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Duality theorems for constrained convex optimization

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We show that if F, X are two locally convex spaces, $\Gamma: X \to 2^F$ a convex multifunction, $h: F \to \bar{R}$ a convex functional and $x_0 \in X$, then, under suitable assumptions on Γ and h, the computation of $\inf_{y \in \Gamma(x_0)} h(y)$ can be reduced to the computation of the infimum of h on some larger convex subsets of F, defined with the aid of functionals $\Phi \in X^*$, $\Phi \neq 0$. We give some applications to the cases of linear systems and convex systems $(F \xrightarrow{u} X)$, studied in [8].

1. Introduction

We recall the following general problem of constrained optimization, which encompasses a large variety of special cases: Let X, F be two sets, $\Gamma: X \to 2^F$ a multifunction (where 2^F denotes the collection of all subsets of F, including the empty set $\emptyset$), $x_0 \in X$ and $h: F \to \bar{R} = [-\infty, +\infty]$ a functional. Find convenient formulae for

$$\inf_{y \in \Gamma(x_0)} h(y), \tag{1.1}$$

under suitable assumptions on X, F, Γ and h. Here Γ is also called a *constraints multifunction* (see [2]), since $\{\Gamma(x)\}_{x \in X}$ is a family of constraints; in particular, $\Gamma(x_0)$ is the constraint corresponding to the index $x_0 \in X$. Problem (1.1) has been studied by S. Dolecki and S. Kurcyusz [3] (see also [2] and the references therein), who have reduced it to a problem of unconstrained optimizataion, with the aid of an associated Lagrange function.

In the present paper we shall give some duality theorems of a different (non-Lagrangian) type, for the problem of *constrained convex optimization*, that is, for problem (1.1) in which it is assumed that X, F are (real) locally convex spaces and that the multifunction $\Gamma: X \to 2^F$ and the functional $h: F \to \bar{R}$ are convex. Let us recall that if X, F are two linear spaces, a multifunction $\Gamma: X \to 2^F$ is said to be *convex* (see e.g. [6]), if

$$\text{Graph } \Gamma = \{(x, y) \in X \times F \mid y \in \Gamma(x)\} \tag{1.2}$$

is a convex subset of $X \times F$, or, equivalently, if we have

$$\lambda \Gamma(x_1)+(1-\lambda)\,\Gamma(x_2) \subset \Gamma(\lambda x_1+(1-\lambda)\,x_2) \tag{1.3}$$

for all $x_1, x_2 \in X$ and all λ with $0 \leqslant \lambda \leqslant 1$. Note that if $\Gamma\colon X \to 2^F$ is convex, then $\Gamma(x)$ is a convex subset of F for each $x \in X$, since (1.3) for $x_1 = x_2 = x \in X$ and $0 \leqslant \lambda \leqslant 1$ yields

$$\lambda \Gamma(x)+(1-\lambda)\,\Gamma(x) \subset \Gamma(x). \tag{1.4}$$

Thus, since the set $\Gamma(x_0)$ is convex, it might seem natural to apply to problem (1.1) the known duality theorems for the minimization of the convex functional h on the convex subset $\Gamma(x_0)$ of F. For example, from a "strip theorem" of [8] (see [8], Theorem 2.2) it follows that if $\Gamma(x_0)$ and the (possibly empty) level sets $\{y \in F \mid h(y) \leqslant r\}$ $(r \in R)$ are closed for a topology τ on F, weaker than or equal to the initial topology on F, and either $\Gamma(x_0)$ or sets S_r $(r \in R)$ are compact for τ, then

$$\inf_{y \in \Gamma(x_0)} h(y) = \sup_{0 \neq \Psi \in F^*} \inf_{\substack{y \in F \\ \Psi(y) \in \Psi(\Gamma(x_0))}} h(y). \tag{1.5}$$

However, since this result of duality involves only functionals $\Psi \in F^*$ (the set of all continuous linear functionals on F) and since we want to obtain, similarly to the particular case of linear systems and convex systems (see [8]), formulae for (1.1) involving functionals $\Phi \in X^*$, we shall use a different approach. Namely, considering the whole family of optimization problems

$$\inf_{y \in \Gamma(x)} h(y) \quad (x \in X) \tag{1.6}$$

(with the usual conventions $\inf \emptyset = +\infty$, $\sup \emptyset = -\infty$, to be used throughout this paper), we shall apply, to the associated primal functional $f\colon X \to \bar{R}$, defined by

$$f(x) = \inf_{y \in \Gamma(x)} h(y) \quad (x \in X), \tag{1.7}$$

the following duality theorems of [7], [8] (let us observe that one could also apply other duality theorems, but here we shall consider only these):

