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Numerical solution of the c-observation problem for linear nonstationary systems

by

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The practical effective method is developed to solve the c-observation problem for the linear nonstationary systems.

This method requires that only auxiliary systems of differential equations with known initial conditions and system of the algebraic equations are solved for determining the current state of the considered systems.

The block diagram realization of the indirect c-observer for the unforced and forced linear nonstationary systems has been presented.

1. Introduction

When stabilizing controlled systems, or designing optimal feedback controllers, etc., it is necessary to have sufficiently complete information on the current state of the systems in the phase space. In many control situations, however, direct measurements of some state coordinates are difficult and, sometimes, even impossible. In these cases, the problem concerned with determining the current state vector from a complete knowledge of the system's input and output history on some finite time interval becomes important to be studied.

In what follows we shall refer to this problem as the c-observation problem.

The following definitions are used in sequel.

Definition 1. *A dynamical system which permits the reconstruction of the current state vector from a complete knowledge of the system's input and output history on some finite time interval is called c-observable.*

Definition 2. *A computing system which performs the calculation of the current state vector from a complete knowledge of the object's input and output history on some finite time interval is referred to as the indirect c-observer.*

Kalman first has considered the problem of determining the conditions which a linear dynamical system must satisfy in order that it be c-observable [1]. Assuming that the system's input is zero and that a complete knowledge of the system's output on some finite interval is available, he has obtained necessary and sufficient conditions of c-observability for both continuous and discrete time linear dynamical systems.

Kalman investigations were further developed and extended by Gilbert [2], Krasovskii [3], and other authors [4, 5].

In the present paper, the practical effective method is proposed to solve the c-observation problem for the linear nonstationary systems.

The effectiveness and practicality of the method lies in the fact that only auxiliary systems of differential equations with known initial conditions and system of algebraic equations are solved for determining the current state vector of the considered system from a complete knowledge of the system's input and output history on some finite time interval.

The block diagram realization of the indirect c-observer for the unforced and forced linear nonstationary systems has been presented.

2. Numerical solution of the c-observation problem for the unforced linear nonstationary systems

2.1. Problem statement. Consider first a system described by linear differential equations in the following matrix form

$$\frac{dx}{dt}=A(t)\,x \tag{2.1}$$

where x is an n-dimensional vector representing the phase state of the considered system at time t, $A(t)$ is a known $(n\times n)$-dimensional matrix.

Let us assume that the phase state $x(t)$ is inaccessible to direct observation, only an m-dimensional output vector

$$z(t)=Q(t)\,x(t), \quad \vartheta-h\leqslant t\leqslant\vartheta, \tag{2.2}$$

is accessible to noise-free measurement, where $m<n$, $Q(t)$ is a known $(m\times n)$-dimensional matrix.

Then, the c-observation problem for the unforced linear system (2.1) consists in the following: it is required to find the unknown state vector $x(t)$ at the present time $t=\vartheta$ from a complete knowledge of the output vector $z(t)$ on the finite time interval $\vartheta-h\leqslant t\leqslant\vartheta$ for some $h>0$, where $\vartheta-h$ is some past time $(\vartheta-h<\vartheta)$ because, since $m<n$, equation (2.2) does not allow immediate finding of $x(t)$ from $z(t)$.

2.2. Solution technique. It is well known that the solution of differential equation (2.1) can be presented in the following form

$$x(t)=X(t)\,X^{-1}(\vartheta)\,x(\vartheta), \tag{2.3}$$

where $X(t)$ is an $(n\times n)$-dimensional fundamental matrix and $X^{-1}(\cdot)$ is an inverse with respect to the matrix $X(\cdot)$.

