verification of the general convergence results.

For this purpose we define in Section 2 one-dimensional example (P) of a vibrating string. The problem consists in minimizing, within a given time, an integral of the vibration energy including the control cost in the functional.

Problem (P) is obtained by a simple modification of some boundary-control problem for vibrating string considered in [4] and [7]. In both cases the control constraints of amplitude type were assumed, whereas the control cost component did not appear in the functional. The solutions in an analytical form were obtained.

Using the same technique it is shown that (P) can be reduced into an equivalent Fredholm integral equation of the second kind, which can be solved numerically with arbitrarily high accuracy. Basing on such a solution, rate of convergence for a finite-dimensional approximation can be calculated.

In Section 3 some regularity conditions for the optimal state variables are established provided that the optimal control is properly smooth. These conditions, closely related to the results of [2], are formulated for a general calss (G) containing (P) as a special case.

In Section 4 the discrete-time Galerkin approximation applied to (G) is introduced. Utilizing regularity conditions the rate of convergence of approximation is estimated.

Furthermore the results of a numerical test are presented. For the parameters of discretization sufficiently small it can be seen that the convergence of a finite-dimensional approximation is close to linear, what confirms the previous general results.

The fundamental notation used throughout the paper is mainly based on [5] and [6].

2. The problem statement

In the rectangle $Q=(0, 1)\times(0, T)$, where $T\leqslant 2$, we consider vibrations of a homogeneous string governed by the one-dimensional wave equation

$$\frac{\partial^2 y(x, t)}{\partial t^2}-\frac{\partial^2 y(x, t)}{\partial x^2}=xu(t), \quad (x, t)\in Q \tag{2.1}$$

with the homogeneous boundary conditions

$$y(0, t)=y(1, t)=0, \quad t\in[0, T] \tag{2.1a}$$

and the initial conditions

$$y(x, 0)=y_1(x), \quad \frac{\partial y}{\partial t}(x, 0)=y_2(x), \quad x\in[0, 1]. \tag{2.1b}$$

Assume that the control $u(\cdot)$ appearing in the right side of (2.1) belongs to $U=L^2(0, T)$. Consider the following

Problem (P). *Find $u^o\in U$ such that*

$$I(u^o)\leqslant I(u) \quad \forall u\in U,$$

where

$$I(u)=J(u, y(u))=$$

$$=\frac{1}{2}\gamma\int_0^T u^2(t)\,dt+\frac{1}{2}\int_0^T\int_0^1\left[\left(\frac{\partial y}{\partial t}\right)^2(x, t)+\left(\frac{\partial y}{\partial x}\right)^2(x, t)\right]dx\,dt, \tag{2.2}$$

provided that $y(x, t)$ is the solution of (2.1), and $\gamma>0$ is a given constant. ■

Utilizing technique developed in [1, 4] we can reduce (P) to a certain integral equation,which can be solved numerically with arbitrarily high accuracy.

Let us start with transforming of the cost functional to the form depending only on the control and the initial data. To this end note [4] that for any $(x, t) \in Q$ the following relation holds:

$$\frac{1}{2}\frac{\partial}{\partial t}[y_t^2(x,t)+y_x^2(x,t)]-\frac{\partial}{\partial x}[y_t(x,t)\cdot y_x(x,t)]= \\ =y_t(x,t)\,[y_{tt}(x,t)-y_{xx}(x,t)]. \tag{2.3}$$

Taking into consideration (2.3) and the state equation (2.1) we can express the vibration energy at the time t as

$$E(t,u)=\frac{1}{2}\int_0^1\left[\left(\frac{\partial y}{\partial t}\right)^2(x,t)+\left(\frac{\partial y}{\partial x}\right)^2(x,t)\right]dx= \\ =\frac{1}{2}\int_0^1\left[\left(\frac{\partial y}{\partial t}\right)^2(x,0)+\left(\frac{\partial y}{\partial x}\right)^2(x,0)\right]dx+\int_0^t\int_0^1\frac{\partial y}{\partial \tau}(x,\tau)\,x\cdot u(\tau)\,dx\,d\tau, \tag{2.4}$$

since by (2.1a)