THEOREM 1.1 ([7], Corollary 2.1). *Let X be a locally convex space, $f\colon X \to \bar{R}$ a lower semi-continuous convex functional and $x_0 \in X$. Then*

$$f(x_0) = \sup_{0 \neq \Phi \in X^*} \inf_{\substack{x \in X \\ \Phi(x) = \Phi(x_0)}} f(x). \tag{1.8}$$

THEOREM 1.2 ([8], Theorem 4.1). *Let X be a locally convex space, $f\colon X \to \bar{R}$ a convex unctional and $x_0 \in X$, such that the set*

$$A_{x_0} = \{x \in X \mid f(x) < f(x_0)\} \tag{1.9}$$

is non-empty and open. Then we have (1.8) *and there exists $\Phi_0 \in X^*$, $\Phi_0 \neq 0$, such that*

$$f(x_0) = \inf_{\substack{x \in X \\ \Phi_0(x) = \Phi_0(x_0)}} f(x) \tag{1.10}$$

(*i.e., for which the* sup *in* (1.8) *is attained*).

In §2 we shall give some sufficient conditions on Γ and h in order that the primal functional (1.7) satisfy the assumptions of Theorem 1.1 or Theorem 1.2.

In §3 we shall give the main duality theorems, which reduce the computation of (1.1) to the computation of the infimum of h on some larger convex subsets of F, defined with the aid of functionals $\Phi \in X^*$, $\Phi \neq 0$. We shall also show that these theorems admit natural geometric interpretations.

Finally, in §4 we shall observe that one can associate with the "linear systems" and with the "convex systems" $(F \xrightarrow{u} X)$, studied in [8], some natural convex multifunctions $\Gamma: X \to 2^F$, to which we shall then apply the results of §2 and §3.

2. Two lemmas on the primal functional

LEMMA 2.1. *Let F, X be two linear spaces, $\Gamma: X \to 2^F$ a convex multifunction and $h: F \to \bar{R} = [-\infty, +\infty]$ a convex functional. Then the functional $f: X \to \bar{R}$, defined by*

$$f(x) = \inf_{y \in \Gamma(x)} h(y) \quad (x \in X) \tag{2.1}$$

is convex.

Proof. Let $x_1, x_2 \in X$ and $0 \leq \lambda \leq 1$, and let $\varepsilon > 0$. Then, by (2.1), there exist $y_i \in \Gamma(x_i)$ such that $h(y_i) \leq f(x_i) + \varepsilon$ $(i = 1, 2)$. But then, since Γ is convex,

$$\lambda y_1 + (1-\lambda) y_2 \in \lambda \Gamma(x_1) + (1-\lambda) \Gamma(x_2) \subset \Gamma(\lambda x_1 + (1-\lambda) x_2),$$

whence, by (2.1) and since h is convex, we obtain

$$\begin{aligned} f(\lambda x_1 + (1-\lambda) x_2) &\leq h(\lambda y_1 + (1-\lambda) y_2) \\ &\leq \lambda h(y_1) + (1-\lambda) h(y_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2) + \varepsilon, \end{aligned}$$

which, since $\varepsilon > 0$ was arbitrary, proves that f is convex. This completes the proof of lemma 2.1. ■

In order to apply Theorem 1.1 to the functional $f: X \to \bar{R}$ defined by (2.1), we need to ensure that f is lower semi-continuous. Here we shall use only one lemma in this direction, giving a sufficient condition which is expressed in terms of simple and natural properties of Γ and h (for some other conditions ensuring the lower semi-continuity of f, which could be also used, see [4]). We recall that if X, F are two topological spaces, a multifunction $\Gamma: X \to 2^F$ is said to be *upper semi-continuous*, respectively *lower semi-continuous*, if the set

$$A_G = \{x \in X \mid \Gamma(x) \cap G \neq \emptyset\} \tag{2.2}$$

is closed for each closed subset G of F, respectively open for each open subset G of F.

