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Pattern recognition properties of multilinear response functions. Part II

by

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5. Examples

We shall now work out in detail two simple examples. They will illuminate several aspects of the abstract constructions discussed so far. In each case, the field K will be the real numbers R. We shall denote polynomials in $R[z_1]$ by π and polynomials in $R[z_2]$ by σ; more explicitly, these polynomials are written as $\pi(z_1) = \sum_{k \geqslant 0} \pi_k z_1^k$ and $\sigma(z_2) = \sum_{k \geqslant 0} \sigma_k z_2^k$.

Example 1. Consider the bilinear function f defined by the formal power series

$$\sum_{k, l, m > 0} [a^{k-1} z_l^{-k}] [b^{l-1} z_2^{-l}] [c^{m-1} (z_1 z_2)^{-m}], \quad a, b, c \in R. \tag{5.1}$$

This power series corresponds to the transfer function

$$\frac{1}{(z_1 - a)(z_2 - b)(z_1 z_2 - c)}. \tag{5.2}$$

We recognize at once that f may be realized by the tensor product of the system $1/(z-a)$ with the system $1/(z-b)$, followed by the system $1/(z-c)$. We shall deduce this fact via systematic calculations.

a. *The equivalence relation* E_{12}. We find that there are precisely two R-independent equivalence classes, which may be represented by $[1]_{12}$ and $[(z_1-a)(z_2-b)]_{12}$.

To prove this, we note first that $\pi(z_1) E_{12} \pi(a)$ and $\sigma(z_2) E_{12} \sigma(b)$; this is immediate from (5.1). Moreover (see the denominator of (5.2))

$$OE_{12}(z_1 z_2 - c)(z_1 - a)(z_2 - b),$$

$$OE_{12} z_1^k z_2^l (z_1 - a)(z_2 - b) \quad \text{if} \quad k \neq l.$$

Consequently, we get by induction

$$(z_1 z_2)^k (z_1 - a)(z_2 - b) E_{12} c^k (z_1 - a)(z_2 - b)$$

for all $k, l \geqslant 0$. Thus, if we write,

$$\pi(z_1)\sigma(z_2) = [\pi(a) + (z_1 - a)\hat{\pi}(z_1)][\sigma(b) + (z_2 - b)\hat{\sigma}(z_2)]$$

(where $\hat{\pi}$ is the polynomial $\hat{\pi}(z_1) = (\pi(z_1) - \pi(a))/(z_1 - a)$ and $\hat{\sigma}$ is defined similarly), it follows that

$$\pi(z_1)\sigma(z_2) E_{12} (\xi_1 + (z_1 - a)(z_2 - b)\xi_2),$$

where the coordinate functions are qiven by

$$\xi_1(\pi\sigma) = \pi(a)\sigma(b) \tag{5.3}$$

and

$$\xi_2(\pi\sigma) = \sum_{k \geqslant 0} c^k \hat{\pi}_k \hat{\sigma}_k. \tag{5.4}$$

Formulas (5.3–4) completely determine the vector-space structure of X_{12} relative to the basis $[1]_{12}$ and $[(z_1 - a)(z_2 - b)]_{12}$. Remember that $X_{12} = X_{f_\otimes}$.

b. *The equivalence relation* E_1. There is a single R-independent equivalence class; we represent it by $[1]_1$. Clearly

$$\pi(z_1) E_1 \pi(a);$$

in other words, the coordinate function is

$$\eta_1(\pi) = \pi(a). \tag{5.5}$$

c. *The equivalence relation* E_2. The analysis is the same as for E_1. We have, therefore,

$$\zeta_1(\sigma) = \sigma(b). \tag{5.6}$$

Clearly, $N = \dim X_{12} + \dim X_1 + \dim X_2 = 4$.

d. *The map* ρ: $R[z_1] \times R[z_2] \to X_{12} \oplus X_1 \oplus X_2 = R^N$. From (5.3–6) we get

$$\rho: (\pi, \sigma) \mapsto (\pi(a)\sigma(b), \xi_2(\pi\sigma), \pi(a), \sigma(b)).$$

