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Theorem on the Existence of a ρ-Satisfying Suboptimal Control for Linear-quadratic Problems Repetitively Controlled

by

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A definition of a repetitive control has been formulated. A continuity of performance index with respect to the repetitive control has been proved for linear-quadratic problems. The continuity implies existence of a suboptimal control arbitrarily close to optimal.

1. Introduction

Repetitive mode of controlling is one of the most efficient methods of managing objects with a long time horizon. The choice of lengths of a repetition period and of a planning horizon is based in majority of cases on intuition. A more objective approach has been proposed in [1]. The crucial point of that approach consists in a theorem on existence of a ρ-satisfying suboptimal control (derived for linear-quadratic problems) which may be understood in terms of a continuity of performance index with respect to suboptimal control. This theorem will be formulated and proved in the paper.

2. Repetitive control definition

A notion of a repetitive control although commonly known has been rarely defined in a formal way. For the purpose of the article following definition will be introduced (Figure presents the repetitive control idea).

DEFINITION. The mode of controlling an object will be called *repetitive* iff

a control u^* consists of the sequence $\{u_k^*\}_{k=1}^N$, whereas u_k^* is found for each (repetition) period $[T_{k-1}, T_k]$ by solving the problem:

$$u_k^* = \arg\min_{u_k \in U} \left\{ Q_k(x_k, u_k) = \int_{T_{k-1}}^{T_{k-1}+T^*} f_0\left(x_k(t), u_k(t), t\right) dt \right\} \tag{1}$$

with constraints [1])

$$\dot{x}_k(t)=f(x_k(t), u_k(t), z(t), t) \qquad t\in[T_{k-1}, T_{k-1}+T^*], \tag{2}$$

$$x_k(T_{k-1})=x_{k-1}(T_{k-1}),$$

$$T_0=0,$$

$$x_0(0)=x^o,$$

$$k=1, 2, \ldots, N \underset{\text{e.g.}}{\overset{\Delta}{=}} \frac{T-T^*}{T_p}+1, ^{2)}$$

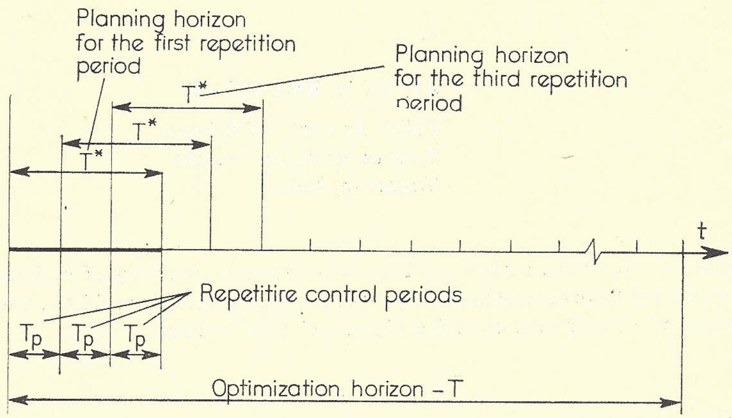

Illustration of the repetitive control

where

$x_k(t)\in R^n$, $u_k(t)\in R^m$, $z(t)\in R^z$, $z\in C^{(z)}$ is a non-controlled input (a deterministic disturbance — [3]) known for $t\in[T_{k-1}, T_{k-1}+T^*]$;

T^* is a *planning horizon* i.e. a period for which the problem (1) is to be solved;

$T_p \overset{\Delta}{=} T_k-T_{k-1}$ is a *repetition period* i.e. a period after which a new problem (1) is to be solved, thus the control u_k^* obtained from (1) is applied to the process (2) for an interval $[T_{k-1}, T_k]$;

$T\in\mathcal{T}\overset{\Delta}{=}[T_{min}, T_{max}]$ is an *optimization horizon* whereas T_{min} and T_{max} result e.g. from a technical realization of the control process, thus $0<T_{min}\leqslant T_p\leqslant T^*\leqslant T\leqslant T_{max}<$ $<\infty$.

For convinience parameters α and β which are reduced — with respect ot the optimization horizon T — lengths of T^* and T_p will be defined as:

$$\alpha\overset{\Delta}{=}T^*/T \quad \text{and} \quad \beta\overset{\Delta}{=}T_p/T$$

(obviously $0<\sigma_{min}\leqslant\beta\leqslant\alpha\leqslant1$ where σ_{min} corresponds to T_{min}).