LEMMA 2.2 ([1], Theorem 2.3). *Let F, X be two topological spaces, $\Gamma: X \to 2^F$ an upper semi-continuous multifunction and $h: F \to \bar{R}$ a lower semi-continuous functional. Then the functional $f: X \to \bar{R}$, defined by* (2.1), *is lower semi-continuous.*

Proof. Since in [1] there is given only a proof of a localized version of Lemma 2.2, let us give here, for the sake of completeness, a direct proof of the above global version. Let $r \in R$ and let $\{x_\delta\}_{\delta \in \Delta}$ be a generalized sequence in

$$S_r = \{x \in X \mid f(x) \leqslant r\}, \tag{2.3}$$

and assume that $x_\delta \to x_0$. Then, given $\varepsilon > 0$, by $x_\delta \in S_r$ and (2.1) there exist $y_\delta \in \Gamma(x_\delta)$ with $h(y_\delta) \leqslant r + \varepsilon$ $(\delta \in \Delta)$. Thus, $y_\delta \in \Gamma(x_\delta) \cap G_\varepsilon$ $(\delta \in \Delta)$, where

$$G_\varepsilon = \{y \in F \mid h(y) \leqslant r + \varepsilon\}. \tag{2.4}$$

But, since h is lower semi-continuous, $G_\varepsilon \subset F$ is closed and hence, by the upper semi-continuity of Γ, so is the set

$$A_{G_\varepsilon} = \{x \in X \mid \Gamma(x) \cap G_\varepsilon \neq \emptyset\}; \tag{2.5}$$

consequently, since $x_\delta \in A_{G_\varepsilon}$ $(\delta \in \Delta)$ and $x_\delta \to x_0$, it follows that $x_0 \in A_{G_\varepsilon}$. Thus, there exists $y_\varepsilon \in \Gamma(x_0) \cap G_\varepsilon$, that is, $y_\varepsilon \in \Gamma(x_0)$ with $h(y_\varepsilon) \leqslant r + \varepsilon$. Therefore, by (2.1), $f(x_0) \leqslant h(y_\varepsilon) \leqslant r + \varepsilon$, whence, since $\varepsilon > 0$ was arbitrary, $f(x_0) \leqslant r$, so S_r is closed. This completes the proof of Lemma 2.2. ■

3. Duality theorems

Along with a multifunction $\Gamma: X \to 2^F$, we shall also use the inverse multifunction $\Gamma^{-1}: F \to 2^X$, defined by

$$\Gamma^{-1}(y) = \{x \in X \mid y \in \Gamma(x)\} \quad (y \in F). \tag{3.1}$$

For the set A_G defined by (2.2), we have, clearly,

$$A_G = \bigcup_{g \in G} \{x \in X \mid g \in \Gamma(x)\} = \bigcup_{g \in G} \Gamma^{-1}(g) = \Gamma^{-1}(G).$$

THEOREM 3.1. *Let F, X be two locally convex spaces, $\Gamma: X \to 2^F$ an upper semi-continuous convex multifunction, $h: F \to \bar{R}$ a lower semi-continuous convex functional, and $x_0 \in X$. Then we have*

$$\inf_{y \in \Gamma(x_0)} h(y) = \sup_{0 \neq \Phi \in X^*} \inf_{\substack{y \in F \\ \Phi(x_0) \in \Phi(\Gamma^{-1}(y))}} h(y). \tag{3.2}$$

Proof. Define a functional $f: X \to \bar{R}$ by (2.1). Then, by Lemma 2.1 and Lemma 2.2, f is convex and lower semi-continuous. Hence, by Theorem 1.1, we obtain

$$\inf_{y \in \Gamma(x_0)} h(y) = f(x_0) = \sup_{0 \neq \Phi \in X^*} \inf_{\substack{x \in X \\ \Phi(x) = \Phi(x_0)}} f(x) = \sup_{0 \neq \Phi \in X^*} \inf_{\substack{x \in X \\ \Phi(x) = \Phi(x_0)}} \inf_{y \in \Gamma(x)} h(y). \tag{3.3}$$