Substituting (2.3) into (2.2) we have

$$z(t)=Q(t)\,X(t)\,X^{-1}(\vartheta)\,x(\vartheta). \tag{2.4}$$

Multiply (2.4) by $[X^{-1}(\vartheta)]' X'(t) Q'(t)$ from the left and integrate from $\vartheta-h$ to ϑ to obtain

$$G_h(\vartheta)\, x(\vartheta) = \int_{\vartheta-h}^{\vartheta} [X^{-1}(\vartheta)' X'(t) Q'(t) z(t)\, dt, \tag{2.5}$$

where

$$G_h(\vartheta) = [X^{-1}(\vartheta)]' D_h(\vartheta) X^{-1}(\vartheta), \tag{2.6}$$

$$D_h(\vartheta) = \int_{\vartheta-h}^{\vartheta} X'(t) Q'(t) Q(t) X(t)\, dt. \tag{2.7}$$

Here the prime denotes the transposition.

From Eq. (2.4) we see that a necessary condition for the unforced linear system (2.1) to be c-observable on the interval $\vartheta-h \leqslant t \leqslant \vartheta$ is that the columns of the matrix $Q(t) X(t) X^{-1}(\vartheta)$ be linearly independent on this interval, otherwise, there exists a state vector $x(\vartheta)$ such that $z(t) \equiv 0$, $\vartheta-h \leqslant t \leqslant \vartheta$. This condition of linear independence, expressed mathematically by

$$Q(t) X(t) X^{-1}(\vartheta)\, \lambda \neq 0, \quad \forall t \in [\vartheta-h, \vartheta] \quad \text{for each } \lambda \neq 0 \text{ in } R^n \tag{2.8}$$

is also a sufficient one.

In fact, if the condition (2.8) is fulfilled for the unforced linear system (2.1), then, defining an m-dimensional vector $v_\lambda(t, \vartheta)$ by

$$v_\lambda(t, \vartheta) \triangleq Q(t) X(t) X^{-1}(\vartheta)\, \lambda \neq 0, \quad \forall t \in [\vartheta-h, \vartheta] \text{ for each } \lambda \neq 0 \text{ in } R^n$$

we have

$$\int_{\vartheta-h}^{\vartheta} v'_\lambda(t, \vartheta)\, v_\lambda(t, \vartheta)\, dt = \lambda' G_h(\vartheta)\, \lambda > 0 \quad \text{for each } \lambda \neq 0 \text{ in } R^n.$$

The last inequality implies that the Gramian matrix $G_h(\vartheta)$ is positive definite. Hence, in this case $det\, G_h(\vartheta) \neq 0$ and the state vector $x(\vartheta)$ is defined uniquely from algebraic equation (2.5), i.e. system (2.1) is c-observable on the interval $\vartheta-h \leqslant t \leqslant \vartheta$.

Thus, the unforced linear system (2.1) is c-observable on the interval $\vartheta-h \leqslant t \leqslant \vartheta$ if and only if the condition (2.8) holds, or equivalently, if and only if the Gramian matrix $G_h(\vartheta)$ is nonsingular.

In what follows we shall assume that the condition (2.8) is fulfilled for the considered system.

Now, we pass to the computational procedure of determining the state vector $x(\vartheta)$.

First, we look at the problem of evaluating the Gramian matrix $G_h(\vartheta)$. Note that the formula (2.6) requires to compute the inverse matrix $X^{-1}(\vartheta)$ in order to obtain the Gramian matrix $G_k(\vartheta)$.

It turns out, as well known, that the Gramian matrix can be obtained without above-mentioned requirement, namely (see Ref. [6])

$$G_h(\vartheta) = \Phi(\vartheta)\, D_h(\vartheta)\, \Phi'(\vartheta), \tag{2.9}$$

where $\Phi(t)$ is an $(n \times n)$-dimensional fundamental matrix of the adjoint differential equation

$$\frac{d\varphi}{dt} = -A'(t)\,\varphi\,.$$

Thus, if ϑ and h are given, then the Gramian matrix can be predetermined by (2.7) and (2.9).