[1]) A more general formulation of constraints is possible as well.

[2]) The repetition period number — N — follows e.g. from an assumption that after the moment T–T^* has been reached the control is to be realized continuously; it is assumed that T–T^* is divisible by T_p.

The performance index for the whole control period $[0, T]$, in case of repetitively controlled objects, consists of the sum of indices computed for each period $[T_{k-1}, T_k]$:

$$Q^* \triangleq \sum_{k=1}^{N} Q_k(x_k^*, u_k^*) \tag{3}$$

where $x_k^* \in C^n$ is a trajectory corresponding to $u_k^* \in U^*$ (U^* is in general a set of acceptable controls).

It is easy to see that the index (3) is a function of parameters α and β which characterize the planning horizon T^* and the repetition period T_p. These parameters describe two aspects of the repetitive control:

α — the smaller α the shorter the interval for which the non-controlled input $z(t)$ must be known (e.g. disturbance forecast) but, the worse quality of controlling (in terms of Q^*) is obtained;

β — the larger β the fewer repetitions of computing (and of controlling, of course) are to be performed, but, the longer is the period in which one uses non-reliabe (might be) disturbance forecast.

A user (decision maker) trying to achieve the best global effect of controlling, should be interested in knowing properties of the relationship $Q^*(\alpha, \beta)$: e.g. if this characteristic is flat he can choose such (α, β) that economic costs (computers memory and time, disturbance forecast length etc.) might be sufficently low while the performance index Q^* does not differ very much from an optimal value.

In the paper a simple but important problem of continuity of the performance index (3) with respect to α and β will be considered. A theorem proving it will be derived.

Importance of the considered continuity consists in some guaranted area of changes of α and β which do not imply large variation of Q^*. Of course, the bigger is this area the better for the user; however the problem of differentiability or finding the Lipschitz constants (solution of them could analitically clarify sensitivity of Q^* with respect to α and β changes) surpasses the scope of the paper. The attention has been paid to the continuity which is a primal notion for the sensitivity mentioned above.

In order to achieve a reference level, in the paper the optimal control for the process $\dot{x}(t) = f(x(t), u(t), z(t)), x(0) = x^o, t \in [0, T]$ will be understood as

$$\hat{u} = \arg\min_{u \in U} \left\{ Q(u, x) = \int_0^T f_0(x, (t), u(t), t)\, dt \right\} \tag{4}$$

where $z(t)$ is a non-controlled input known for the whole optimization period $[0, T]$.

The continuity of Q^* with respect to α and β may be reformulated in terms of existence of a suboptimal control (repetitive $T_p = \beta T$, based on planning horizon $T^* = \alpha T$) close to optimal. In the paper existence of such a control determined by α and β will be proved.

DEFINITION. The control $\tilde{u} \in U$ is called to be ρ-satisfying iff

$$Q(\tilde{u}) - Q(\hat{u}) < \rho$$

where: $Q(\hat{u})$ — optimal performance index value, $Q(\tilde{u})$ — performance index value obtained for a suboptimal control $\tilde{u}$.

3. The theorem on existence of a ρ-satisfying suboptimal control

In this section a theorem generalizing the idea of continuity of performance index with respect to control will be formulated and proved.

In order to obtain a concise and analytical proof the problem will be formulated in terms of linear-quadratic problem. Thus instead of (1) and (2) one has — respectively —

$$u_k^* = \operatorname{argmin} \left\{ Q_k(x_k, u_k) = \int_{T_{k-1}}^{T_{k-1}+T^*} \left(x_k^T(t) P(t) x_k(t) + u_k^T(t) R(t) u_k(t) \right) dt \right\} \quad (1a)$$

$$\dot{x}_k(t) = A(t) x_k(t) + B(t) u_k(t) + z(t). \quad (2a)$$

The rest of formulae remains unchanged.

THEOREM. For any linear process.

$$\dot{x}(t) = A(t) x(t) + B(t) u(t) + z(t); \; t \in [t_1, t_2] \triangleq \Theta,$$

$$x(0) = x^o$$

with quadratic performance index

$$Q(x, u) = \frac{1}{2} \int_{t_1}^{t_2} \left(x^T(t) P(t) x(t) + u^T(t) R(t) u(t) \right) dt$$

there exists a (suboptimal) control u^*, repetitive with a repetition period $T_p = \beta T$, found for a planning horizon $T^* = \alpha T$ $(0 < \sigma_{min} \leqslant \beta \leqslant \alpha < 1)$, which is arbitrarily close to optimal; hence u^* is a ρ-satisfying control. Moreover the performance index Q is continuous with respect to α and β.