We shall show that for each $\Phi \in X^*$ we have

$$\bigcup_{\substack{x \in X \\ \Phi(x) = \Phi(x_0)}} \Gamma(x) = \{y \in F \mid \Phi(x_0) \in \Phi(\Gamma^{-1}(y))\}, \tag{3.4}$$

which, together with (3.3), will yield (3.2), completing the proof. Indeed, if $x \in X$, $\Phi(x) = \Phi(x_0)$, $y \in \Gamma(x)$, then $\Phi(x_0) = \Phi(x) \in \Phi(\Gamma^{-1}(y))$. Conversely, if $y \in F$, $\Phi(x_0) \in \Phi(\Gamma^{-1}(y))$, then $\Phi(x_0) = \Phi(x')$ for some $x' \in \Gamma^{-1}(y)$, so $y \in \Gamma(x')$ and therefore y belongs to the left-hand side of (3.4), which proves (3.4). This completes the proof of Theorem 3.1. ∎

REMARK 3.1. a) Formula (3.2) admits the following geometric interpretation:

$$\inf_{y \in \Gamma(x_0)} h(y) = \sup_{\substack{H \in \mathscr{H} \\ x_0 \in H}} \inf_{\substack{y \in F \\ \Gamma^{-1}(y) \cap H \neq \emptyset}} h(y), \tag{3.5}$$

where $\mathscr{H}$ denotes the collection of all hyperplanes in X. Indeed, if $\Phi \in X^*$, $\Phi \neq 0$, then the hyperplane

$$H = H_{x_0, \Phi} = \{x \in X \mid \Phi(x) = \Phi(x_0)\} \tag{3.6}$$

contains x_0 and we have

$$\{y \in F \mid \Phi(x_0) \in \Phi(\Gamma^{-1}(y))\} = \{y \in F \mid \Gamma^{-1}(y) \cap H \neq \emptyset\}, \tag{3.7}$$

so (3.2) implies (3.5); conversely, if (3.5) holds, then, since every $H \in \mathscr{H}$ with $x_0 \in H$ is of the form (3.6) for some $\Phi \in X^*$, $\Phi \neq 0$, from (3.7) we obtain (3.2).

b) If $\Gamma: X \to 2^F$ is a convex multifunction (as in the case of Theorem 3.1), then *for each* $y \in F$ *and each* $\Phi \in X^*$, $\Phi \neq 0$, *the sets* $\Gamma^{-1}(y)$ *and* (3.7) *are convex*. Indeed, if $x_1, x_2 \in \Gamma^{-1}(y)$ and $0 \leqslant \lambda \leqslant 1$, then, by (1.3),

$$y = \lambda y + (1-\lambda) y \in \lambda \Gamma(x_1) + (1-\lambda) \Gamma(x_2) \subset \Gamma(\lambda x_1 + (1-\lambda) x_2),$$

so $\lambda x_1 + (1-\lambda) x_2 \in \Gamma^{-1}(y)$. On the other hand, if $\Gamma^{-1}(y_i) \cap H \neq \emptyset$, say $x_i \in \Gamma^{-1}(y_i) \cap$ $\cap H$ $(i = 1, 2)$ and if $0 \leqslant \lambda \leqslant 1$, then, by (1.3),

$$\lambda y_1 + (1-\lambda) y_2 \in \lambda \Gamma(x_1) + (1-\lambda) \Gamma(x_2) \subset \Gamma(\lambda x_1 + (1-\lambda) x_2),$$

so $\lambda x_1 + (1-\lambda) x_2 \in \Gamma^{-1}(\lambda y_1 + (1-\lambda) y_2)$ and, clearly, $\lambda x_1 + (1-\lambda) x_2 \in H$; therefore, $\Gamma^{-1}(\lambda y_1 + (1-\lambda) y_2) \cap H \neq \emptyset$.

THEOREM 3.2. *Let F, X be two locally convex spaces, $\Gamma: X \to 2^F$ a lower semi-continuous convex multifunction, satisfying $\bigcup_{x \in X} \Gamma(x) = F$, h a finite and continuous convex functional on F and $x_0 \in X$ such that*

$$\inf_{y \in F} h(y) < \inf_{y \in \Gamma(x_0)} h(y). \tag{3.8}$$

Then we have (3.2) *and there exists $\Phi_0 \in X^*$, $\Phi_0 \neq 0$, such that*

$$\inf_{y \in \Gamma(x_0)} h(y) = \inf_{\substack{y \in F \\ \Phi_0(x_0) \in \Phi_0(\Gamma^{-1}(y))}} h(y) \tag{3.9}$$

(*i.e., for which the* sup *in* (3.2) *is attained*).

