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On minimally interconnected subnetworks of a network

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A connection between terminal capacities and minimal subsets of a line weighted graph is presented. The connection reduces the computing work of minimal subsets when the terminal capacities are known. It is also shown that the minimal subsets constitute a hierarchical clustering of the points of a line weighted graph. Thus the determination problem here is a part of a general problem: the clustering of the points of a graph.

1. Minimal cutsets and terminal capacities

Let $G=(V, E)$ be a connected undirected graph without loops and multiple lines. Such a graph with weights on lines and points is called a network. In the following we shall consider graphs with nonnegative weights w on lines only. Let $A, B \subset V$, $A \cap B = \emptyset$ and w_{ij} be the weight of the line joining the points i and j. $f(A, B)$ is a brief notation for the sum $\sum_{i,j} w_{ij}$, where $i \in A$ and $j \in B$. A nonempty subset $S \subset V$ is called minimal if for every nonempty susbset R of S, $R \neq S$, it holds:

$$f(R, V \setminus R) > f(S, V \setminus S).$$

Moreover, each point of G constitute a minimal subset of G.

Kacprzyk and Stańczak have derived a great number of properties of minimal subsets (called groups) of graphs in their papers [1] and [2]. They have also developed further the algorithm of Luccio and Sami [4] for enumerating all the minimal subsets of a given graph G. The purpose here is to show how the enumeration work can be reduced when it is known the terminal capacity matrix M_{Gt} of G and a path (a linear tree) realizing that matirx.

As well known (see e.g. the monograph [3] of Kim and Chien), the terminal capacity t_{ij} between the points i and j equals the minimum sum of the weights of the lines in a line cut separating the points i and j in G. Such a set is called a minimum cutset separating i and j, denoted by $C(i,j)$, and the corresponding weight is $w(C(i,j))$. t_{ii} is defined to be ∞. Thus the terminal capacities can be collected into a matrix $M_{Gt}=[t_{ij}]$, the terminal capacity matrix of G. If $[t_{ij}]$ is the terminal capacity matrix of a graph G, there exist also trees having $[t_{ij}]$ as their terminal capacity matrix [3; Part V, Thm. 14 and Cor. 5]: in fact, there is a path realizing $[t_{ij}]$.

THEOREM 1. *Let $G=(V,E)$ be a given graph and $M_{Gt}=[t_{ij}]$ its terminal capacity matrix. If S is a minimal subset of G, then S is a minimal subset of each path realizing M_{Gt}.*

Proof. We shall show that the points of S constitute a subpath of each path P_{Gt} realizing M_{Gt}. Then there can be at most two lines joining the points of $V\setminus S$ to the points of S in a P_{Gt}. The weights of these lines in a P_{Gt} are of interest when proving S to be a minimal subset in P_{Gt}. We concentrate first on proving S to be a subpath of P_{Gt}.

If the lines between S and $V\setminus S$ constitute a minimal cutset between a point i from S and j from $V\setminus S$, there is nothing to prove. So we assume that the lines between S and $V\setminus S$ do not constitute a minimal cutset for any pair i, j of points, $i\in S$ and $j\in V\setminus S$.

Let $i\in S$ and $j\in V\setminus S$, and let the minimal cutset separating i and j contain lines joining a point of S to another point of S (see Figure 1, where the symbols $a, b, \ldots, g$ denote weights on lines). In Figure 1 we see that the cutset separating

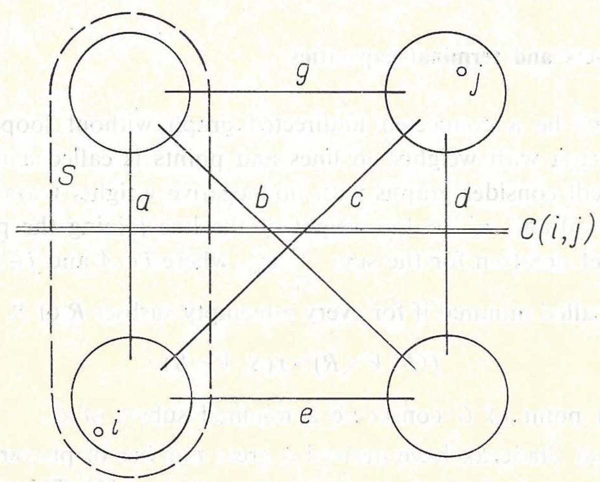

Fig. 1

i and j has the weight $a+b+c+d$. According to the properties of minimal sets [2, Lemma1], $f(R, S\setminus R)>f(S\setminus R, V\setminus S)$ for each $R\subset S$, $R\neq S$. Thus, according to the symmetry, $a>g+b$ and $a>c+e$. But then, because the weights are nonnegative, $a+b+c+d>g+c+d$. Thus the minimal cutset separating i and j, $i\in S$ and $j\in V\setminus S$, does not contain lines joining two points of S.

