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Existence of optimal decisions

by

SJUR D. FLÅM

The Chr. Michelsen Institute, Dept. of Science and Technology
Bergen, Norway

Sufficient conditions for the existence of optimal one-stage decisions are given.

Introduction

Let S be the non-empty state space of some system and A the non-empty action space of the single player. 2^A denotes the power set of A. The function $D: S \dashrightarrow 2^A$ specifies the set $D(s)$ of admissible actions when the state is s. On $D(s)$ there is given a binary relation $<_s$. We are concerned with one-stage decisions.

A decision is a function $d: S \dashrightarrow A$ such that $d(s) \in D(s)$ for all $s \in S$. A decision is optimal if $d(s)$ is maximal in $D(s)$ with respect to $<_s$ for each $s \in S$. Denote by $\hat{D}(s)$ the set of $<_s$ maximal elements in $D(s)$. $(S, A, D, <)$ is called a decision problem [8].

Let $(S, \mathscr{S})$ be a measurable (topological) space and A a topological space. $\mathscr{S}$ denotes the family of measurable (resp. closed) subsets of S. $\varphi: S \to 2^A$ is said to be upper measurable (upper semi-continuous) if

$$\varphi^{-1}(B) = \{s \in S: \varphi(s) \cap B \neq \emptyset\} \in \mathscr{S} \quad \text{for every closed } B \subseteq A. \tag{1}$$

When the condition (1) is valid for every open $B \subseteq A$, φ is said to be lower measurable (lower semi-continuous when $\mathscr{S}$ denotes the collection of open sets). If φ is both upper and lower semi-continuous it is defined to be continuous. Whenever a space is both measurable and topological it is understood that the σ-field is invariably the one generated by the open sets.

The selection theorem of Kuratowski, Ryll-Nardzewski [6] gives immediately

THEOREM 1. *If A is a separable metric space and $s \to \hat{D}(s) \neq \emptyset$ is a lower measurable function with complete values then there exists a measurable decision for the problem $(S, A, D, <)$.*

A Polish space is a separable, complete metric space. The selection theorem of Aumann [1] implies directly

THEOREM 2. *If* $(S, \mathscr{S}, \mu)$ *is a complete σ-finite measure space, A is a Borel subset of a Polish space and* $s \to \hat{D}(s) \neq \emptyset$ *has measurable graph then there exists a measurable function* $d: S \to A$ *such that* $d(s) \in \hat{D}(s)$ *a.e.*

REMARK. Let $(S, \mathscr{S}, \mu)$ be a complete measure space. Let $s \to \hat{D}(s)$ have closed values and A be separable, metric and the continuous image of a Polish space. Then $\hat{D}$ is lower measurable iff $\hat{D}$ has a measurable graph. [5] Theorem 3.5, pp. 57–58.

Our problem is to give sufficient conditions for the existence of optimal decisions. Theorem 1 and Theorem 2 with the remark tell that the important thing is to study when $\hat{D}$ is non-void and lower measurable.

PROPOSITION 1. *Let* $D: S \to 2^A$ *be given. Suppose S and A are topological spaces and for each* $s \in S$ $<_s$ *is an irreflexive, transitive relation on* $D(s)$ *such that*

$$\begin{gathered} \{(s, a, a') \in S \times D^2(s) : a \nless_s a'\} \\ \text{is closed in } S \times A^2. \end{gathered} \tag{2}$$

If $D(s)$ *is compact and non-empty for some s then* $\hat{D}(s) \neq \emptyset$. *If D is continuous at s then* $\hat{D}$ *is upper semi-continuous there.*

Proof. See [4], p 29.

REMARK. When (2) is satisfied we say that $\nless$ is closed.

COROLLARY 1. *If A is seprable metric, S is a topological space, D is continuous with non-empty values and* $\nless$ *is closed then there exists a measurable optimal decision for* $(S, A, D, <)$.

Proof. We only need to show that $\hat{D}$ is lower measurable. But $\hat{D}$ being upper semi-continuous is evidently upper measurable. Since A is perfectly normal upper measurability implies lower measurability. Q.E.D. ∎

Suppose hereafter that $\leqq_s$ is a preorder (i.e. a reflexive and transitive relation) on $D(s)$. We define $<_s$ by $a <_s a'$ iff $a \leqq_s a'$ and $a' \nleqq_s a$. a is said to be maximal if $a \leqq_s a'$ implies $a' \leqq_s a$. By a utility with respect to $\leqq_s$ we mean a real function u_s with domain containing $D(s)$ such that

$$a \leqq_s a' \Rightarrow u_s(a) \leqq u_s(a') \text{ and}$$

$$a <_s a' \Rightarrow u_s(a) < u_s(a').$$

Observe that when $\leqq_s$ is total (i.e. $a \leqq_s a'$ or $a' \leqq_s a$ for any pair $a, a' \in D(s)$) the requirements on u_s can be condensed to

$$a \leqq_s a' \Leftrightarrow u_s(a) \leqq u_s(a).$$

Also note that if u_s is a utility with respect to $\leqq_s$ and $u_s(a) = \sup u_s(D(s))$ then a is maximal. We would like to have the preference $\leqq_s$ represented by a well behaved utility because of the following