$$\frac{\partial y}{\partial t}(0,t)=\frac{\partial y}{\partial t}(1,t)=0.$$

It is known [4, 7] that the solution of (2.1) can be expressed in the form

$$y(x,t)=y^*(x,t)+\hat{y}(x,t), \tag{2.5}$$

where

$$y^*(x,t)=2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(k\pi)^2}\int_0^t u(s)\sin k\pi(t-s)\,ds\cdot\sin k\pi x \tag{2.6}$$

is the solution of (2.1) for homogeneous initial conditions, whereas

$$\hat{y}(x,t)=\frac{1}{2}\left[\hat{y}_1(x+t)+\hat{y}_1(x-t)+\int_{x-t}^{x+t}\hat{y}_2(s)\,ds\right] \tag{2.7}$$

is the solution of (2.1) for a homogeneous right-hand side. The functions appearing in (2.7) are defined as follows

$$\hat{y}_i(x)=\begin{cases} y_i(x) & \text{for} \quad 0\leqslant x\leqslant 1 \\ -y_i(-x) & \text{for} \quad -1\leqslant x\leqslant 0 \end{cases} \tag{2.8}$$

and $i=1, 2$. They are extended to all $x \in R^1$ as periodic ones with the period equal 2.

Substituing (2.5) into (2.4) we have

$$E(t,u)=\frac{1}{2}\int_0^1[(y_1'(x))^2+(y_2(x))^2]\,dx+\int_0^t\int_0^1\frac{\partial y^*}{\partial\tau}(x,\tau)\cdot x\cdot u(\tau)\,dx\,d\tau+ \\ +\int_0^t\int_0^1\frac{\partial\hat{y}}{\partial\tau}(x,\tau)\cdot x\cdot u(\tau)\,dx\,d\tau. \tag{2.9}$$

Now we shall modify the last two components of the sum (2.9). First of all denote

$$F(\tau)=\int_0^1 x\frac{\partial \hat{y}}{\partial \tau}(x,\tau)\,dx=\int_0^1\int_x^1 \frac{\partial \hat{y}}{\partial \tau}(\xi,\tau)\,d\xi\,dx. \tag{2.10}$$

Differentiating the series (2.6) with respect to time we obtain [4]

$$\frac{\partial y^*}{\partial t}(x,t)=2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k\pi}\int_0^t u(\tau)\cos k\pi(t-\tau)\,d\tau\,\sin k\pi x. \tag{2.11}$$

Hence, applying (2.11) we derive by some elementary calculations

$$\int_0^1 x\frac{\partial y^*}{\partial \tau}(x,\tau)\,dx=2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k\pi}\int_0^{\tau} u(s)\cos k\pi(\tau-s)\,ds\int_0^1 x\sin k\pi x\,dx=$$

$$=2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(k\pi)^2}\int_0^{\tau} u(s)\cos k\pi(\tau-s)\,ds\cdot(-\cos k\pi)=$$

$$=2\sum_{k=1}^{\infty}\frac{1}{(k\pi)^2}\int_0^{\tau} u(s)\cos k\pi(\tau-s)\,ds=2\int_0^{\tau} u(s)\sum_{k=1}^{\infty}\frac{\cos k\pi(\tau-s)}{(k\pi)^2}\,ds=$$

$$=\int_0^{\tau} u(s)\left[\frac{1}{3}-(\tau-s)+\frac{1}{2}(\tau-s)^2\right]ds. \tag{2.12}$$

We have utilized in the last step the known formula

$$\sum_{k=1}^{\infty}\frac{\cos k\pi x}{k^2}=\frac{\pi^2}{6}-\frac{\pi x}{2}+\frac{x^2}{4}$$

satisfied for any $0\leqslant x\leqslant 2\pi$.