As well known, a minimal cutset divides G into two connected nonempty subgraphs. When cutting G we proceed as follows: Let a minimal cutset separating $i \in S$ and $j \in V \setminus S$ divide G into two connected subgraphs $G(i)$ and $G(j)$. As shown above, the points $V(j)$ of $G(j)$ belong to $V \setminus S$. We choose i and j such that the number $|V(j)|$ of points in $V(j)$ is maximal (i.e. for other pairs s, k of points, $s \in S$ and $k \in V \setminus S$, it holds: $|V(j)| \geqq |V(k)|$). After choosing such a point j we denote it by j_1 and choose another $j = j_2$ according to the same criterion from the set $V \setminus S \setminus V(j_1)$; $i \in S$. Because a minimal cutset separates the points $V(j_1)$ from other points of G, $V(j_1) \subset V(j_2)$ or $V(j_1) \cap V(j_2) = \varnothing$ [3, Part V, Cor. 6]. The case $V(j_1) \subset V(j_2)$ is impossible, because then $|V(j_1)|$ were not maximal as the criterion requires. Thus $V(j_1) \cap V(j_2) = \varnothing$ and as above, $V(j_2) \subset V \setminus S$. We continue the choosing of points $j_1, \dots, j_m$ until $V \setminus S \setminus V(j_1) \setminus \dots \setminus V(j_m) = \varnothing$; assume that $V \setminus S \setminus V(j_1) \setminus \dots \setminus V(j_m) = \varnothing$ but $V \setminus S \setminus V(j_1) \setminus \dots \setminus V(j_{m-1}) \neq \varnothing$.

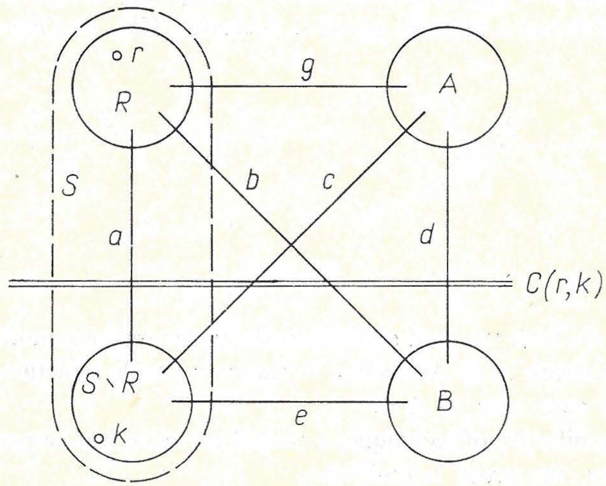

Fig. 2

Let us separate now two points k and r of S by a minimal cutset. Such a cutset separates G into two disjoint connected subgraphs $G(r)$ and $G(k)$ with pointsets $V(r)$ and $V(k)$, respectively. Because minimal cutsets separate the sets $V(j_1)$, $V(j_2)$, ..., $V(j_m)$ from G, then either $V(j_h) \subset V(r)$ or $V(j_h) \cap V(r) = \varnothing$ (and thus either $V(j_h) \cap V(k) = \varnothing$ or $V(j_h) \subset V(k)$ because $V(r) \cap V(k) = \varnothing$ and $V(r) \cup V(k) = V$), $h = 1, \dots, m$. This minimal cutset $C(r, k)$ may so contain lines joining a point of S to a point of $V \setminus S$ as well as lines joining two points of $V \setminus S$. We denote by R the set $S \cap V(r)$ and by $S \setminus R$ the set $S \cap V(k)$. Accordingly, the weight $w\big(C(r, k)\big) = t_{rk}$ of the cutset satisfies the inequality: $w\big(C(r, k)\big) \geqq f(R, S \setminus R)$. The separation of the points of S can now be continued in $R = V(r) \cap S$ and in $S \setminus R = V(k) \cap S$ until all terminal capacities t_{rk} of pairs, $r, k \in S$ are determined.