Further, it is easy to show that the right-hand side of the equation (2.5) can be evaluated by

$$\psi(\vartheta) = \int_{\vartheta-h}^{\vartheta} [X^{-1}(\vartheta)]'\,X'(t)\,Q'(t)\,z(t)\,dt\,, \tag{2.10}$$

where $\psi(t)$ is the solution of the following differential equation

$$\frac{d\psi}{dt} = -A'(t)\,\psi + Q'(t)\,z(t) \tag{2.11}$$

subject to the initial condition

$$\psi(\vartheta-h) = 0\,. \tag{2.12}$$

Clearly, it is very much easier to solve the equation (2.11) with the initial condition (2.12) than to evaluate the integral on the right-hand side of the equation (2.5).

Combining the above results, the formulation of the computational procedure is obtained. It entails the following steps.

Step 0. Predetermine the Gramian matrix $G_h(\vartheta)$ by (2.7) and (2.9).

Step 1. Solve forward equation (2.11) subject to (2.12) from $\vartheta-h$ to ϑ and at the end of the forward integration obtain vector $\psi(\vartheta)$.

Step 2. Solve algebraic equation

$$G_h(\vartheta)\,x(\vartheta) = \psi(\vartheta) \tag{2.13}$$

in order to find the state vector $x(\vartheta)$.

2.3. Block diagram realization of the indirect c-observer for the unforced linear nonstationary systems

From the previous development it is obvious that the indirect c-observer can be constructed for the unforced linear nonstationary systems. The block diagram realization of this observer is shown in Fig. 1, where the block AC denotes an analog computer designed for solving differential equation (2.11) subject to (2.12) and the block DC denotes a digital computer designed for solving algebraic equation (2.13).

Thus, it is concluded that the constructed above indirect c-observar representing the special purpose hybrid computer connected in parallel with a considered object provides automatical calculation of the state vector and can effectively surmount the difficulties associated with a control design when the state is inaccessible to direct observation.

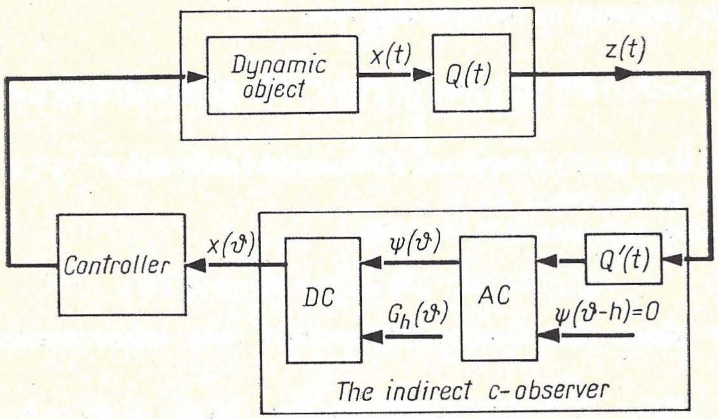

Fig. 1. Block diagram realization of c-observer for the unforced linear systems

3. Numerical solution of the c-observation problem for the forced linear nonstationary systems

3.1. Problem statement. We have previously derived a numerical solution of the c-observation problem for the unforced linear nonstationary systems. Now, we consider a linear forced dynamical system represented by the differential equation in the following form

$$\frac{dx}{dt}=A(t)\,x+B(t)\,u(t)+v(t), \tag{3.1}$$

where x is an n-dimensional vector representing the phase state of the considered system at time t, $A(t)$ and $B(t)$ are a known, respectively, $(n\times n)$- and $(n\times m)$-dimensional matrices, $u(t)$ is an m-dimensional vector-valued function representing the control parameter and $v(t)$ is a given n-dimensional vector-valued forced function.

Let us assume that the system's input $u(t)$ and forced function $v(t)$ are either known a priori or can be measured exactly and that phase state $x(t)$ is inaccessible to direct observation, only an m-dimensional output vector

$$z(t)=Q(t)\,x(t), \quad \vartheta-h\leqslant t\leqslant\vartheta \tag{3.2}$$

is accessible to noise-fiee measurement, where $m<n$, $Q(t)$ is a known $(m\times n)$-dimensional matrix.