It follows from (2.9), (2.10) and (2.12) that the functional (2.2) and the following one

$$I(u)=\int_0^T\left\{\frac{1}{2}\gamma u^2(t)+(T-t)\left[F(t)+\int_0^t u(\tau)\left(\frac{1}{3}-(t-\tau)+\frac{1}{2}(t-\tau)^2\right)d\tau\right]u(t)\right\}dt \tag{2.13}$$

are equivalent having the same minimizer. Since the functional (2.13) is convex — by convexity of the problem (P) — the solution u^o is determined by the condition

$$\delta I(u;h)=\int_0^T\left\{\gamma u(t)\cdot h(t)+(T-t)F(t)\cdot h(t)+(T-t)\int_0^t u(\tau)\left(\frac{1}{3}-(t-\tau)+\right.\right.$$

$$\left.\left.+\frac{1}{2}(t-\tau)^2\right)d\tau\cdot h(t)+(T-t)u(t)\int_0^t\left(\frac{1}{3}-(t-\tau)+\frac{1}{2}(t-\tau)^2\right)h(\tau)\,d\tau\right\}dt=0$$

$$\forall h\in L^2(0,T). \tag{2.14}$$

It can be easily shown, by a simple modification of the last term, that (2.14) is equivalent to the following integral equation:

$$\int_0^T \left\{ \gamma u(t) + (T-t) F(t) + \int_0^t u(\tau)(T-t)\left(\frac{1}{3} - (t-\tau) + \frac{1}{2}(t-\tau)^2\right) d\tau + \right.$$
$$\left. + \int_t^T u(\tau)(T-\tau)\left(\frac{1}{3} + (t-\tau) + \frac{1}{2}(t-\tau)^2\right) d\tau \right\} h(t)\, dt = 0,$$

which should be satisfied for any $h \in L^2(0, T)$. This finally yields

$$\gamma u(t) + \int_0^t u(\tau)(T-t)\left(\frac{1}{3} - (t-\tau) + \frac{1}{2}(t-\tau)^2\right) d\tau + \int_t^T u(\tau)(T-\tau) \times$$
$$\times \left(\frac{1}{3} + (t-\tau) + \frac{1}{2}(t-\tau)^2\right) d\tau = -(T-t) F(t) \qquad (2.15)$$

for almost all $t \in [0, T]$.

If we denote

$$K(t, \tau) = \begin{cases} (T-t)\left(\frac{1}{3} - (t-\tau) + \frac{1}{2}(t-\tau)^2\right) & \text{for} \quad \tau \leqslant t, \\ (T-\tau)\left(\frac{1}{3} + (t-\tau) + \frac{1}{2}(t-\tau)^2\right) & \text{for} \quad \tau > t, \end{cases} \qquad (2.16)$$

then (2.15) can be expressed as the following Fredholm integral equation of the second kind:

$$u(t) + \frac{1}{\gamma} \int_0^T K(t, \tau)\, u(\tau)\, d\tau = -\frac{1}{\gamma}(T-t) F(t). \qquad (2.17)$$

There exists a unique solution of (2.17) by unique solvability of (P) and equivalence of these two problems . This solution can be found numerically with arbitrarily high accuracy. The last fact will be utilized in the sequel for the rate of convergence testing of the finite-dimensional approximation applied directly to (P).

3. General regularity conditions

3.1. Characterization of the optimal variables

Regularity of the optimal solutions is an important piece of information on the problem. The crucial role is played here by regularity of the optimal control which is responsible for the smoothness of the state variables.

In the case (P) regularity of u^o follows from equivalent integral equation. To show it observe that (2.15) can be expressed in a general form

$$\gamma u(t)+\sum_{i=1}^{3}\left[g_i^1(t)\int_0^t g_i^2(\tau)\,u(\tau)\,d\tau+g_i^3(t)\int_0^T g_i^4(\tau)\,u(\tau)\,d\tau\right]=G(t), \tag{3.1}$$

where

$$g_i^k,\ G\in C^\infty[0,T] \quad \text{for} \quad i=1,2,3, \quad k=1,\ldots,4. \tag{3.1a}$$

Note that $u(\cdot)$ is continuous by continuity of the kernel (2.16). Hence, differentiating (3.1) with respect to time and using (3.1a) we arrive at $u\in C^1[0,T]$.