Let t_{rk} be the least terminal capacity between two points r and k of S. Then, according to the properties of S, $a = f(R, S \setminus R) > g + b$, $c + e$ (see Figure 2). On the other hand, $t_{rk} = a + b + c + d > g + c + d$, $b + d + e$. Moreover, for each $w\big(C(i, j_h)\big)$,

$j_h \in A$, $g+c+d \geq w(C(i, j_h))$, and similarly, $d+b+e \geq w(C(i, j_h))$ for each $w(C(i, j_h))$, $j_h \in B$, because the line sets $C(i, j_h)$ are minimal cutsets separating a point $i \in S$ from a point $j_h \in V \setminus S$. But then also the least $t_{rk} \geq w(C(i, j_h))$, $h=1, ..., m$, and so the points of S constitute a subpath of each path realizing M_{Gt}.

As shown above, every point of $V \setminus S$ belongs to one of the pointdisjoint sets $V(j_1), ..., V(j_m)$ separated from G by a minimal cutset $C(i, j_h)$, $i \in S$ and $j_h \in V \setminus S$, $h=1, ..., m$. But then the points of $V \setminus S$ are joined in every path realization P_{Gt} of M_{Gt} to the subpath of S by a line having the capacity of at most $w(C(i, j_h))$, $i \in S$ and $j_h \in V \setminus S$, $h=1, ..., m$. Because the least capacity of a line in the subpath realizing the points of S is properly greater than the capacities of the lines joining the points of S to the points of $V \setminus S$ in every path realization P_{Gt}, S is a minimal set in every P_{Gt} of M_{Gt}. This completes the proof. ■

The converse does not hold. This can be seen from the graphs of Figure 3, where G is the given tree and P_{Gt} a path realization of M_{Gt}. The set $\{a, b, c\}$ is a minimal set in P_{Gt}, but it is not a minimal set of G, because $f(\{b\}, V \setminus \{b\}) = 4 = f(\{a, b, c\},$

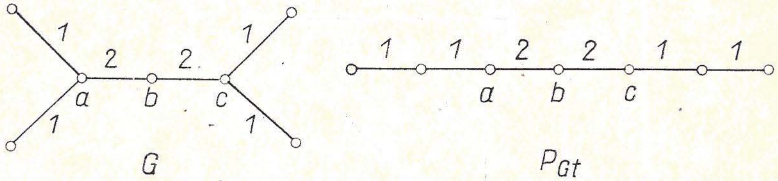

Fig. 3

$V \setminus \{a, b, c\})$ in G. Thus the path realization shows the sets, the minimality of which is reasonable to check in G. According to the very simple structure of paths, their minimal sets is very easy to find.

By using a similar proof technique, a partial converse can be proved; according to the similarity, the proof is omitted.

THEOREM 2. *If S is a minimal set in every tree realization T_{Gt} of the terminal capacity matrix M_{Gt} of a given graph, then S is a minimal set in G.*

By denoting in Figure 3 the path P_{Gt} as a given graph G^* and G as a tree realization T_{G^*t} of the matrix M_{G^*t} one sees that the converse of Theorem 2 does not hold: $\{a, b, c\}$ is a minimal set of the given graph G^* but it is not a minimal set in every tree realization of M_{G^*t}.

2. A new interpretation

We like to point out a character not noted before when considering minimal sets of a graph: The minimality property induce a hierarchical clustering of the points of G.

Let i and j be two points of G. We define a binary operation t on V as follows:

$$t(i, j) = \{k \mid \text{the terminal capacities } t_{ik}, t_{jk} \geq t_{ij}\}.$$

Moreover, we define that $t(i, i)=\{i\}$ for every point $i \in V$; this is consistent with the definition of $t(i, j)$ if the graph G in question contains lines with finite weights only. The definition implies that $i, j \in t(i, j)$. When I and J are two nonempty subsets of V

$$t(I, J)=\{k \mid k \in t(i, j) \text{ for some pair } i \text{ and } j,\ i \in I \text{ and } j \in J\}.$$

A subset I is t-closed when $t(I, I)=I$.

According to the definition every point of G is t-closed and trivially, $t(V, V)=V$, because $t(i, i)=\{i\}$ and $i \in V$. The definition implies also that when I and J are t-closed subsets of V and $I \cap J=K \neq \varnothing$, then K is t-closed. But then there is for any two t-closed subsets I and J of V a least t-closed subset containing I and J; this is denoted by $I \vee J$ and it is $\bigcap \{M \mid t(M, M)=M \text{ and } I, J \subset M\}$. Thus the t-closed subsets of G constitute a joinsemilattice H.

LEMMA 1. *Each minimal set S of a graph G is t-closed.*

Proof. We should show that $t(S, S)=S$. Because $t(i, i)=\{i\}$ for each $i \in S$, $S \subset t(S, S)$, and thus it remains to prove that $t(S, S) \subset S$, i.e. if $i, j \in S$ then $t(i, j) \subset S$.