Proposition 2. *Let A be a separable metric space, S a measurable space and $u: S \times A \to R$ a function such that $u(s, \ )$ is continuous for each $s \in S$ and $u(\ , a)$ is measurable for each $a \in A$.*

If $D: S \to 2^A - \emptyset$ is upper measurable with compact values then there exist a measurable $d: S \to A$ such that $d(s) \in D(s)$ for all $s \in S$ and $\sup u(s, D(s)) = u(s, d(s))$.

Proof. $s \to u(s, D(s))$ is lower measurable by [5] Th. 6.5. Further $s \to \sup u(s, D(s))$ is measurable [O.C. Th. 6.6] and $\sup u(s, D(s)) \in u(s, D(s))$ since $u(s, \ )$ is continuous and $D(s)$ is compact. Then the conclusion follows from [5] Th. 7.1.

Because of this last proposition we turn to the problem of representing the preference by some utility function.

Definition 1. The decision problem $(S, A, D, \leqq)$ is said to be lower measurable iff

(i) A is compact and metric, – and

(ii) the correspondence $s \to U(s) = \{u_s \in C^*(A): u_s$ is a utility with respect to $\leqq_s\}$ is lower measurable with complete values. Here $C^*(A)$ denotes the set of bounded, continuous $f: A \to R$.

Theorem 3. *Let $(S, A, D, \leqq)$ be a lower measurable decision problem such that D has connected, closed, non-void values. Suppose $\leqq_s$ is a total preorder and $\{a \in D(s): a' \leqq_s a\}$ and $\{a' \in D(s): a \leqq_s a'\}$ are both closed for every $s \in S$. Then there exists a measurable optimal decision.*

Proof. By [2] there exists a continuous utility $u_s: D(s) \to R$ with respect to $\leqq_s$. Extend u_s continuously to A. So $U(s) \neq \emptyset$ for every $s \in S$ and U admits a measurable selection u. [5] Th. 5.6. Then an appeal to Proposition 5 completes the proof. ∎

When $\leqq_s$ is not total for all $s \in S$ we have

Theorem 4. *Let $(S, D, A, \leqq)$ be a lower measurable decision problem such that D has closed non-void values. Suppose $\leqq_s$ is closed as a subset of $D^2(s)$. Then there exists a measurable optimal decision.*

The proof is a duplicate of foregoing. We rely on the following definition and lemma.

Definition 2. When $\leq$ is a preorder on a set $\mathscr{A}$ we say that A^+ is increasing if $a \in A^+$, $a \leqq a'$ implies $a' \in A^+$. The complement of an increasing set is called decreasing. We say that the topological space $\mathscr{A}$ is normally preordered by $\leq$ if whenever A^+ closed, increasing and A^- closed, decreasing are disjoint there exist G^+ open, decreasing $\supseteq A^+$ and G^- open, decreasing $\supseteq A^-$ such that $G^+ \cap G^- = \emptyset$. [7], Ch. I, §2.

LEMMA 1. *Suppose $\mathscr{A}$ is locally compact, Hausdorff and normally preordered by $\leqq$ which is closed regarded as a subset of $\mathscr{A}^2$. If $\mathscr{A}^2 - \leqq$ is Lindelöf then $\leqq$ admits a representation by a continuous bounded utility u.*

Proof. When $a \nleq a'$ let V_a, $V_{a'}$ be disjoint, compact neighbourhoods of a and a' respectively such that $V_a \nleq V_{a'}$. Now $(V_a \times V_{a'})_{a \nleq a'}$ is a cover of $\mathscr{A}^2 - \leqq$ and has a countable subcover $(V_i \times V_{i'})_{i=1}^{\infty}$. Define

$$u_i = \begin{cases} 1 \text{ on } V_i, \\ 0 \text{ on } V_{i'}. \end{cases}$$

Then u_i is continuous and increasing on its compact domain $V_i \cup V_{i'}$. Extend u_i to the entire space so as to remain continuous and increasing [7]. Th. 1.6 with values in [0, 1].

Then $u = \sum_{i=1}^{\infty} \frac{u_i}{2^i}$ is a continuous, bounded utility. Q.E.D. ∎

Remarks

Measurable selection of extremal elements has been studied by Dolecki [3] in a Banach space setting with ordering relations compatible with the algebraic and topological structure of the space.

By specializing the state and action space we could also apply selection theorems proved in [9].

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Istnienie optymalnych decyzji

Podano warunki dostateczne istnienia decyzji optymalnych dla zadań jednoetapowych.

Существование оптимальных решений

Представленые досгаточные условия существования оптимальных решений для одно этапных задач.