Proof. There have been shown in [1] several continuities in C-spaces:

1° The solution of Riccati equation $K(t; T) \in R^{n \times n}$

$$\dot{K}(t; T) = -K(t; T) A(t) - A^T(t) K(t; T) + K(t; T) S(t) K(t; T) - P(t) \quad \text{for} \quad t \in [t_1, t_1 + T]$$

$$K(T; T) = 0$$

$$K(t; T) = 0 \quad \text{for} \quad t \in \{\Theta \setminus [t_1, t_1 + T]\}$$

$$T \in [T_{min}, t_2 - t_1] \triangleq \mathcal{T}$$

is continuous on rectange $\Theta \times \mathscr{T}$ and moreover $K(\cdot\,;T) \in C_{\Theta}^{n\times n}$ is continuous on interval $\mathscr{T}$ (with respect to $T \in \mathscr{T}$); $C_{\Theta}^{n\times n}$ is an $n \times n$ product of C -spaces of continuous functions over Θ; $S(t) \triangleq B(t)\,R^{-1}(t)\,B^T(t)$.

2° The solution of linear differential equation:

$$\dot{h}(t;T,K) = -\left(A(t) - S(t)\cdot K(t;T)\right)^T h(t;T,K) + K(t;T)\cdot z(t) \quad \text{for} \quad t \in [t_1, t_1+T]$$

$$h(T;T,K) = 0$$

$$h(t;T,K) = 0 \quad \text{for} \quad t \in \{\Theta \setminus [t_1, t_1+T]\}$$

$h(\cdot\,;T,K) \in C_{\Theta}^{(n)}$ is continuous with respect to (T, K) on product $\mathscr{T} \times \mathscr{K}$, where $\mathscr{K} \subset C_{\Theta}^{(n\times n)}$.

3° The solution of state equation:

$$\dot{x}(t;T,K,h) = \begin{cases} \left(A(t) - S(t)\,K(t;T)\right) x(t;T,K,h) + S(t)\cdot h(t;T,K) + z(t) & \text{for} \quad t \in [t_1, t_1+T] \\ A(t)\cdot x(t;T,K,h) + z(t) & \text{for} \quad t \in \{\Theta \setminus [t_1, t_1+T]\} \end{cases}$$

$$x(t_1;T,K,h) = x^o$$

$x(\cdot\,;T,K,h) \in C_{\Theta}^{(n)}$ is continuous with respect to (T, K, h) on product $\mathscr{T} \times \mathscr{K} \times \mathscr{H}$ $(\mathscr{H} \subset C_{\Theta}^{(n)})$.

4° A mapping transforming the set of lengths of horizons — $\mathscr{T} \subset \mathscr{R}^1$ — into a set of controls $U \subset C_{\Theta}^{(m)}$, defined as follows

$$u(\cdot\,;T) = R^{-1}(\cdot)\,B^T(\cdot)\cdot\left(h(\cdot\,;T,K) - K(\cdot\,;T)\,x(\cdot\,;T,h)\right)$$

$$u_T \triangleq u(\cdot\,;T) \in U \subset C_{\Theta}^{(m)}$$

$$u_T : \mathscr{T} \to U$$

is continuous, where

$$U \triangleq \{u_T : u_T(t;T) = R^{-1}(t)\,B^T(t) \times [h(t;T,K) - K(t;T)\,x(t;T,K,h)]\ t \in \Theta,\ T \in \mathscr{T}\}.$$

Basing on a well known theorem that every continuous mapping transforms a compact set into a compact set, now the theorem will be proved since one shows that:

5° A mapping Q^* transforming a set of controls $u^* \in U^* \subset L_2^{(m)}[t_1, t_2]$ into R^1, defined as follows:

$$Q^* : U^* \to R^1$$

$$Q^*(u^*) = \frac{1}{2} \sum_{k=1}^{N} \int_{T_{k-1}}^{T_k} \left(x_k^{*T}(t)\,P(t)\,x_k^*(t) + u_k^{*T}(t)\,R(t)\,u_k^*(t)\right) dt$$

is continuous, where $x_k^*(t)$ is a trajectory corresponding to the control $u_k^*(t)$.