Applying just the same arguments we prove inductively that the solution of (P)

$$u^o\in C^\infty[0,T]. \tag{3.2}$$

It will be shown in the next paragraph that regularity of y^o follows from (3.2) and from the state equation (2.1). In order to obtain the result of this type concerning the adjoint state variable p^o, we derive the adjoint equation applying the standard optimality conditions [5]

$$\int_0^T (\delta_y' L(u^o,y^o,p^o)(t),\,y(t))\,dt=0, \tag{3.3}$$

$$\forall y\in L^2(0,T;H_0^1(\Omega)), \qquad y(0)=\frac{dy}{dt}(0)=0, \tag{3.3a}$$

$$\int_0^T (\delta_u' L(u^o,y^o,p^o)(t),\,u(t))\,dt=0 \qquad \forall u\in L^2(0,T;H^0(\Omega)), \tag{3.4}$$

where

$$L(u,y,p)\overset{\text{df}}{=}J(u,y(u))+\int_0^T\left(p(t),\ \frac{\partial^2 y}{\partial t^2}(t)-\frac{\partial^2 y}{\partial x^2}(t)\right)dt+$$
$$-\int_0^T (p(t),(Bu)(t))\,dt, \tag{3.5}$$

$$(Bu)(x,t)\overset{\text{df}}{=}x\cdot u(t).$$

It follows from (2.1a), (2.2) and (3.3a) that

$$\int_0^T (\delta_y' J(u^o,y^o)(t),\,y(t))\,dt=-\int_0^T\left(\frac{\partial^2 y^o}{\partial t^2}(t)+\right.$$
$$\left.+\frac{\partial^2 y^o}{\partial x^2}(t),\,y(t)\right)dt+\left(\frac{\partial y^o}{\partial t}(T),\,y(T)\right). \tag{3.6}$$

On the other hand, integrating twice by parts with respect to time and using (3.3a), we obtain

$$\int_0^T \left(p^o(t), \frac{\partial^2 y}{\partial t^2}(t)\right) dt = \int_0^T \left(\frac{\partial^2 p^o}{\partial t^2}(t), y(t)\right) dt + \left(p^o(T), \frac{\partial y}{\partial t}(T)\right) - \left(\frac{\partial p^o}{\partial t}(T), y(T)\right), \tag{3.7}$$

whereas applying the Green's formula and taking advantage of (3.3a) again, we get

$$\int_0^T \left(p^o(t), \frac{\partial^2 y}{\partial x^2}(t)\right) dt = \int_0^T \left(\frac{\partial^2 p^o}{\partial x^2}(t), y(t)\right) dt + \int_0^T p^o(1, t) \times \frac{\partial y}{\partial x}(1, t)\, dt - \int_0^T p^o(0, t) \cdot \frac{\partial y}{\partial x}(0, t)\, dt. \tag{3.8}$$

Substituting (3.6)–(3.8) to (3.3) we arrive at the following adjoint mixed problem:

$$\frac{\partial^2 p^o(x, t)}{\partial t^2} - \frac{\partial^2 p^o(x, t)}{\partial x^2} = \frac{\partial^2 y^o(x, t)}{\partial t^2} + \frac{\partial^2 y^o(x, t)}{\partial x^2}, \quad (x, t) \in Q, \tag{3.9}$$

$$p^o(0, t) = p^o(1, t) = 0, \quad t \in [0, T], \tag{3.9a}$$

$$p^o(x, T) = 0, \quad \frac{\partial p^o}{\partial t}(x, T) = \frac{\partial y^o}{\partial t}(x, T), \quad x \in [0, 1]. \tag{3.9b}$$

Observe that by (3.5)

$$(B^* p^o)(t) = \int_0^1 x p^o(x, t)\, dx,$$

hence according to (3.4) we obtain [5] the gradient of the functional $J(\cdot, \cdot)$ in the form

$$\delta'_u J(u, y(u)) = \gamma u + \int_T^1 x p^o(x, t)\, dx. \tag{3.10}$$

Equations (2.1), (3.9) together with $\delta'_u J(u, y(u)) = 0$ characterize the optimal solution of (P).

3.2. Regularity of the optimal states

The main result of this section will be formulated for a certain general class of problems containing (P) as a special case. We start with introducing of this class.

Let be given a bounded domain $\Omega \subset R^n$ situated locally on one side of the properly regular boundary $\partial\Omega$ and let T be a fixed time.

In $Q=\Omega\times(0,T)$ we consider the following weak formulated hyperbolic problem:

$$\left(\frac{d^2y(t)}{dt^2}, v\right)+a\left(y(t), v\right)=\left(f(t), v\right) \qquad \forall v\in V=H_0^1(\Omega), \qquad t\in(0,T), \tag{3.11}$$

$$y(0)=y^1, \qquad \frac{dy}{dt}(0)=y^2, \tag{3.11a}$$

where symmetric, bilinear form $a(\cdot,\cdot)$ defined on $H_0^1(\Omega)\times H_0^1(\Omega)$ is assumed to be continuous and coercive.