We see easily that $\xi_2(\pi\sigma) = \xi_2(\hat{\pi}\hat{\sigma})$; that is, ξ_2 is independent of $\pi(a)$ and $\sigma(b)$. Hence the only relation which exists between the coordinates is the obvious one:

$$\xi_1(\pi\sigma) = \eta_1(\pi)\zeta_1(\sigma). \tag{5.7}$$

In particular, we have now proved: *The variety V_f of f is the 3-dimensional hypersurface in R^4 corresponding to the prime principal ideal* $R[z_1, z_2, z_3, z_4](z_1 - z_3 z_4) = I(V_f)$.

e. *The internal equations of motion of the system.* In X_1 and X_2 they are obtained by inspection. We replace the polynomials π, σ by

$$\pi \mapsto (z_1 \pi + p),$$

$$\sigma \mapsto (z_2 \sigma + s).$$

where p and s respectively are the new input symbols occurring in Channels 1 and 2 at $t=0$. In one unit of time we have then the state transitions

$$[\pi]_1 \mapsto [z_1 \pi + p]_1 = (a\pi(a) + p)[1]_1,$$

$$[\sigma]_2 \mapsto [z_2 \sigma + s]_1 = (b\sigma(b) + s)[1]_2.$$

In X_{12} the calculations are more intricate:

$$\begin{aligned}
[\pi\sigma]_{12} &\mapsto [(z_1 \pi + p)(z_2 \sigma + s)]_{12} \\
&= [z_1 z_2 \pi\sigma]_{12} + [z_1 \pi s]_{12} + [z_2 \sigma p]_{12} + [ps]_{12}, \\
&= z_1 z_2 [\pi\sigma]_{12} + s z_1 [\pi]_{12} + p z_2 [\sigma]_{12} + [ps]_{12}, \\
&= \xi_1 [z_1 z_2]_{12} + \xi_2 [z_1 z_2 (z_1 - a)(z_2 - b)]_{12} + \\
&\quad + (as\,\pi(a) + bp\,\sigma(b) + ps)[1]_{12}, \\
&= ((a\eta_1 + p)(b\zeta_1 + s) + ab(\xi_1 - \eta_1 \zeta_1))[1]_{12} + \\
&\quad + (\xi_1 + c\xi_2)[(z_1 - a)(z_2 - b)]_{12}.
\end{aligned}$$

The read-out function, which has played no rôle in these computations, is given by

$$[\pi\sigma]_{12} \mapsto \xi_2(\pi\sigma) = (f_\otimes(\pi\sigma))_1 \in R. \tag{5.8}$$

f. *The reduced internal equations.* In view of the algebraic relation (5.7) and the fact that ξ_1 is not involved in the computation of the output (see (5.8)), we can write a set of minimal equations for the system in the form

$$\left\{\begin{array}{ll}
\text{(a)} & \xi_2(t+1) = c\,\xi_2(t) + \eta_1(t)\,\zeta_1(t), \\
\text{(b)} & \eta_1(t+1) = a\,\eta_1(t) + p, \\
\text{(c)} & \zeta_1(t+1) = b\,\zeta_1(t) + s, \\
\text{(d)} & y(t) = \xi_2(t).
\end{array}\right. \tag{5.9}$$

The "mechanization" of these equations requires precisely one multiplier (in (5.9a)). Notice that the realization provided by (5.9) corresponds to what we expected to obtain, given the special form of the transfer function (5.2). The system is shown in Fig. 5.1. This realization is characterized by the fact that two stored (state) variables occur *before* the multiplier and one *after*.