6′ A mapping u^* given as follows:

$$u^*: \mathscr{R}^1 \times \mathscr{R}^1 \to L\,[0, T]$$

$$u^* \in U^* \triangleq \left\{ u^*: u^*(t; \alpha, \beta) \triangleq \left\{ \begin{array}{lll} u_1(t; \alpha T) & \text{for} & t \in [0, \beta T] \\ \vdots & & \\ u_k(t; \alpha T) & \text{for} & t \in [(k-1)\beta T, \beta T] \\ \vdots & & \\ u_N(t; \alpha T) & \text{for} & t \in [T - \alpha T, T] \end{array} \right\} \right\}$$

$$\alpha T \in \mathscr{T} \subset \mathscr{R}^1, \quad \beta T \in \mathscr{T} \subset \mathscr{R}^1, \quad \alpha \geqslant \beta, \quad u_k(\cdot, \alpha T) \subset C^{(m)}_{[(k-1)\beta T, \beta T]},$$

$$u^* \in U^* \subset L_2^{(m)}\,[0, T]$$

is continuous.

The proof of 5° is elementary; the 6° will be proved in Appendix. Thus to conclude the proof of the theorem it is sufficient to notice that:

(a) because of the contunuity of mappings Q and u^* and of the compactness of $\mathscr{T} \times \mathscr{T}$ the mapping $Q(u^*): R^1 \times R^1 \to R^1$ is continuous also;

(b) because of the latter continuity and the compactness of $\mathscr{T} \times \mathscr{T}$

$$\forall \rho > 0 \exists \delta > 0 \text{ s.t. } \sqrt{(\alpha-1)^2+\beta^2} < \delta \Rightarrow |Q(u^*(\cdot; \alpha, \beta)) - Q(u^*(\cdot; 1, 0))| < \rho. \quad \blacksquare$$

4. Conclusing remarks

The theorem proved above states that the repetitive mode of controlling may be as good as a continuous one when disturbance forecasts are known sufficiently well for each planning horizon. Thus the paper has examined interrelationships between the dynamics of object as well as non-stochastic disturbance and lengths of planning horizon and repetition period only.

The importance of that theorem consists in a possibility of a decomposition in time of a control problem. They are in general two groups of user's needs for which the theorem might turn out useful:

1° to spare memory of a computer;

2° to adjust in proper moments the control according to revised disturbance forecasts.

Ad 1°. The theorem may lead to determination — following [1] — of lengths of T^* and T_p for which the loss of optimality is to be neglected, sparing memory of a computer.

Ad 2°. In case investigated in the paper the theorem guarantees continuity of performance index; it makes in turn possible to choose such a pair of (T^*, T_p) reasonable from the economical point of view and giving sufficiently good Q^*. In practice disturbance forecast become worse with time thus one applies for a new repetition period a control based on revised forecast. As it has been stated yet the problem of taking into account a stochastic character of disturbance surpasses the scope of the paper; however it is easy (intuitively) to see that making assumption on forecast weaker (e.g. there is known its covariance and the latter increases with time) should even facilitate the proof of a similar theorem formulated in stochastic terms. In [1] an algorithm finding satisfying (T^*, T_p) is presented; it takes into account various aspects of a real control process. Some practical results obtained on the base of the presented approach are to be found in [1] and [2].

Next steps of research into sensitivity of performance index to suboptimal control and inclusion of the stochastic character of disturbance should become subject of further investigation.

Appendix

A mapping u^* given in 6° is continuous.

Proof. The statement will be proved since one shows that

$$\forall \varepsilon>0 \quad \exists \delta>0 \quad \text{s.t.}$$

$$\sqrt{(\alpha-\alpha')^2+(\beta-\beta')^2}<\delta \Rightarrow \|u^*(\cdot\,;\alpha,\beta)-u^*(\cdot\,;\alpha',\beta')\|<\varepsilon .$$

Because of the compactness of $\mathscr{T}\times\mathscr{T}\ni(\alpha,\beta)$ the problem boils down into two independent:

(a) $\forall \varepsilon>0 \quad \exists \delta>0 \quad \text{s.t.} \quad |\alpha-\alpha'|<\delta \Rightarrow \|u^*(\cdot\,;\alpha,\beta)-u^*(\cdot\,;\alpha',\beta)\|<\varepsilon;$