We shall assume on the sequel that the initial data satisfy the conditions

$$\begin{aligned} &y^1\in H^{11/2}(\Omega), \qquad y^2\in H^{9/2}(\Omega),\\ &y^1=y^2=0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{3.12}$$

which imply in particular [3, 6] that the solution of (3.11) belongs to

$$Y=\left\{y\in C\left([0,T]; H^1(\Omega)\right):\ \frac{dy}{dt}\in C\left([0,T]; H^0(\Omega)\right)\right\}.$$

Let $\mathscr{H}$ be some Hilbert space and $\mathscr{H}\ni z_d$ be a given element. Denote by $U=L^2(0,T; H^0(\Omega))$ the space of controls with the usual scalar product $((\cdot,\cdot))$ and consider the cost functional of the form

$$J(u, y(u))\overset{\mathrm{df}}{=}\|Cy-z_d\|^2_{\mathscr{H}}+((Nu,u)), \tag{3.13}$$

where

$$C\in\mathscr{L}\left(H^{1,1}(Q), \mathscr{H}\right), \tag{3.14}$$

whereas

$$N\in\mathscr{L}(U,U) \tag{3.15}$$

satisfies the following condition of coercivity:

$$\exists\alpha>0 \qquad ((Nu,u))\geqslant\alpha\|u\|^2 \qquad \forall u\in U. \tag{3.15a}$$

Furthermore, let be given linear operators

$$B\in\mathscr{L}\left(H^r(\Omega), H^r(\Omega)\right) \quad \text{for} \quad r=0,\dots,4, \tag{3.16a}$$

$$D_1\in\mathscr{L}\left(H^{1,1}(Q), C\left([0,T]; H^1(\Omega)\right)\right), \tag{3.16b}$$

$$D_2\in\mathscr{L}\left(H^{1,1}(Q), C\left([0,T]; H^0(\Omega)\right)\right). \tag{3.16c}$$

Consider the following problem of optimal control.

Problem (G). *Find $u^o\in U$ such that*

$$J(u^o, y(u^o))\leqslant J(u, y(u)) \qquad \forall u\in U,$$

where $y(u)$ is the solution of the state equation

$$\left(\frac{d^2 y(t)}{dt^2}, v\right) + a(y(t), v) = (Bu(t), v) \quad \forall v \in H_0^1(\Omega), \quad t \in (0, T) \qquad (3.17)$$

along with the initial data

$$y(0) = y^1, \quad \frac{dy}{dt}(0) = y^2 \qquad (3.17a)$$

satisfying conditions (3.12). ■

Assume furthermore that the optimal adjoint state is characterized by the set

$$\left(\frac{d^2 p^o(t)}{dt^2}, v\right) + a(p^o(t), v) = \left(C^*(Cy(u^o) - z_d), v\right),$$

$$\forall v \in H_0^1(\Omega), \quad t \in (0, T), \qquad (3.18)$$

$$p^o(T) = (D_1 y(u^o))(T), \quad \frac{dp^o}{dt}(T) = (D_4 y(u^o))(T). \qquad (3.18a)$$

Basing on [6] one can get (modifying the proof of Lemma 1.3 in [3]) the following result concerning reguarity of the solution to the problem (3.11).

LEMMA 3.1. *If for some $r \geqslant 1$ the following conditions hold*

$$f \in H^{2r-1, 2r}(Q), \qquad (3.19a)$$

$$\frac{d^i f}{dt^i}(x, 0) \in H^{\frac{4r-1}{2} - i}(\Omega) \quad \text{for} \quad i = 0, 1, \ldots, 2r-1, \qquad (3.19b)$$

$$y^1 \in H^{\frac{4r+3}{2}}(\Omega), \quad y^2 \in H^{\frac{4r+1}{2}}(\Omega), \qquad (3.19c)$$

$$y^1 = y^2 = 0 \quad \text{on } \partial\Omega \qquad (3.19d)$$

then the solution of (3.11)

$$y \in H^{2r+1, 2r+1}(Q). \qquad (3.20)$$ ■

Applying the last result we can formulate regularity conditions related to the solution of the problem (G).