Because S is a minimal set of G, it is a minimal set in every path realization P_{Gt} of M_{Gt}. Let $k \in V \setminus S$. In a path realization P_{Gt} the points of S are joined by at most two lines to the points of $V \setminus S$; let the weights of these lines be t_a and t_b. Because $k \in V \setminus S$, $t_{ik}, t_{jk} \leq \max\{t_a, t_b\}$. As shown in the proof of Theorem 1, $t_{ij} > t_a, t_b$ for any two points $i, j \in S$. Thus $V \setminus S \cap t(i, j)=\varnothing$ and so $t(i, j) \subset S$. This completes the proof.

Kacprzyk and Stańczak [1] proved that if S_1 and S_2 are two minimal sets of a graph G, then $S_1 \cap S_2=\varnothing$ or one of the sets contains another. But then the minimal sets of G constitute a substructure $T(S)$ of the joinsemilattice H of t-closed subsets in G, which is a tree (two elements have a common lower bound only if one contains another). This tree $T(S)$ is a hierarchical clustering of the points of G, because the greatest element of $T(S)$ is V and every point i of V belongs to the lowest level of $T(S)$. The points of a set $R \subset V$ constitute a cluster whenever R is a minimal set of G.

As a final observation we prove another lemma which shows that two t-closed subsets have a common lower bound only if one contains another. Accordingly H is a tree and it determines also a hierarchical clustering of the points of G. $T(S)$ is usually a proper substructure of H: e.g. in the graph G of Figure 3, the set $\{a, b, c\}$ is t-closed but not minimal. On the other hand, as the definition of t-closed subsets shows (they are determined by terminal capacities), all t-closed subsets can easily be determined by any path P_{Gt} realizing the capacity matrix of the given graph G. Thus there must always be good reasons for choosing the hierarchical clustering of V determined by minimal sets and not that determined by t-closed subsets because t-closed subsets can be detected easily and directly by means of termial capacities but the minimal sets not.

LEMMA 2. *Let I and J be two t-closed subsets of a graph G such that $I \cap J \neq \varnothing$. Then either $I \subset J$ or $J \subset I$.*

Proof. The terminal capacity matrix M_{Gt} of G has a path realization P_{Gt}. In this P_{Gt} any two points i and j are joined by a path; let it be $i=h_0, h_1, h_2, ..., h_m, h_{m+1}=j$. Because the path is unique, $t_{ij}=\min\{t_{h_p h_{p+1}} | p=0, ..., m\}$.

Let $c \in I \cap J$. If $I=\{c\}$ or $J=\{c\}$, there is nothing to prove, whence we assume that $I, J \neq \{c\}$. There is a unique path of P_{Gt} joining two outermost points of I and every point of I is on this path. Thus there is a point $i^* \in I$ such that $t_{ci^*} = \min\{t_{is} | i, s \in I\}$; similarly, there is a point $j^* \in J$ such that $t_{cj^*} = \min\{t_{jr} | j, r \in J\}$. Because t_{cj^*} and t_{ci^*} are real numbers, $t_{ci^*} \leqslant t_{cj^*}$ (or $t_{cj^*} \leqslant t_{ci^*}$). But then $t(c, i^*) = \{k | k \in V$ and $t_{ki^*}, t_{kc} \geqslant t_{ci^*}\}$ contains every point k from I and every point k from J, and thus $J \subset t(c, i^*)$. Because $t(I, I)=I$, $t(c, i^*) \subset I$ and consequently, $J \subset I$. If $t_{cj^*} \leqslant t_{ci^*}$, $I \subset J$. This completes the proof. ■

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O zespołach minimalnych sieci

Przedstawia się zależność między pojemnościami końcowymi i zespołami minimalnymi liniowego grafu ważonego. Zależność ta zmniejsza nakład obliczeń przy wyznaczaniu zespołów minimalnych, gdy znane są pojemności końcowe. Pokazuje się także, że zespoły minimalne stanowią hierarchiczne zgrupowanie punktów liniowego grafu ważonego. A zatem, zagadnienie wyznaczania zespołów minimalnych jest częścią ogólnego problemu grupowania punktów grafu.

О минимальных комплексах сетей

Представлена зависимость между конечными ёмкостями и минимальными комплексами линейного взвешенного графа.

Эта зависимость снижает количество вычислений при определении минимальных комплексов, когда известны конечные ёмкости. Показано также, что минимальные комплексы являются иерархической группой точек линейного взвешенного графа. Таким образом, задача определения минимальных комплексов является частью общей проблемы группирования точек графа.