Then, the c-observation problem for the forced linear system (3.1) consists in the following: it is required to find the unknown state vector $x(t)$ at the present time $t=\vartheta$ from a complete knowledge of the system's input $u(t)$, forced function $v(t)$ and output vector $z(t)$ on the finite time interval $\vartheta-h\leqslant t\leqslant\vartheta$ for some $h>0$, where $\vartheta-h$ is some past time $(\vartheta-h<\vartheta)$ because, since $m<n$, equation (3.2) does not allow immediate finding of $x(t)$ from $z(t)$.

3.2. Solution technique. It is well known that the solution of the differential equation (3.1) can be presented in the following form

$$x(t)=X(t)X^{-1}(\vartheta)x(\vartheta)-\int_t^{\vartheta}X(t)X^{-1}(\xi)[B(\xi)u(\xi)+v(\xi)]\,d\xi, \tag{3.3}$$

where $X(t)$ is an $(n\times n)$-dimensional fundamental matrix and $X^{-1}(\cdot)$ is an inverse with respect to the matrix $X(\cdot)$.

Substituting (3.3) into (3.2) we have

$$Q(t)X(t)X^{-1}(\vartheta)x(\vartheta)=z(t)-Q(t)\lambda(t), \tag{3.4}$$

where $\lambda(t)$ denotes the following n-dimensional vector-valued function

$$\lambda(t)=-\int_t^{\vartheta}X(t)X^{-1}(\xi)[B(\xi)u(\xi)+v(\xi)]\,d\xi,\quad \vartheta-h\leqslant t\leqslant\vartheta. \tag{3.5}$$

Multiply (3.4) by $[X^{-1}(\vartheta)]'X'(t)Q'(t)$ from the left and integrate from $\vartheta-h$ to ϑ to obtain

$$G_h(\vartheta)x(\vartheta)=\int_{\vartheta-h}^{\vartheta}[X^{-1}(\vartheta)]'X'(t)Q'(t)[z(t)-Q(t)\lambda(t)]\,dt, \tag{3.6}$$

where $G_h(\vartheta)$ is the Gramian matrix which was defined by (2.7) and (2.9).

If the condition (2.8) is fulfilled for the considered system (3.1) or equivalently, if $\det G_h(\vartheta)\neq 0$, then the state vector $x(\vartheta)$ is defined uniquely from algebraic equation (3.6).

From (3.5) and (3.6) it is plain that computer implementation requires the computation of the $(n\times n)$-dimensional matrix $X^{-1}(t)$. In many cases, however, analytical evaluation of the inverse matrix $X^{-1}(t)$ is difficult and, sometimes, impossible too. Moreover, for determining the state vector we must take definite integral in the right-hand side of the equation (3.6) and indefinite integral (3.5) that is not convenient and requires a lot of computer time. Therefore, it is of interest to find the practical effective computer procedure which avoids these difficulties.

It is easy to show that the vector-valued function $\lambda(t)$ defined by (3.5) is the solution of the following equation

$$\frac{d\lambda}{dt}=A(t)\lambda+B(t)u(t)+v(t) \tag{3.7}$$

subject to

$$\lambda(\vartheta)=0. \tag{3.8}$$

Since here only the terminal condition (3.8) is known, in order to evaluate $\lambda(t)$ on the interval $\vartheta-h\leqslant t\leqslant\vartheta$ we first must find the corresponding missing initial condition for $\lambda(t)$ under which the terminal condition (3.8) is satisfied. Furthermore, it is required that a process of determining this condition must be finished at time $t=\vartheta-h$.