Proof. Define a functional $f: X \to \bar{R}$ by (2.1). Then, by (3.8) and since h is finite and continuous, the set

$$N_0 = \{y \in F \mid h(y) < f(x_0)\} \tag{3.10}$$

is non-empty and open. Hence, by $\bigcup_{x \in X} \Gamma(x) = F$ and since Γ is lower semi-continuous, the set

$$A_{N_0} = \{x \in X \mid \Gamma(x) \cap N_0 \neq \emptyset\} \tag{3.11}$$

is non-empty and open We claim that

$$A_{N_0} = \{x \in X \mid f(x) < f(x_0)\}. \tag{3.12}$$

Indeed, if $x \in A_{N_0}$, then, by (3.11) and (3.10), there exists $y \in \Gamma(x)$ such that $h(y) < f(x_0)$, whence $f(x) = \inf_{y' \in \Gamma(x)} h(y') \leqslant h(y) < f(x_0)$. Conversely, if $x \in X$, $f(x) = \inf_{y' \in \Gamma(x)} h(y') < f(x_0)$, then there exists $y \in \Gamma(x)$ such that $h(y) < f(x_0)$, whence $y \in \Gamma(x) \cap N_0 \neq \emptyset$, so $x \in A_{N_0}$, which proves the claim (3.12). Thus, the set $\{x \in X \mid f(x) < f(x_0)\}$ is non-empty and open, whence, by Theorem 1.2 and (3.4), there exists $\Phi_0 \in X^*$, $\Phi_0 \neq 0$, such that

$$\inf_{y \in \Gamma(x_0)} h(y) = f(x_0) = \inf_{\substack{x \in X \\ \Phi_0(x) = \Phi_0(x_0)}} f(x) = \inf_{\substack{x \in X \\ \Phi_0(x) = \Phi_0(x_0)}} \inf_{y \in \Gamma(x)} h(y) =$$

$$= \inf_{\substack{y \in F \\ \Phi_0(x_0) \in \Phi_0(\Gamma^{-1}(y))}} h(y). \tag{3.13}$$

Thus, (3.9) holds, which, together with the obvious inequality $\geqslant$ in (3.2), yields (3.2). This completes the proof of Theorem 3.2. ■

REMARK 3.2. The conclusion of Theorem 3.2 admits the following geometric interpretation: There exists a hyperplane H_0 in X such that $x_0 \in H_0$ and that

$$\inf_{y \in \Gamma(x_0)} h(y) = \inf_{\substack{y \in F \\ \Gamma^{-1}(y) \cap H_0 \neq \varnothing}} h(y). \tag{3.14}$$

4. The cases of linear systems and convex systems

We recall (see [8]) that a triple $(F \xrightarrow{u} X)$ consisting of two (real) locally convex spaces F, X and a continuous linear mapping u of F into X is called a *linear system* (this generalizes the terminology of [5], where it has been assumed that X, F are Banach spaces). Similarly (see [8]), a triple $(F \xrightarrow{u} X)$ consisting of a locally convex space F, a partially ordered locally convex space X and a convex mapping u of F into X (i.e., such that $u(\lambda y_1 + (1-\lambda) y_2) \leqslant \lambda u(y_1) + (1-\lambda) u(y_2)$ for all $y_1, y_2 \in F$ and all λ with $0 \leqslant \lambda \leqslant 1$), is called a *convex system.*

REMARK 4.1. *If $(F \xrightarrow{u} X)$ is a linear system and Ω a convex subset of X or if $(F \xrightarrow{u} X)$ is a convex system and $\Omega = \{x \in X \mid x \leqslant 0\}$, the convex cone of all non-positive elements in X, then the multifunction $\Gamma: X \to 2^F$ defined by*

$$\Gamma(x) = u^{-1}(x + \Omega) = \{y \in F \mid u(y) \in x + \Omega\} \quad (x \in X) \tag{4.1}$$

is convex. Indeed, if $(F \xrightarrow{u} X)$ is a linear system and $\Omega \subset X$ is convex, then for any $x_1, x_2 \in X$ and $0 \leqslant \lambda \leqslant 1$ we have

$$\begin{aligned} &\lambda \Gamma(x_1) + (1-\lambda)\Gamma(x_2) \\ &\quad = \{\lambda y_1 + (1-\lambda) y_2 | y_i \in F,\, u(y_i) \in x_i + \Omega \ (i=1,2)\} \\ &\quad \subset \{y \in F | u(y) \in \lambda x_1 + (1-\lambda) x_2 + \Omega\} \\ &\quad = \Gamma(\lambda x_1 + (1-\lambda) x_2). \end{aligned}$$

