Control
and Cybernetics
VOL. 9 (1980) No. 3

Convergence of Galerkin type approximation to control constrained problems for hyperbolic systems

by

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A discrete time Galerkin approximation of a class of convex, control constrained optimal control problems for second order hyperbolic system is considered. The optimal solutions regularity in the original problem is investigated. Basing on [1], the respective estimates for the state approximation error are derived. Exploiting the obtained results and applying Lagrange formalism technique, a linear convergence of approximating sequence to the optimal control is proved.

1. Introduction

The paper deals with convergence of the standard Galerkin approximation [1,8] applied to an optimal control problem for a linear second order hyperbolic system with locally constrained distributed control and the cost functional of integral type.

A linear convergence of approximation was proved in [8] for an analogous parabolic case, as well as in [4] for an unconstrained hyperbolic problem, with the use of Lagrange formalism technique. The presence of control constraints in hyperbolic case restricts considerably [5] regularity of the optimal solutions that makes difficult a direct adaptation of the previous methods.

In the paper the problem of regularity is handled taking advantage of some additional, but unrestrictive assumptions concerning the functional. It enables us to demonstrate in Section 3 that the primal and dual optimal state variables are in $H^{2,2}(Q)$ and $H^{3,3}(Q)$, respectively.

In Section 4 a discrete time Galerkin approximation to the original control problem is introduced by means of the known two-level form of the discrete state equation. Applying techniques of [1], $L^\infty(H^0)$ — estimates of the state approximation error are derived with essentially weakened requirements on regularity of the solution. Moreover, the obtained $L^\infty(H^1)$ — estimates are slightly stronger in comparison with the analogous result of [2].

The main theorem estimating the rate of convergence of approximation for the optimal controls by $0(\tau+h+k)$ is derived in Section 5 using Lagrange formalism technique.

The notation used is based on [6]. In particular, for $s\geq 0$, $H^s(\Omega)$ will denote the Sobolev space of real-valued functions on Ω with a norm $\|\cdot\|_s$. As a consequence, $H^0(\Omega)=L^2(\Omega)$ with the inner product

$$(u,v)=\int_\Omega u\cdot v\,dx, \quad u,v\in L^2(\Omega).$$

For a Banach space X with norm $\|\cdot\|_X$ we shall consider the spaces of Lebesgue measurable functions $v\colon [0,T]\to X$ denoted, by

$$L^p(0,T;X)=\{v\colon \|v\|_{L^p(X)}<\infty\}, \quad p=2,\infty$$

with the norms

$$\|v\|^2_{L^2(X)}=\int_0^T \|v(\cdot,t)\|^2_X\,dt$$

and

$$\|v\|_{L^\infty(X)}=\sup_{0\leq t\leq T}\|v(\cdot,t)\|_X,$$

respectively.

For $Q=\Omega\times(0,T)$, $H^{r,s}(Q)$ will denote the Sobolev space of functions $v\in L^2(0,T;H^r(\Omega))$ such that

$$\frac{d^s v}{dt^s}\in L^2(0,T;H^0(\Omega)).$$

Throughout the rest of the paper all constants appearing in the error estimates will be denoted by the generic C.

2. The optimal control problem

Let be given a bounded domain $\Omega\subset R^n$ situated locally on one side of the properly regular boundary $\partial\Omega$.

In the cylinder $Q=\Omega\times(0,T)$ we consider a system governed by the following weak formulated hyperbolic equation

$$\left(\frac{d^2y(t)}{dt^2},v\right)+a(y(t),v)=(f(t),v) \quad \forall v\in H^1(\Omega),\ t\in(0,T), \tag{2.1}$$

along with the initial data

$$y(0)=y^1\in H^2(\Omega), \quad \frac{dy}{dt}(0)=y^2\in H^1(\Omega). \tag{2.1a}$$