The requirements indicated above can be fulfilled by integrating equation (3.7) backward in an accelerated time, i.e. introducing a new independent variable τ by means of the following relation

$$t=\rho(\tau)\triangleq\vartheta-h-\frac{1}{\varepsilon}[\tau-(\vartheta-h)],\quad \varepsilon>0. \tag{3.9}$$

Form (3.7) and (3.9) we obtain the following equation

$$\varepsilon\frac{\mathrm{d}\tilde{\lambda}}{\mathrm{d}\tau}=-\tilde{A}(\tau)\tilde{\lambda}-\tilde{B}(\tau)\tilde{u}(\tau)-\tilde{v}(\tau), \tag{3.10}$$

where

$$\tilde{\lambda}(\tau)=\lambda(\rho(\tau)),\quad \tilde{A}(\tau)=A(\rho(\tau)),\quad \tilde{B}(\tau)=B(\rho(\tau)),\quad \tilde{u}(\tau)=u(\rho(\tau)),\quad \tilde{v}(\tau)=v(\rho(\tau)).$$

Solving equation (3.10) forward from $\tau_0=\vartheta-h-\varepsilon h$ to $\tau_f=\vartheta-h$ subject to the following initial condition

$$\tilde{\lambda}(\tau_0)=0 \tag{3.11}$$

we obtain at the end of the forward integration the vector $\tilde{\lambda}(\tau_f)$ which is a requisite missing initial condition for $\lambda(t)$.

Now, the vector-valued function $\lambda(t)$ can be evaluated by solving forward equation (3.7) from $\vartheta-h$ to ϑ subject to the following initial condition

$$\lambda(\vartheta-h)=\tilde{\lambda}(\tau_f). \tag{3.12}$$

Further, it is easy to show that the right-hand side of the equation (3.6) can be then determined by

$$\psi(\vartheta)=\int_{\vartheta-h}^{\vartheta}[X^{-1}(\vartheta)]'X'(t)Q'(t)[z(t)-Q(t)\lambda(t)]\,dt, \tag{3.13}$$

where $\psi(t)$ is the solution of the following differential equation

$$\frac{d\psi}{dt}=-A'(t)\psi+Q'(t)[z(t)-Q(t)\lambda(t)] \tag{3.14}$$

subject to

$$\psi(\vartheta-h)=0. \tag{3.15}$$

Combining the above results, the formulation of the computational procedure is obtained. It entails the following steps:

Step 0. Determined the Gramian matrix $G_h(\vartheta)$ by (2.7) and (2.9).

Step 1. Solve forward equation (3.10) subject to (3.11) from $\tau_0=\vartheta-h-\varepsilon h$ to $\tau_f=\vartheta-h$ and at the end of the forward integration obtain the vector $\tilde{\lambda}(\tau_f)$.

Step 2. Solve forward equation (3.7) subject to (3.12) from $\vartheta-h$ to ϑ. During the forward integration evaluate vector-valued function $\lambda(t)$, $\vartheta-h\leqslant t\leqslant\vartheta$.

Simultaneously solve forward equation (3.14) subject to (3.15) from $\vartheta-h$ to ϑ and to end of the forward integration obtain vector $\psi(\vartheta)$.

Step 3. Solve algebraic equation

$$G_h(\vartheta)\, x(\vartheta)=\psi(\vartheta) \tag{3.16}$$

in order to find the state vector $x(\vartheta)$.

3.3 Block diagram realization of the indirect c-observer for the forced linear nonstationary systems

From the above developments it is plain that the indirect c-observer can be constructed for the forced linear nonstationary systems. The block diagram realization of this observer is shown in Fig. 2, where the block AC-1 denotes an analog