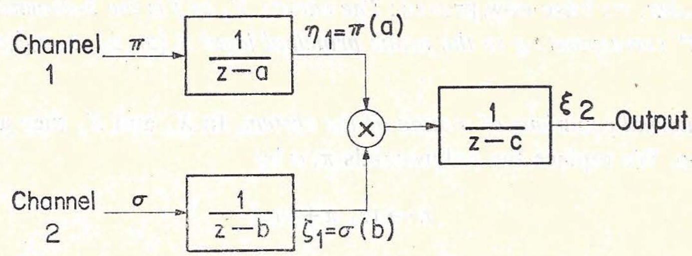

Figure 5.1

Under certain circumstances, a minimal realization may not be unique; in this example, the property mentioned in the last sentence above is not preserved when $b=0$. For, if $b=0$, we can write an alternate set of state equations:

$$\left\{\begin{array}{ll}\text{(a)} & \xi_1(t+1)=(a\,\eta_1(t)+p)\,s, \\ \text{(b)} & \xi_2(t+1)=c\,\xi_2(t)+\xi_1(t), \\ \text{(c)} & \eta_1(t+1)=a\,\eta_1(t)+p, \\ \text{(d)} & y(t)=\xi_2(t).\end{array}\right. \tag{5.10}$$

This system is shown in Fig. 5.2. Notice that the transfer function for Fig. 5.1 (with $b=0$) is

$$\left(\frac{1}{z_1-a}\right)\left(\frac{1}{z_2}\right)\cdot\left(\frac{1}{z_1\,z_2-c}\right),$$

whereas for Fig. 5.2 it is

$$\left(\frac{z_1}{z_1-a}\right)\cdot\left(\frac{1}{z_1\,z_2}\right)\left(\frac{1}{z_1\,z_2-c}\right).$$

(The dot indicates the position of the multiplier.)

In the second case a cancellation $(z_1/z_1\,z_2)$ takes place *across* the multiplier. Because of the possibility of such cancellations (reminiscent of similar difficulties in linear systems), the "structural" properties of multilinear systems relative to the position of the multipliers are not unique, even for minimal realizations.

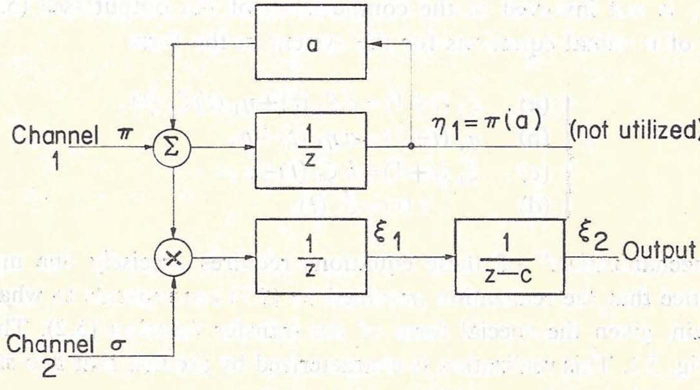

Figure 5.2

Example 2. We consider now a bilinear function which is the product of two linear functions. Here the expected form of the realization is obvious and yet the details of the systematic calculation will reveal some surprising features. The transfer function of the input/output function is given by

$$\left[\frac{z_1^2+z_1+1}{z_1^3+2z_1^2-z_1-2}\right]\left[\frac{z_2}{z_2^2-1}\right]=\left[\frac{1}{z_1+2}+\frac{1}{z_1^2-1}\right]\left[\frac{z_2}{z_2^2-1}\right]. \tag{5.11}$$

(Since no terms involving $z_1 z_2$ appear in the denominator, this is obviously a tensor product of two input/output functions. The function in Channel 1 has a 3-dimensional minimal realization, while the function in Channel 2 has a 2-dimensional minimal realization. These facts should be kept in mind in doing the deductive computations which follow.)

a. *The equivalence relation* E_{12}. The output is a linear combination of four basic sequences, as follows

$$\begin{aligned}\alpha&=(1,\ 0,\ 1,\ 0,\ 1,\ 0,\ \ldots),\\ \beta&=(2^0,0,\ 2^2,0,\ 2^4,0,\ \ldots),\\ \gamma&=(0,\ 1,\ 0,\ 1,\ 0,\ 1,\ \ldots),\\ \delta&=(0,\ 2^1,0,\ 2^3,0,\ 2^5,\ \ldots).\end{aligned}$$