Here and in the sequel $a(\cdot,\cdot)$ denotes a symmetric bilinear form defined on $H^1(\Omega)\times H^1(\Omega)$ by

$$a(\varphi,\psi)\overset{\mathrm{df}}{=}\int_{\Omega}\left[\sum_{i,j=1}^{n}a_{ij}(x)\frac{\partial\varphi(x)}{\partial x_i}\cdot\frac{\partial\psi(x)}{\partial x_j}+a_0(x)\,\varphi(x)\,\psi(x)\right]dx, \tag{2.2}$$

where the functions $a_0(\cdot)$ and $a_{ij}(\cdot)=a_{ji}(\cdot)$ are assumed to be sufficiently regular and such that

$$\sum_{i,j=1}^{n}a_{ij}(x)\,\xi_i\,\xi_j\geqslant\alpha_0\sum_{i=1}^{n}\xi_i^2\ \ \forall x\in\Omega;\ \xi_i,\xi_j\in R^1, \tag{2.2a}$$

$$a_0(x)\geqslant\alpha_0\qquad\forall x\in\Omega \tag{2.2b}$$

for some constant $\alpha_0>0$.

It follows from (2.2) that $a(\cdot,\cdot)$ is continuous and coercive on $H^1(\Omega)$, i.e. there exists a constant $\alpha_1>0$ such that

$$a(\varphi,\psi)\leqslant\alpha_1\,\|\varphi\|_1\cdot\|\psi\|_1\qquad\forall\varphi,\psi\in H^1(\Omega), \tag{2.3}$$

$$a(\varphi,\varphi)\geqslant\alpha_0\|\varphi\|_1^2\qquad\forall\varphi\in H^1(\Omega). \tag{2.4}$$

Associated with (2.2) is an operator $A\in\mathscr{L}(H^1(\Omega),(H^1(\Omega))')$ defined by

$$(Ay)(x)=\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial y}{\partial x_i}(x)\right)-a_0(x)\,y(x) \tag{2.5}$$

along with homogeneous Neumann boundary conditions on y

$$\frac{\partial y}{\partial\nu_A}(x,t)=0,\qquad(x,t)\in\partial\Omega\times[0,T]. \tag{2.5a}$$

Certain general regularity results [6,4] are essentially important in the sequel.

LEMMA 2.1. *If* (2.3), (2.4) *hold and moreover*

$$f\in H^{0,1}(Q)$$

then the solution of (2.1)

$$y\in H^{2,2}(Q).$$

LEMMA 2.2. *If* (2.3), (2.4) *hold and*

$$f\in H^{1,2}(Q), \tag{2.6a}$$

$$f(0)\in H^{3/2}(\Omega),\qquad\frac{df}{dt}(0)\in H^{1/2}(\Omega), \tag{2.6b}$$

$$y^1\in H^{7/2}(\Omega),\qquad y^2\in H^{5/2}(\Omega), \tag{2.6c}$$

$$\frac{\partial y^1}{\partial\nu_A}(0)=0,\qquad\frac{\partial y^2}{\partial\nu_A}(0)=0 \tag{2.6d}$$

then the solution of (2.1)

$$y\in H^{3,3}(Q). \tag{2.7}$$

In order to state the optimal control problem let us introduce the space of control functions

$$U=L^2(0, T; H^0(\Omega)) \tag{2.8}$$

containing the set of admissible controls

$$U_a \overset{\text{df}}{=} \{u \in U: \Psi(u) \leqslant 0 \quad \text{a.e. in } Q\} \neq \Phi, \tag{2.9}$$

where

$$\Psi^T(u)=[u^1-u, \; -u^2+u] \tag{2.9a}$$

for a given

$$u^1, u^2 \in H^{1,1}(Q). \tag{2.9b}$$

We define the cost functional of integral type

$$J(u, y) \overset{\text{df}}{=} \int_0^T \int_\Omega \varphi(u(x, t), \; y(x, t)) \, dx dt, \tag{2.10}$$

where $\varphi(\cdot, \cdot)$ is a convex, twice differentiable function satisfying the conditions

$$|\varphi''_{u,u}(u, y)|, \; |\varphi''_{y,y}(u, y)|, \; |\varphi'''_{y,y,y}(u, y)| \leqslant C, \tag{2.11a}$$

$$\varphi''_{u,u}(u, y) \geqslant \alpha > 0 \quad \forall u, y \in R^1, \tag{2.11b}$$

$$\varphi''_{u,y} = \varphi''_{y,u} = 0 \text{ a.e. in } Q. \tag{2.11c}$$

Assume moreover that there is given an operator

$$B \in \mathscr{L}(H^r(\Omega), H^r(\Omega)) \quad \text{for } r=0,1. \tag{2.12}$$

The following control — constrained optimal control problem will be discussed in the sequel.