COROLLARY 3.1. *If*

$$u^o \in H^{4,4}(Q), \qquad (3.21)$$

then

$$y^o \in H^{5,5}(Q). \qquad (3.22)$$

If moreover

$$C^* \in \mathscr{L}\left(\mathscr{H}', H^{2,2}(Q)\right), \tag{3.23a}$$

$$C^* C \in \mathscr{L}\left(H^{5,5}(Q), H^{2,2}(Q)\right), \tag{3.23b}$$

$$D_1 \in \mathscr{L}\left(H^{5,5}(Q), C\left([0, T]; H^{7/2}(\Omega)\right)\right), \tag{3.23c}$$

$$D_2 \in \mathscr{L}\left(H^{5,5}(Q), C\left([0, T]; H^{5/2}(\Omega)\right)\right), \tag{3.23d}$$

then

$$p^o \in H^{3,3}(Q). \tag{3.24}$$

Proof. Let $r=2$. Conditions (3.19a–b) follow from (3.21) and (3.16a), hence taking (3.12) into account we obtain (3.22) by Lemma 3.1.

Now let $r=1$. Consider the adjoint equation (3.18) and apply Lemma 3.1 for the reverse direction of time $\Theta=T-t$. To prove (3.24) it is enough to check that the right-hand side in (3.18) satisfies (3.19a) as well as (3.19b) at $\Theta=0$. To this end observe that by (3.22) and (3.23a–b)

$$f=C^*(Cy^o-z_d) \in H^{2,2}(Q),$$

hence assumptions (3.19a–b) are satisfied. Now (3.24) follows from Lemma 3.1 since (3.19c–d) are implied by (3.23c–d) and (3.22). ■

Note that in the case (P) we have $z_d=0$, $D_1=0$, $D_2=1$ and in particular by (3.9)

$$C^* C=\left(\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}\right) \in \mathscr{L}\left(H^{5,5}(Q), H^{2,2}(Q)\right).$$

Hence, if conditions (3.12) are satisfied then Corollary 3.1 applies to Problem (P).

4. Rate of convergence estimates for the finite-dimensional approximation

4.1. A discrete-time Galerkin approximation

We shall define a finite-dimensional approximation of (G). Let $\tau=T/N(\tau)$ be a parameter of time discretization converging to zero, where $N(\tau)$ is a natural number. By $\chi_n(t)$ we denote the characteristic function of subinterval $[n\tau, (n+1)\tau)$. Now introduce the space $E_\tau(r_1\tau, (r_2+1)\tau; X)$ of step functions

$$v_\tau(t)=\sum_{n=r_1}^{r_2} v_\tau(n\tau)\,\chi_n(t), \qquad v_\tau(n\tau) \in X,$$

where $0 \leqslant r_1 < r_2 \leqslant N(\tau)$ and X is a given Banach space.

Let for the parameters of discretization h, k converging to zero — V_h be a linear finite-element subspace of $V=H_0^1(\Omega)$, and W_k be a constant finite-element subspace of $W=H^0(\Omega)$ (see [3] for details).

In the sequal we shall approximate Y by $E_\tau(0, T+\tau; V_h)$ and U by $U_{k,\tau}=$ $=E_\tau(0, T; W_k)$, respectively.

Using the standard finite-difference operators

$$(\partial_\tau v_\tau)(t)=[v_\tau(t+\tau)-v_\tau(t)]/\tau,$$
$$(\partial_\tau^2 v_\tau)(t)=[v_\tau(t+\tau)-2v_\tau(t)+v_\tau(t-\tau)]/\tau^2,$$
$$v_{\tau/4}(t)=\frac{1}{4}v_\tau(t+\tau)+\frac{1}{2}v_\tau(t)+\frac{1}{4}v_\tau(t-\tau), \tag{4.1}$$

for $v_\tau \in E_\tau(0, T+\tau; V_h)$, we introduce the following discrete-time Galerkin approximation of the optimal control problem (G):