Easy calculations show which input polynomials (in $R[z_1, z_2]$) correspond to which output sequences. We find, for all $k\geqslant 0$,

$$\left\{\begin{aligned}&z_1^{2k}/4^k\Rightarrow\beta,\\ &z_1^{2k+1}\Rightarrow\alpha-2\cdot 4^k\beta,\\ &z_1^{2k}\,z_2\Rightarrow\gamma-4^k\delta,\\ &z_1^{2k+1}\,z_2/4^k\Rightarrow 2\delta.\end{aligned}\right. \tag{5.12}$$

It is clear that $\dim X_{12}=4$. Notice that we have a *degeneracy* here; if we realize each factor in (5.11) *separately* and then take their tensor product, we would get a 6-dimensional system. The trouble is that the latter system is not a minimal realization of the (abstract linear) function $f_\otimes$.

An obvious (and useful) choice of coordinates for X_{12} is one in which each coordinate gives rise to exactly one of the output sequences $\alpha, \ldots, \delta$. So we let

$$x=\xi_1\,[z_1+2]_{12}+\xi_2\,[1]_{12}+\xi_3\left[\frac{1}{2}(z_1+2)\,z_2\right]_{12}+\xi_4\left[\frac{1}{2}z_1\,z_2\right]_{12}.$$

To compute the coordinate map $[\pi\sigma]_{12}\mapsto(\xi_1,\ldots,\xi_4)$ we write

$$\pi(z_1)\,\sigma(z_2)=\sum_{k,l\geqslant 0}\left\{a_{kl}^{(1)}\,z_1^{2k}\,z_2^{2l}\,(z_1+2)+a_{kl}^{(2)}\,z_1^{2k}\,z_2^{2l}/4^k+\right.$$
$$\left.+\frac{1}{2}\,a_{kl}^{(3)}\,z_1^{2k}\,z_2^{2l}\,z_2\,(z_1+2)+\frac{1}{2}\,a_{kl}^{(4)}\,z_1^{2k}\,z_2^{2l}\,z_1\,z_2/4^k\right\}. \tag{5.13}$$

(The reader should verify that the coefficients $a_{kl}^{(1)}, \ldots, a_{kl}^{(4)}$ are in fact uniquely determined by $\pi\sigma$.) With the aid of this special representation of the elements of $R[z_1, z_2]$, we find immediately that

$$\zeta_i(\pi\sigma) = \sum_{k,l\geqslant 0} a_{kl}^{(i)}(\pi\sigma), \quad i = 1, \ldots, 4. \tag{5.14}$$

b. *The equivalence relation E_2.* It is very easy to see that X_2 is 2-dimensional; everything is explained by the relations

$$z_2^{2k} E_2 1 \quad \text{and} \quad z_2^{2k+1} E_2 z_1.$$

In other words

$$[\sigma]_2 = \zeta_1 [1]_2 + \zeta_2 [z_2]_2,$$

where

$$\sigma(z_2) = \sum_{k\geqslant 0} (\sigma_{2k} z_2^{2k} + \sigma_{2k+1} z_2^{2k+1}), \tag{5.15a}$$

$$\zeta_1(\sigma) = \sum_{k\geqslant 0} \sigma_{2k}, \quad \zeta_2(\sigma) = \sum_{k\geqslant 0} \sigma_{2k+1}. \tag{5.15o}$$

c. *The equivalence relation E_1.* This is much more complicated. Let us note first that

$$(z_1^2 - 1)(z_1 + 2) E_1 0;$$

or, more usefully,

$$z_1^2 (z_1 + 2) E_1 (z_1 + 2),$$

and

$$z_1^2 (z_1^2 - 4) E_1 (z_1^2 - 4).$$

This suggests using the basis $[1]_1$, $[z_1+2]_1$, and $[z_1^2-4]_1$ for X_1. If we now write

$$\pi(z_1) = \sum_{k\geqslant 0} \{a_0^{(1)} + a_k^{(2)} z_1^{2k}(z_1+2) + a_k^{(3)} z_1^{2k}(z_1^2-4)\} \tag{5.16a}$$