PROBLEM (P). *Find* $u^0 \in U_a$ *such that*

$$J(u^0, y(u^0)) \leqslant J(u, y(u)) \quad \forall u \in U_a,$$

where $y(u)$ *is the solution of the state equation*

$$\left(\frac{d^2 y(t)}{dt^2}, v\right) + a(y(t), v) = (Bu(t), v) \quad \forall v \in H^1(\Omega), \; t \in (0, T)$$

along with the initial conditions (2.1a).

The convexity of φ and (2.11b) imply in particular that the Hessian $\delta^2 J$ satisfies the condition

$$(\delta^2 J(u, y); v, z; v, z) \geqslant \alpha \|v\|^2_{L^2(H^0)} \quad \forall u, y, v, z \in L^2(0, T; H^0(\Omega)). \tag{2.14}$$

Furthermore, since $I(u)=J(u, y(u))$ is strictly convex and continuous, while U_a is weakly compact in $U-(P)$ admits a unique solution.

Exploiting techniques developed in [7] a Lagrange characterization of this solution can be given. For this purpose let us define a Lagrange functional

$$L(u, y, \lambda, p) \overset{\text{df}}{=} J(u, y) + \left(\left(\frac{d^2 y}{dt^2}, p\right)\right) + \mathfrak{a}(y, p) - ((Bu, p)) + ((\psi u, \lambda)), \tag{2.15}$$

where

$$\mathfrak{a}(y, p) \overset{\text{df}}{=} \int_0^T a(y(t), p(t))\, dt \tag{2.15a}$$

and

$$p \in L^2(0, T; H^1(\Omega)),\ \lambda \in L^2(0, T; H^0(\Omega)).$$

By convexity of the problem (P) its solution can be characterized [3] as the saddle point of the Lagrangian (2.15). This fact can be expressed [5,7] in the equivalent form of Kuhn-Tucker conditions.

LEMMA 2.3. *The elements u^0 and $y^0 = y(u^0)$ are the solutions of (P) iff there exist uniquely defined Lagrange multipliers*

$$\lambda^0 \in L^2(0, T; H^0(\Omega)), \quad \lambda^0 \geqslant 0 \tag{2.16}$$

and p^0 satisfying the adjoint equation

$$\left(\frac{d^2 p^0(t)}{dt^2}, v\right) + a(p^0(t), v) = -(\delta_y J(u^0, y^0)(t), v) \quad \forall v \in H^1(\Omega), t \in (0, T), \tag{2.17}$$

$$p^0(T) = \frac{dp^0}{dt}(T) = 0 \tag{2.17a}$$

such that

$$\delta_u J(u^0, y^0) + B^* p^0 + ((\lambda^0, \delta_u \psi(u^0))) = 0 \tag{2.18}$$

and

$$((\lambda^0, \psi(u^0))) = 0. \tag{2.19}$$

Regularity of the optimal solutions u^0, y^0 and p^0 is essential in investigating of the rate of convergence for the finite-dimensional approximation to (P). These regularity properties are formulated in the following