(b) $\forall \varepsilon>0 \quad \exists \delta>0 \quad \text{s.t.} \quad |\beta-\beta'|<\delta \Rightarrow \|u^*(\cdot\,;\alpha,\beta)-u^*(\cdot\,;\alpha,\beta')\|<\varepsilon.$

The norm of $u^*(\cdot\,;\alpha,\beta)$ in $L_2^{(m)}[0,T]$ is of the form:

$$\|u^*(\cdot\,;\alpha,\beta)\|=\sum_{k=1}^{N}\|u_k^*(\cdot\,;\alpha,\beta)\|=\sum_{k=1}^{N}\int_{T_{k-1}}^{T_k}\big(u_k(t;\alpha,\beta)\big)^2\,dt$$

thus because of $N<\infty$ it is sufficient to consider continuities (a) and (b) in an interval $[T_{k-1}, T_k]$.

The validity of (a) follows from lemmae proved in [1] and recalled in the paper as 1°, 2°, 3° and 4° and from the theorem on continuity of a differential equation with respect to initial conditions. Indeed

$$\begin{aligned}\|u_k^*(\cdot\,;\alpha,\beta)-u_k^*(\cdot\,;\alpha',\beta)\|=\int_{T_{k-1}}^{T}\big(R^{-1}(t)\,B^T(t)\,\big((h_k(t;\alpha T,K_k)-\\ -K_k(t;\alpha T)\,x_k(t;\alpha T,K_k,h_k)\big)+R^{-1}(t)\,B^T(t)\,\big(h_k(t;\alpha' T,K_k')-\\ -K_k(t;\alpha' T)\,x_k(t;\alpha' T,K_k',h_k')\big)\big)^2\,dt\leqslant(T_k-T_{k-1})\|R^{-1}(\cdot)B^T(\cdot)\|\times\\ \times(\|h_k-h_k'\|+\|K_k-K_k'\|+\|x_k-x_k'\|)\xrightarrow[\alpha\to\alpha']{}0 \qquad \text{(A1)}\end{aligned}$$

where for convenience arguments have been omitted and with, perturbated values have been denoted $\big($e.g. $h_k'\triangleq h_k(\cdot\,;\alpha' T, K_k(\cdot\,;\alpha' T))\big)$.

In case of proving the validity of (b) one should notice that the interval $[T_{k-1}, T_k]$ consists of the two following subintervals (except the last interval which is close to optimal by definition):

$$[T_{k-1}, T_k]=[(k-1)\,\beta T, k\beta' T]\cup[k\beta' T, k\beta T].$$

Thus the norm in that case may be bounded

$$\|u_k^*(\cdot\,;\alpha,\beta)-u_k^*(\cdot\,;\alpha,\beta')\|\leqslant(k\beta'-(k-1)\beta)\,T\cdot\max_t|u_k^*(t;\alpha,\beta)-u_k^*(t;\alpha,\beta')|+$$
$$+k(\beta-\beta')\,T\cdot\max_t|u_k^*(t;\alpha,\beta)-u_k^*(t;\alpha,\beta')|<\varepsilon. \tag{A2}$$

When $\beta'\to\beta$ the first term in (A2) goes to zero because of 4° (the intervals (horizons) $[(k-1)\,\beta T, (k-1)\,\beta T+\alpha T]$ and $[(k-1)\,\beta' T, (k-1)\,\beta' T+\alpha T]$ for which both controls have been found become close one to another) and because of the continuity of solution of the differential equation with respect to initial conditions; the second term goes to zero when $\beta'\to\beta$ because both controls are continuous in $[k\beta' T, k\beta T]$. ■

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Twierdzenie o istnieniu sterowania suboptymalnego spełniającego warunek ρ-dokładności dla układów liniowo-kwadratowych sterowanych repetycyjnie

Podano definicję sterowania repetycyjnego. Udowodniono ciągłość wskaźnika jakości względem tego sterowania dla układów liniowo-kwadratowych. Ciągłość ta implikuje istnienie sterowania suboptymalnego dowolnie zbliżonego do sterowania optymalnego.

Теорема о существовании субоптимального управления удовлетворяющего условия ρ-точности для линейно--квадратных систем повторяемо управляемых

Дается определение повторяемого управления. Доказана непрерывность показателя качества по отношению к этому управлению для линейно-квадратных систем. Из этой непрерывности следует существование субоптимального управления произвольно близкого оптимальному управлению.