On the other hand, if $(F \xrightarrow{u} X)$ is a convex system and $\Omega = \{x \in X | x \leqslant 0\}$, then

$$\begin{aligned} &\lambda \Gamma(x_1) + (1-\lambda)\Gamma(x_2) \\ &\quad = \{\lambda y_1 + (1-\lambda) y_2 | y_i \in F,\, u(y_i) \leqslant x_i \quad (i=1,2)\} \subset \\ &\quad \subset \{y \in F | u(y) \leqslant \lambda x_1 + (1-\lambda) x_2\} = \Gamma(\lambda x_1 + (1-\lambda) x_2), \end{aligned}$$

which proves our assertions. Thus, we can apply the results of §3, to the situation of Remark 4.1. Note that for the multifunction $\Gamma : X \to 2^F$ defined in Remark 4.1, the optimization problem (1.1) becomes

$$\inf_{\substack{y \in F \\ u(y) \in x_0 + \Omega}} h(y) \tag{4.2}$$

if $(F \xrightarrow{u} X)$ is a linear system and Ω a convex subset of X, respectively

$$\inf_{\substack{y \in F \\ u(y) \leqslant x_0}} h(y) \tag{4.3}$$

if $(F \xrightarrow{u} X)$ is a convex system and $\Omega = \{x \in X | x \leqslant 0\}$. In these cases it is convenient to assume that $x_0 = 0$ in (4.2) and (4.3) (see [8]).

REMARK 4.2. *In the cases considered in Remark* 4.1, *we have*

$$\Gamma^{-1}(y) = u(y) - \Omega \quad (y \in F), \tag{4.4}$$

whence

$$\{y \in F |\ \Phi(x_0) \in \Phi(\Gamma^{-1}(y))\} = \{y \in F | \Phi(u(y)) \in \Phi(x_0 + \Omega)\}. \tag{4.5}$$

Indeed,

$$\Gamma^{-1}(y) = \{x \in X |\ y \in u^{-1}(x + \Omega)\} = \{x \in X |\ u(y) \in x + \Omega\} = u(y) - \Omega \quad (y \in F),$$

so (4.4) holds. Consequently,

$$\begin{aligned} \{y \in F |\ \Phi(x_0) \in \Phi(\Gamma^{-1}(y))\} &= \{y \in F |\ \Phi(x_0) \in \Phi(u(y)) - \Phi(\Omega)\} \\ &= \{y \in F |\ \Phi(u(y)) \in \Phi(x_0 + \Omega)\}, \end{aligned}$$

so we have (4.5). In the case when $(F \xrightarrow{u} X)$ is a linear system and $\Omega \subset X$ is convex, (4.5) yields

$$\{y \in F |\ \Phi(x_0) \in \Phi(\Gamma^{-1}(y))\} = \{y \in F |\ u^*(\Phi)(y) \in \Phi(x_0 + \Omega)\}, \tag{4.6}$$

which is a "strip" in F (indeed, since Ω is convex, $\Phi(x_0+\Omega)$ is an interval in R, finite or infinite, closed or open at either end). On the other hand, if $(F \xrightarrow{u} X)$ is a convex system, $\Omega=\{x\in X|x\leqslant 0\}$ and $\Phi\geqslant 0$, then (4.5) becomes

$$\{y\in F|\Phi(x_0)\in\Phi(\Gamma^{-1}(y))\}=\{y\in F|\Phi(u(y))\leqslant\Phi(x_0)\}. \tag{4.7}$$

Thus, in both cases, the conclusions of Theorems 3.1 and 3.2 above, with $x_0=0$, are the same as those of the corresponding results in [8].