(the $a_k^{(i)}$ are clearly unique!) then

$$\begin{aligned} \eta_1(\pi) &= a_0^{(1)}, \\ \eta_3(\pi) &= \sum_{k\geqslant 0} a_k^{(2)}, \\ \eta_3(\pi) &= \sum_{k\geqslant 0} a_k^{(3)}. \end{aligned} \tag{5.16b}$$

Important remark: Notice (from (5.12)) that $[z_1^2-4]_{12} = 0$ but $[z_1^2-4]_1 \neq 0$! In other words, the module X_1 has a more complicated structure than the corresponding submodule $\{[\pi]_{12} : \pi \in R[z_1]\} \subset X_{12}$; the projection $[\pi]_1 \mapsto [\pi]_{12}$ is a nontrivial operation; the module structure of X_1 gives more information than the module structure of X_{12}.

Clearly, $N = 9$.

d. *The map* $\rho: R[z_1] \times R[z_2] \to X_{12} \oplus X_1 \oplus X_2 = R^N$. Formally, we have that

$$\rho: (\pi, \sigma) \mapsto (\xi_1, \ldots, \xi_4; \; \eta_1, \eta_2, \eta_3; \; \zeta_1, \zeta_2).$$

Here the relationships between coordinates are rather complex. We shall utilize some special tricks related to the definitions of the various bases. Notice that $z_1^{2k}[z_1^2-4]_{12}=0$ and $z_1^{2k}[(z_1^2-4)\,z_2]_{12} = -3 \cdot \frac{1}{2} z_1^{2k}[(z_1+2)\,z_2]_{12}$ for all $k \geqslant 0$. Remembering this, we see by easy calculations from (5.13), (5.15), and (5.16) that

$$\begin{cases} \xi_1(\pi\sigma) = \eta_2(\pi)\,\zeta_1(\sigma), \\ \xi_2(\pi\sigma) = \eta_1(\pi)\,\zeta_1(\sigma), \\ \xi_3(\pi\sigma) = (\eta_1(\pi) + 2\eta_2(\pi) - 3\eta_3(\pi))\,\zeta_2(\sigma), \\ \xi_4(\pi\sigma) = -\eta_1(\pi)\,\zeta_2(\sigma). \end{cases} \tag{5.17}$$

So our variety V_f is $9-4=5$-dimensional, as expected. Moreover, the first 4 coordinates may be eliminated: the map is realizable as a product of linear systems. Conclusion: *the ideal* $I(V_f)$ *of the variety of* f *is generated by*

$$\{z_1 - z_6 z_8, \; z_2 - z_5 z_8, \; z_3 - (z_5 + 2z_6 - 3z_7)\,z_9, \; z_4 + z_6 z_9\}.$$

e. *The internal equations of motion of the system.* Since the readout function is

$$\begin{aligned} [\pi\sigma]_{12} \mapsto \xi_1(\pi\sigma) + \xi_2(\pi\sigma) &= f_{\otimes}(\pi\sigma)_1, \\ &= [\eta_1(\pi) + \eta_2(\pi)]\zeta_1(\sigma), \end{aligned}$$

it suffices (and it is necessary) to mechanize the computation of $\eta_1(\pi) + \eta_2(\pi)$ and $\zeta_1(\sigma)$.

The linear system in Channel 2

$$f_2: \sigma \mapsto \sum_{j=1}^{\infty} \zeta_1(z_2^{j-1}\sigma)\,z^{-j}$$

is 2-dimensional and described by

$$z_2: \begin{cases} \zeta_1 \mapsto \zeta_2, \\ \zeta_2 \mapsto \zeta_1. \end{cases}$$

Since the output of this system is given by $y_2 = \zeta_1(t)$, its realization can be seen by inspection to be

$$z_2 \approx F_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad H_2 = [1 \;\; 0].$$

The corresponding transfer function is

$$H_2(z_2 I - F_2)^{-1} G_2 = \frac{z_2}{z_2^2 - 1}.$$

This is the same as the second factor in (5.11). Note that we have discovered this factor by a strictly deductive procedure, using only the power series (5.11)[16]).