THEOREM 2.1. *The solutions u^0, y^0 of (P) satisfy the regularity conditions*

$$u^0 \in H^{1,1}(Q), \tag{2.20a}$$

$$y^0 \in H^{2,2}(Q). \tag{2.20b}$$

If moreover

$$n \leqslant 3 \quad \text{or} \quad \varphi'''_{y,y,y}(\cdot, \cdot) = 0 \text{ in } R^2 \tag{2.21}$$

then

$$p^0 \in H^{3,3}(Q). \tag{2.20c}$$

Proof. To prove (2.20a) we use the optimality condition (2.18). First define an auxiliary function $\bar{u}(x, t)$ given by

$$\delta_u l(\bar{u}, y^0, B^* p^0) = 0 \quad \text{a.e. in } Q, \tag{2.22}$$

where

$$l(u, y, B^* p) \stackrel{\text{df}}{=} \varphi(u, y) + <B^* p, u>. \tag{2.23}$$

Note that by (2.11b)

$$|[\delta_{u,u} l(u, y, B^* p)]^{-1}| \leqslant \frac{1}{\alpha} < \infty \forall u, y, B^* p \in R^1, \tag{2.24}$$

hence it follows from implicite function theorem that there exists a mapping $\bar{u} = \bar{u}(y, B^* p)$ continuous in a neighbourhood of the point $(y^0, B^* p^0)$ and such that (2.22) holds. The mapping $\bar{u}$ is moreover differentiable at $(y^0, B^* p^0)$ that by (2.23), (2.24) and (2.11c) yields

$$\frac{\partial \bar{u}(x, t)}{\partial t} = -[\varphi''_{u,u}(u^0, y^0)]^{-1} \cdot \frac{\partial B^* p^0(x, t)}{\partial t}, \tag{2.25a}$$

$$\frac{\partial \bar{u}(x, t)}{\partial x_i} = -[\varphi''_{u,u}(u^0, y^0)]^{-1} \cdot \frac{\partial B^* p^0(x, t)}{\partial x_i}, \quad (i = 1, 2, ..., n). \tag{2.25b}$$

It is obvious [6] that by (2.1a) $y^0, p^0 \in H^{1,1}(Q)$, thus making use of (2.25) we have

$$\bar{u} \in H^{1,1}(Q). \tag{2.26}$$

The optimal control $u^0(x, t)$ satisfying (2.8a) can be expressed in the form

$$u^0(x, t) = \max\{u^1(x, t), \min\{\bar{u}(x, t), u^2(x, t)\}\}. \tag{2.27}$$

Utilizing (2.9b) it is possible to check [7] that for $\bar{u}$ fulfiling (2.26), the function u^0 defined by (2.27) fulfils (2.20a).

Now (2.20b) is trivially satisfied by Lemma 2.1, since its assumptions hold by (2.20a) and (2.12).

To prove (2.20c) consider the adjoint equation (2.17) and apply Lemma 2.2 for the reversed direction of time. Conditions (2.6c–d) hold by (2.17a). Therefore it is enough to check that the right—hand side of (2.17) satisfies (2.6a) as well as (2.6b) at $t = T$. To this end note that by (2.11c)

$$\delta_y J(u^0, y^0)(x, t) = \varphi'_y(u^0(x, t), y^0(x, t)), \tag{2.28a}$$

$$\frac{d}{dt} \delta_y J(u^0, y^0)(x, t) = \varphi''_{y,y}(u^0(x, t), y^0(x, t)) \cdot \frac{dy^0(x, t)}{dt}, \tag{2.28b}$$

$$\frac{\partial}{\partial x_i} \delta_y J(u^0, y^0)(x, t) = \varphi''_{y,y}(u^0(x, t), y^0(x, t)) \cdot \frac{\partial y^0(x, t)}{\partial x_i}, \quad (i = 1, 2, ..., n), \tag{2.28c}$$

$$\frac{d^2}{dt^2} \delta_y J(u^0, y^0)(x, t) = \varphi''_{y,y}(u^0(x, t), y^0(x, t)) \cdot \frac{d^2 y^0(x, t)}{dt^2} + \\ + \varphi'''_{y,y,y}(u^0(x, t), y^0(x, t)) \cdot \left[\frac{dy^0(x, t)}{dt}\right]^2. \tag{2.28d}$$

Assume that $\varphi'''_{y,y,y}(\cdot,\cdot)=0$. Then (2.6 a–b) follow directly from (2.28) and (2.20b), that by Lemma 2.2 implies (2.20c).