Problem (D). *Find* $u^o_{k,\tau} \in U_{k,\tau}$ *such that*

$$J(u^o_{k,\tau}, y_{h,\tau}(u^o_{k,\tau})) \leqslant J(u_{k,\tau}, y_{h,\tau}(u_{k,\tau})) \qquad \forall u_{k,\tau} \in U_{k,\tau},$$

where $y_{h,\tau}(u_{k,\tau})$ *is the solution of the discrete state equation*

$$(\partial_\tau^2 y_{h,\tau}(t), v_h)+a(y_{h,\tau/4}(t), v_h)=(Bu_{k,\tau}(t), v_h) \qquad \forall v_h \in V_h, \qquad t \in [\tau, T] \tag{4.2}$$

provided that

$$a(y_{h,\tau}(0)-y^1, v_h)=a(\partial_\tau y_{h,\tau}(0)-y^2, v_h)=0 \qquad \forall v_h \in V_h. \tag{4.2a}$$

It can be shown [2, 3] that (D), as a standard finite-dimensional convex programming problem, admits a unique solution $u^0_{k,\tau}$. Applying Corollary 3.1 we can establish, by a simple modification of general convergence results in [3], the following rate of convergence estimate for the finite-dimensional approximation of (G):

Corollary 4.1. *If the assumptions of Corollary* 3.1 *hold, then*

$$\|u^o-u^o_{k,\tau}\|_{L^2(H^0)}=0(\tau+h+k). \qquad ■$$

Recall that the last result applies to the general class of problems containing (P) as a special case. It means that (P) can be used for numerical verification of the thesis in Corollary 4.1. Some results of such a test are presented in the next paragraph.

4.2. A numerical experiment

The computational test was carried over for a different values of parametres. We give below results obtained for

$$y_1(x)=\sin x, \qquad y_2(x)=0, \qquad T=1, \qquad \gamma=0.5.$$

Some other results can be found in [3].

The integral equation (2.17) was solved with a high accuracy using a standard numerical procedure FRH2 [8]. A finite-dimensional approximation of (P) was introduced according to the general scheme (D). A discrete solution of this problem