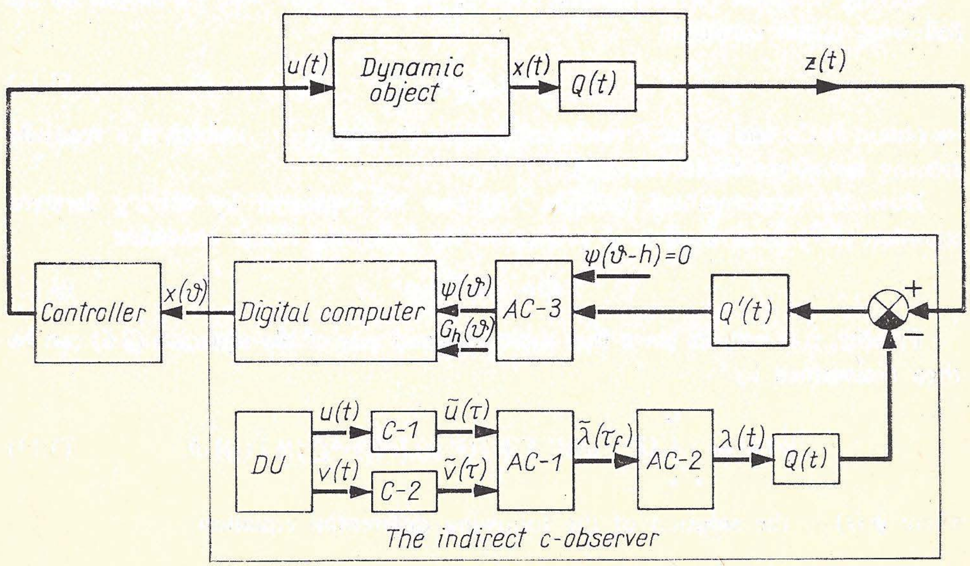

Fig. 2. Block diagram realization of c-observer for the forced linear systems

computer designed for solving differential equation (3.10) subject to (3.11), the block AC-2—an analog computer designed for solving the differential equation (3.7) subject to (3.12), the block AC-3—an analog computer designed for solving the differential equation (3.14) subject to (3.15) and the block DC—a digital computer designed for solving algebraic equation (3.16). Besides in this figure the block DU denotes a driver unit and the blocks C-1 and C-2 denote converters.

Thus, the indirect c-observer for the forced linear nonstationary system (3.1) representing special purpose hybrid computer connected in parallel with a controlled object guarantees automatical calculation of the current state vector from a complete knowledge of the system's input and output history on the finite time interval.

4. Conclusions

The paper proposes the practical effective method by which the current state of the unforced as well as forced linear nonstationary systems can be evaluated from a complete knowledge of the system's input and output history on the finite time interval.

The method requires that only auxiliary systems of differential equations with known initial conditions and system of algebraic equation are solved for determining the current state vector of the considered systems.

From the obtained results we have constructed the indirect c-observers for the unforced as well as forced linear systems. These observers representing the special purpose hybrid computer connected in parallel with a controlled object guarantee automatical calculation of the current state vector and can effectively surmount difficulties associated with control design when the state is inaccessible to direct observation.

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Numeryczne rozwiązanie zagadnienia c-obserwacji dla liniowych układów niestacjonarnych

Przedstawiono efektywną metodę praktyczną rozwiązania zagadnienia c-obserwacji dla liniowych układów niestacjonarnych.

Dla określenia aktualnego stanu rozpatrywanych systemów metoda wymaga rozwiązania

pomocniczych układów równań różniczkowych ze znanymi warunkami początkowymi i układu równań algebraicznych.

Przedstawiono w postaci schematu blokowego realizację c-obserwatora pośredniego dla liniowych układów niestacjonarnych z wymuszeniem i bez wymuszenia.

Численное решение задачи c-наблюдений для линейных нестационарных систем

Представлен эффективный практический метод решения задачи c-наблюдений для линейных нестационарных систем.

С целью определения текущего состояния исследуемых систем метод требует решения вспомогательных систем дифференциальных уравнений с известными начальными условиями, а также системы алгебраических уравнений.

Представлена, а виде блок-схемы, реализация косвенной системы c-наблюдений для линейных нестационарных систем с возмущениями и без возмущений.