In the case $n\leqslant 3$ condition (2.20b) and Sovolev's Lemma [6] yield

$$\frac{dy^0}{dt}\in L^4(Q). \tag{2.29}$$

Taking advantage of (2.11a), (2.20b) and (2.29) in (2.28) we conclude that

$$\delta_y J(u^0, y^0)\in H^{1,2}(Q).$$

On the other hand, (2.28a–b) and (2.20a) imply in particular [6] that

$$\delta_y J(u^0, y^0)(\cdot, T)\in H^{3/2}(\Omega),$$

$$\frac{d}{dt}\delta_y J(u^0, y^0)(\cdot, T)\in H^{1/2}(\Omega).$$

In conclusion, the function $\delta_y J(u^0, y^0)$ satisfies conditions (2.6a–b) that completes the proof of (2.20c). □

3. Approximation of the control problem

A finite — dimensional approximation of (P) is founded upon a discrete-time Galerkin scheme characterized fully in [5]. We recall here briefly some main concepts.

Let V_h be a finite — dimensional subspace of $V=H^1(\Omega)$, depending on the parameter of discretization h converging to zero. The family of pairs $\{V_h, P_h\}_{0<h\leqslant 1}$ is said to be an approximation of V for $P_h\in\mathscr{L}(V, V_h)$ being a projection operator. Assume that there exists a constant $C<\infty$ such that

$$\inf_{v_h\in V_h}[\|v-v_h\|_0+h\,\|v-v_h\|_1]\leqslant C\,h\|v\|_s \qquad \forall v\in H^s(\Omega),\ s=1,2. \tag{3.1}$$

The above properties imply [1, 2] the following elliptic projection result, widely applied in error estimation techniques.

Lemma 3.1. *Let $y\in L^2(0,T;H^s(\Omega))$ for $1\leqslant s\leqslant 2$ be the solution of* (2.1). *Then there exists a unique mapping $\omega_h\in L^2(0,T;V_h)$ which satisfies*

$$a(\omega_h(t), v_h)=a(y(t), v_h) \qquad \forall v_h\in V_h,\ t\in[0,T].$$

Furthermore, if for some integer $k\geqslant 0$

$$\frac{d^k y}{dt^k}\in L^p(0,T;H^s(\Omega)),$$

then

$$\frac{d^k\omega_h}{dt^k}\in L^p(0,T;V_h)$$

and

$$\left\| \frac{d^k}{dt^k} (y-\omega_h) \right\|_{L^p(H^i)} \leqslant C h^{s-i} \left\| \frac{d^k y}{dt^k} \right\|_{L^p(H^s)}, \quad i=0,1,$$

where $p=2, \infty$. □

The interval $[0, T]$ is discretized with the step size $\tau = T/N(\tau)$ for a fixed integer $N(\tau)$. Denoting by $\chi_n(t)$ and $\hat{\chi}_n(t)$ the characteristic functions of subintervals $[n\tau, (n+1)\tau)$ and $(n\tau, n+1)\tau]$, respectively, we define the spaces $E_\tau(n_1\tau, (n_2+1)\tau; X)$ and $E_\tau((n_1-1)\tau, n_2\tau; X)$ of step functions

$$v_\tau(t) = \begin{cases} \sum_{n=n_1}^{n_2} v_\tau(n\tau)\, \chi_n(t), \\ \sum_{n=n_1-1}^{n_2-1} v_\tau((n+1)\tau)\, \hat{\chi}_n(t), \end{cases} \qquad v_\tau(n\tau) \in X \tag{3.2}$$

respectively, for any $0 \leqslant n_1 < n_2 \leqslant N(\tau)$ and a Banach space X.