REMARK 4.3. *In the cases considered in Remark* 4.1, *for any subset* G *of* F *and for the set* A_G *defined by* (2.2) *we have*

$$A_G=u(G)-\Omega. \tag{4.8}$$

Indeed,

$$A_G=\{x\in X|\Gamma(x)\cap G\neq\emptyset\}$$
$$=\{x\in X|\exists y\in G, u(y)\in x+\Omega\}=u(G)-\Omega,$$

so (4.8) holds. Consequently, Γ *is upper semi-continuous, respectively, lower semi-continuous, if and only if* $u(G)-\Omega$ *is closed for each closed subset* G *of* F, *respectively open for each open subset* G *of* F. Let us observe that *if* $u(G)$ *and* Ω *are closed for a topology* τ *on* X *weaker than or equal to the initial topology on* X, *and if one of them is compact for* τ, *then* $u(G)-\Omega$ *is closed for* τ *and hence also for the initial topology on* X. In [8], Lemma 2.2, we have shown that these conditions, only for the sets $G=G_r=\{y\in F|h(y)\leqslant r\}$ $(r\geqslant\inf_{y\in F}h(y))$ and Ω, are already sufficient to ensure that the primal functional f defined by (2.1) is lower semi-continuous, so the conclusion of Theorem 3.1 above holds; moreover, in [8], Lemma 2.2, it has not been assumed a priori that h is lower semi-continuous, i.e., that the sets G_r are closed. Similarly, let us observe that *if either* $u(G)$ *or* Ω *is open, then so is* $u(G)-\Omega$ $=\bigcup_{\omega\in\Omega}(u(G)-\omega)=\bigcup_{g\in G}(u(g)-\Omega)$. In [8], Lemma 4.1, we have shown that this condition, only for the set $G=G_a=\{g\in F|h(g)<\inf_{\substack{y\in F\\ u(y)\in\Omega}}h(y)=a\}$ (provided that $G_a\neq\emptyset$) or Ω, is already sufficient to ensure that for the primal functional f the set A_0 defined by (1.9) with $x_0=0$ is non-empty and open, so in this case the conclusion of Theorem 3.2 above holds, with $x_0=0$ (indeed, $\bigcup_{x\in X}\Gamma(x)=\bigcup_{x\in X}u^{-1}(x+\Omega)=F$, since for any $y\in F$ and any $\omega_0\in\Omega$ we have $y\in u^{-1}(u(y)-\omega_0+\Omega)$); moreover, in [8], Lemma 4.1, it has not been assumed a priori that h is finite and continuous, or that the set G_a is open. Finally, let us note that in the particular case when $F=X$ and $u=I_F$, the identity mapping, (4.1) becomes

$$\Gamma(x)=x+\Omega \quad (x\in X=F), \tag{4.9}$$

and $A_G=u(G)-\Omega=G-\Omega$ is open for each open subset G of F, so Γ is lower semi-continuous and Theorem 3.2 works; if Ω is compact, then Γ is also upper semi-continuous, so Theorem 3.1 applies as well.

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Twierdzenia o dualności dla wypukłych zadań optymalizacji z ograniczeniami

Niech F i X będą dwiema lokalnie wypukłymi przestrzeniami, $\Gamma: X \to 2^F$— wypukłą multifunkcją, a $h: F \to \bar{R}$—funkcjonałem wypukłym i niech $x_0 \in X$.

Pokazuje się, że przy odpowiednich założeniach na Γ i h wyznaczenie $\inf\limits_{y \in \Gamma(x_0)} h(y)$ może być sprowadzone do wyznaczenia infimum h na pewnym większym podzbiorze wypukłym przestrzeni F, który jest zdefiniowany przy wykorzystaniu funkcjonałów $\Phi \in X^*$, $\Phi \neq 0$. Omówione są pewne zastosowania do układów liniowych oraz układów wypukłych $(F \xrightarrow{u} X)$ badanych w [8].

Теоремы о двойственности для выпуклых задач оптимизации с ограничениями

Пусть F и X будут двумя локально выпуклыми пространствами, $\Gamma: X \to 2^F$—выпуклой мультифункцией, а $h: F \to \bar{R}$—выпуклым функционалом и пусть $x_0 \in X$. Показано, что при надлежащих придположениях о Γ и h вычисение инфитута $h(y)$, $(y \in \Gamma(x_0))$, можно свести к вычислению инфитута h на некотором большом выпуклом подмножестве пространства F, которое определяется с помощью функционалов $\Phi \in X^*$, $\Phi \neq 0$. Рассматриваются некоторые применения для случая линейных и выпуклых систем $(F \xrightarrow{u} X)$ исследуемых в [8