The linear system in Channel 1

$$f_1 : \pi \mapsto \sum_{j=1}^{\infty} [\eta_1 (z_1^{j-1} \pi) + \eta_2 (z_1^{j-1} \pi)] \, z^{-j}$$

reguires a more elaborate analysis. On X_1 the transition equations give (with respect to the basis fixed earlier)

$$z_1 \approx F_1 = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 2 & -3 \\ 0 & 1 & -2 \end{bmatrix},$$

$$H_1 = [1 \quad 1 \quad 0],$$

$$G_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

This system is minimal. Hence $\dim f_1 = 3$.

The transfer function corresponding to this system is

$$H_1 (z_1 I - F_1)^{-1} G_1 = \frac{z_1^2 + z_1 + 1}{z_1^3 + 2z_1^2 - z_1 - 2}.$$

Thus we have also deduced the first factor in (5.11).

It is instructive to calculate also the tensor product of $(F_1, -, H_1)$ with $(F_2, -, H_2)$, as a check on the preceding calculation of $f_{\otimes}$. We have

$$H_{\otimes} = H_1 \otimes H_2 = [1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0]$$

and

$$F_{\otimes} = F_1 \otimes F_2 = \begin{bmatrix} 0 & -2 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & -3 \\ 1 & 0 & 2 & 0 & -3 & 0 \\ 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 & -2 & 0 \end{bmatrix}.$$

Forming the observability matrix, we see at once that the system is reducible and that its observable part is of dimension 4. Using as a basis for the dual space the row vectors $H_{\otimes}, \ldots, H_{\otimes} F_{\otimes}^3$, we can recompute $H_{\otimes}$ and $F_{\otimes}$ in reduced form and obtain

$$H_{\otimes}^{\mathrm{red}} = [1 \quad 0 \quad 0 \quad 0]$$

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[15]) This paragraph was added in 1978.

and

$$F_{\otimes}^{\mathrm{red}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 0 & 5 & 1 \end{bmatrix}.$$

This shows that the characteristic polynomial is $\chi_F(xy) = ((xy)^2 - 1)((xy)^2 - 4)$. The same result is obtained if we compute χ_f from (5.13) or from (5.12).[16)]

Conclusion. The *simplest* realization of f is given by $\Sigma_{f1} \times \Sigma_{f2}$, as may have been expected from the form of (5.11).

6. Concluding Remarks

While a deeper mathematical investigation of the results discussed here is yet to be made, it is appropriate to comment on certain features of the situation which have obvious intuitive significance.

Nature of the state set. Our fundamental result is that the state set of a multilinear map f is an n-dimensional algebraic variety f embedded in the affine space K^N, where N is the total number of coordinates (in the ordinary sense of the word) which are sufficient to parametrize the Nerode equivalence relation. The "coordinates" corresponding to the Nerode classes define a point on the variety V_f. Therefore some of them may be expressible in terms of the others via algebraic relationships. In other words, *we may visualize the state of the system as described by N coordinates, of which only n need be stored and the remaining N − n can be quickly computed (if needed) since they are given by simple functions of the first n coordinates*[17)]. While the total memory involves N quantities, the possibility of rapid, real-time computation reduces the actual storage requirement to n quantities. It is tempting to speculate that something similar is happening in the brain. It may be that, instead of storing *everything* in the memory, only essential quantities are stored since the others may be recovered quickly by computation when they are needed.

Note, for instance, that a bilinear machine Σ_f has potentially all the capabilities of the corresponding tensor machine $\Sigma_{f\otimes}$ (since $X_{12} \subset X_f \approx K^N$), and usually much more (as expressed by the other state sets X_1 and X_2 in X_f). However, to utilize explicitly the inherent possibility of the tensor machine it may be necessary to add to the system read-out maps (belonging to $K[z_1, \ldots, z_N]/I(V_f)$). In Example 1, such a read-out map would be given by the relation (5.7); in Example 2, the appropriate

[16)] The last two paragraphs were added in 1978.