We introduce the projection operators

$$(\bar{P}_\tau v)(t) = \bar{v}_\tau(t) \stackrel{\text{df}}{=} \sum_{n=0}^{N} v(n\tau)\, \chi_n(t), \tag{3.3a}$$

$$(\hat{P}_\tau v)(t) = \hat{v}_\tau(t) \stackrel{\text{df}}{=} \sum_{n=0}^{N} v(n\tau)\, \hat{\chi}_n(t), \tag{3.3b}$$

$$(\tilde{P}_\tau v)(t) = \tilde{v}_\tau(t) \stackrel{\text{df}}{=} \sum_{n=0}^{N-1} \left(\frac{1}{\tau} \int_{n\tau}^{(n+1)\tau} v(t)\, dt \right) \chi_n(t) \tag{3.4}$$

defined respectively on $C([0, T]; X)$ in the cases (3.3) and on $L^2(0, T; X)$ in the case (3.4).

It is asserted in Lemma 2.1 that the weak solution of (2.1) is in $C([0, T]; H^1(\Omega))$, hence the approximation for the space of states can be chosen as $\{E_\tau(0, T+\tau; V_h), \bar{P}_\tau \cdot \bar{P}_h\}$, where

$$(\bar{P}_h v)(t) \stackrel{\text{df}}{=} P_h v(t) \qquad \forall t \in [0, T]. \tag{3.5}$$

The space of controls $U = L^2(0, T; H^0(\Omega))$ is approximated by $\{U_{k,\tau}, \tilde{P}_\tau \cdot \tilde{P}_k\}$ for $U_{k,\tau} = E_\tau(0, T; W_k)$, where W_k is a finite — dimensional subspace of $W = H^0(\Omega)$ and

$$(\tilde{P}_k w) \stackrel{\text{df}}{=} P_k w \qquad \text{a.e. in } Q. \tag{3.6}$$

The projection operator $P_k \in \mathcal{L}(W, W_k)$ is defined in the same way as those in [5, 8] satisfying the condition

$$\|w - P_k w\|_0 \leqslant Ck \|w\|_1 \qquad \forall w \in H^1(\Omega). \tag{3.7}$$

In the sequel we shall utilize the standard finite — difference operators defined below

$$(\partial_\tau v_\tau)(t)=[v_\tau(t+\tau)-v_\tau(t)]/\tau,$$
$$(\bar{\partial}_\tau v_\tau)(t)=[v_\tau(t)-v_\tau(t-\tau)]/\tau,$$
$$(\partial_\tau^2 v_\tau)(t)=[v_\tau(t-\tau)-2v_\tau(t)+v_\tau(t+\tau)]/\tau^2,$$
$$v_{\tau\pm 1/2}(t)=[v_\tau(t)+v_\tau(t\pm\tau)]/2,$$
$$v_{\theta\tau}(t)=\theta v_\tau(t-\tau)+(1-2\theta)\,v_\tau(t)+\theta v_\tau(t+\tau)$$

for any $v_\tau \in E_\tau(0, T+\tau;\ V_h)$ and $0\leqslant\theta\leqslant\frac{1}{2}$. To avoid some technicalities, the functions $(\partial_\tau^2 v_\tau)(\cdot)$ and $v_{\theta\tau}(\cdot)$ defined on the interval $[\tau, T)$ will be considered as extended by zero to $[0, T)$.

Some essential properties of the introduced approximation aie collected below as

Lemma 3.2. *If the functions u and y are sufficiently regular, then for any $0\leqslant\theta\leqslant\frac{1}{2}$, $p=2, \infty$*

$$\|y-\bar{y}_\tau\|_{L^p(H^s)}+\|y-\bar{y}_{\theta\tau}\|_{L^p(H^s)}\leqslant C\tau\left\|\frac{dy}{dt}\right\|_{L^p(H^s)} \qquad (s=0, 1, \dots),$$

and for $s=0, 1$

$$\|u-\tilde{u}_{k,\tau}\|_{L^2(H^0)}\leqslant C\left[\tau\left\|\frac{du}{dt}\right\|_{L^2(H^0)}+k\,\|u\|_{L^2(H^1)}\right],$$