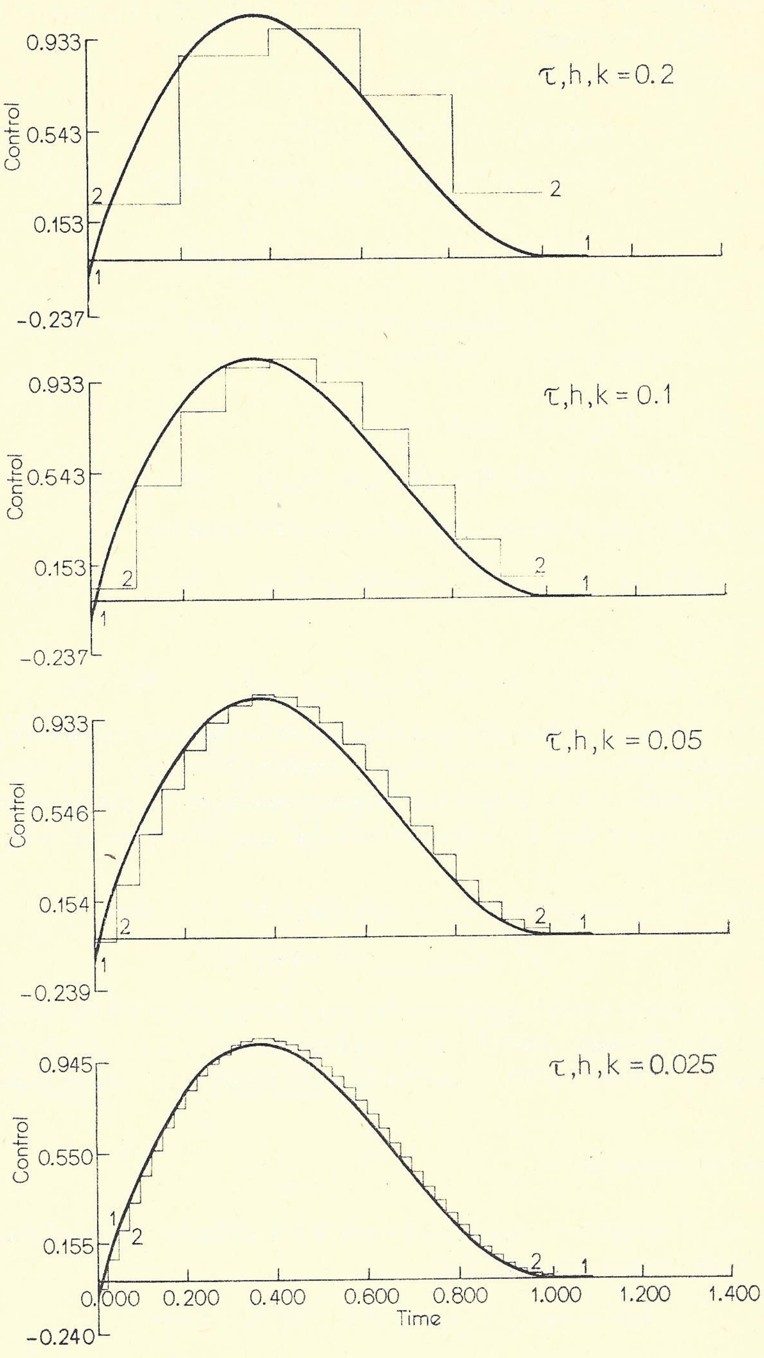

Fig. 1. The approximation error for a different values of discretization parameters
1 — a model solution, *2* — a finite-dimensional solution

was obtained by conjugate gradient optimization algorithm GSPR1 [9], where the gradient was calculated according to formula (3.10).

In Fig. 1 we present in a graphical form the results obtained for a decreasing sequence of the discretization parameters τ, h, $k=\{0.2, 0.1, 0.05, 0.025\}$.

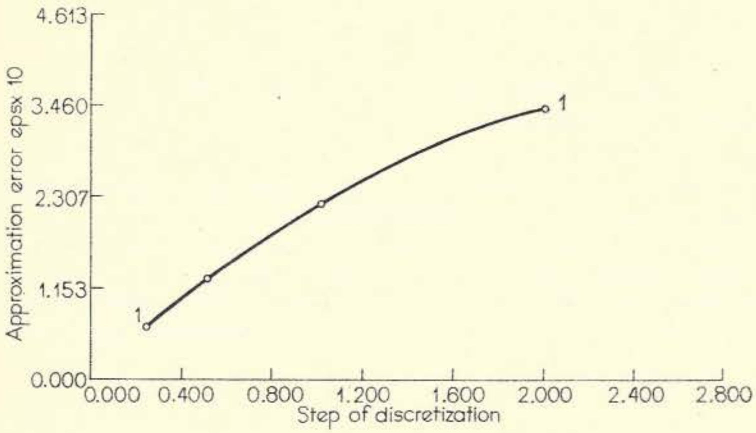

Fig. 2. Convergence of approximation

These results imply that for sufficiently small values of discretization parameters, the convergence to zero of the approximation error

$$\varepsilon=\|u^o-u^o_{k,\tau}\|_{L^2(O,T)}/\|u^o\|_{L^2(O,T)}$$

is close to linear what confirms the thesis of Corollary 4.1. The last fact is illustrated in Fig. 2.

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Liniowo zbieżna aproksymacja kwadratowych zadań sterowania systemami hiperbolicznymi

Rozważana jest klasa zadań sterowania optymalnego systemami opisanymi równaniem hiperbolicznym drugiego rzędu przy kwadratowym wskaźniku jakości. Na bazie rezultatów [2] sformułowano warunki liniowej zbieżności aproksymacji skończenie wymiarowej typu Galerkina zastosowanej do rozważanej klasy zadań. Na jednowymiarowym przykładzie minimalizacji całki z energii drgań struny dokonano weryfikacji numerycznej uzyskanych wyników ogólnych. Wzorcowe sterowanie optymalne uzyskano w tym przypadku rozwiązując numerycznie równoważne równanie całkowe.

Линейно сходимая аппроксимация квадратических задач управления гиперболическими системами

Исследуется класс линейно-квадратических задач оптимального управления системами описываемыми гиперболическим уравнением второго порядка. Используя результаты [2] формулируется условия линейной сходимости конечномерной аппроксимации типа Галеркина примененной к этому классу задач. Общий результат проверен численными методами для одномерной задачи минимизации интеграла кинетической энергии колебаний струны. Образцовое решение этого примера получено применяя очень точные числиенные методы к эквивалентному интегральному уравнению.