[17)] This intuitive description turns out to be essentially correct, although it does not follow immediately from the results presented earlier. It is shown in Kalman [16] that V_f, viewed here as embedded in k^N, is isomorphic with k^n or a Zariski-dense subset of it.

relations are (5.17). In short, our present analysis, while incomplete, already indicates that the capabilities of multilinear machines may be quite wide, provided suitable read-out maps are available from the n basic state variables.

Structural elements of the system. If a polynomial $g \in K[z_1, ..., z_N]$ belongs to the ideal $I(V_f)$, then any (polynomial) read-out map $h: K^N \to K$ may be replaced by the map $g+h$, since $g=0$ on V_f by definition. Consequently the read-out map may be modified at will modulo $I(V_f)$. So *we may interpret* $I(V_f)$ *abstractly as the collection of different operations which can be utilized to build a given system.* Similar comments apply also to the actual mechanization of the equations of motion. Thus the association $f \to V_f$ throws some light on the amount of freedom we have in building a given f and the kind of components which are required. This is, of course, a very basic question of system theory.

Pattern recognition aspects. The examples in Section 5 show that the state coordinates in K^N correspond to rather complicated parameters of the input signals. In particular, a bilinear machine can perform quadratic Fourier analysis (the linear resolution of the product $\pi\sigma$, namely the set X_{12}) as well as linear Fourier analysis (the sets X_1 and X_2). In the multilinear case, the complexity of the situation is of course much greater. If we are interested, for instance, in nonlinear prediction and filtering (where nonlinear = finite sum of multilinear) then *we must not only solve such problems for all* $f_\otimes$ *computed from finite products of linear maps, but must then recombine the results into a minimal system in which the relevent quantities are efficiently stored.* The present theory gives some indications as to how all this might be accomplished. Turning again to biological speculation, it is quite clear that Fourier analysis (of, say, brain-wave patterns) must certainly include the resolution of tensor products *and* their recombination into a minimal system. The first problem seems to be barely understood by experimental investigators at present; as to the second, the author is not aware of any prior suggestions, theoretical or practical, in that direction.

Evidently enormously more remains to be done, but it seems that the correct directions of research are now clear.

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Odpowiedzi wieloliniowych układów dynamicznych a rozpoznawanie obrazów

Teoria liniowych (dyskretnych w czasie) układów dynamicznych została z punktu widzenia algebraicznego w sposób definitywny ustalona dzięki fundamentalnym rezultatom autora [3], które pozwalają identyfikować „stan“ układu dynamicznego „z pamięcią“, jaką układ zachowuje w określonym wejściu. Innymi słowy, „stan“ jest specjalną cechą „obrazu wejściowego“, którą układ jest w stanie zidentyfikować.

Niniejsza praca stanowi naturalną próbę przeniesienia tych rezultatów na teorię układów wieloliniowych.

Podane są zasadnicze elementy algebraicznej teorii układów wieloliniowych, które wskazują na wielką złożoność takich układów. Krótko przedstawiono pewne konsekwencje dotyczące nieliniowej filtracji, rozpoznawania obrazów oraz nieliniowej analizy Fouriera.

Временная характеристика многолинейных динамических систем, а распознавание образов

Теория линейных (дискретных во времени) динамических систем была, с алгебраической точки зрения, окончательно определена благодаря фундаментальным результатам автора [3], позволяющим идентифицировать ,,состояние" динамической системы с ,,памятью", которую сохраняет система об определенном входе. Другими словами ,,состояние" является особой чертой входного ,,образа", которую система может идентифицировать.

Данная работа является естественной попыткой переноса этих результатов в теорию многолинейных систем.

Даются основные элементы алгебраической теории многолинейных систем, которые указывают на значительную сложность таких систем. Кратко представлены некоторые следствия, касающиеся нелинейной фильтрации, распознавания образов и нелинейного анализа Фурье.