$$\|y-\bar{y}_{h,\theta\tau}\|_{L^p(H^s)}\leqslant C\left[\tau\left\|\frac{dy}{dt}\right\|_{L^p(H^s)}+h\left(\|y\|_{L^p(H^{s+1})}+\tau\left\|\frac{dy}{dt}\right\|_{L^p(H^{s+1})}\right)\right],$$

$$\left\|\frac{dy}{dt}-\partial_\tau\bar{y}_{h,\tau}\right\|_{L^2(H^s)}\leqslant C\left[\tau\left\|\frac{d^2y}{dt^2}\right\|_{L^2(H^s)}+h\left\|\frac{dy}{dt}\right\|_{L^2(H^{s+1})}\right],$$

$$\left\|\frac{d^2y}{dt^2}-\partial_\tau^2\bar{y}_{h,\tau}\right\|_{L^2(H^0)}\leqslant C\left[\tau\left\|\frac{d^3y}{dt^3}\right\|_{L^2(H^0)}+h\left\|\frac{d^2y}{dt^2}\right\|_{L^2(H^1)}\right]. \qquad \square$$

For notation simplicity the following symbols have been used: $\bar{y}_{h,\tau}=\bar{P}_\tau\bar{P}_h y$, $\tilde{u}_{k,\tau}=\tilde{P}_\tau\tilde{P}_k u$. Proof of Lemma 3.2, based on some elemenıary but tedious evaluations, can be found in [5].

Let the set of admissible controls U_a be approximated by the finite — dimensional subset

$$U_{ak,\tau}=U_a\cap E_\tau(0, T;\ W_k). \tag{3.8}$$

Then we can introduce the following discrete — time Galerkin approximation of the original optimal control problem (P).

Problem (D). *Find $u^0{}_{k,\tau}\in U_{ak,\tau}$ such that*

$$J(u^0{}_{k,\tau}, y_{h,\tau}(u^0{}_{k,\tau}))\leqslant J(u_{k,\tau}, y_{h,\tau}(u_{k,\tau})) \qquad \forall u_{k,\tau}\in U_{ak\,\tau},$$

where $y_{h,\tau}(u_{k,\tau})$ is a solution of the discrete equation

$$(\partial_\tau z_{h,\tau}(t), v_h)+a(y_{h,\tau+1/2}, v_h)=(f_\tau(t), v_h) \quad \forall v_h \in V_h, \ t \in (\tau, T), \tag{3.9}$$

$$\delta_\tau y_{h,\tau}=z_{h,\tau+1/2}$$

along with $f_\tau=Bu_{k,\tau}$ and the initial data

$$(y_{h,\tau}(0), v_h)=(y^1, v_h) \quad \forall v_h \in V_h, \tag{3.9a}$$

$$z_{h,\tau}(0), v_h)=(y^2, v_h) \quad \forall v_h \in V_h. \quad \square$$

The difference scheme (3.9) is unconditionally stable [1] and (*D*), as a standard finite — dimensional convex programming problem, admits a unique solution.

4. Approximation error of the state variables

In this section we shall be concerned with L^2 — and H^1 — estimates of the state approximation. Let $y_{h,\tau}$, $z_{h,\tau}$ be the respective solutions of the discrete state equation (3.9) and let ω_h be defined by Lemma 3.1 for y being a solution to (2.1). Introduce the functions

$$\xi=y_{h,\tau}-\omega_h, \quad \eta=y-\omega_h, \quad \vartheta=z_{h,\tau}-\frac{d\omega_h}{dt} \tag{4.1}$$

with the time — projections denoted respectively by

$$\xi_\tau=\bar{P}_\tau \xi, \quad \eta_\tau=\bar{P}_\tau \eta, \quad \vartheta_\tau=\bar{P}_\tau \vartheta. \tag{4.1a}$$

The following result is obtained by developing of the error estimation techniques of [1]. A suitable modifications allow to reach a desired result at considerably weakened smoothness requirements on the solution and the right-hand side function.

LEMMA 4.1. *Let*

$$y \in H^{2,2}(Q) \tag{4.2}$$

be the solution of (2.1) *and $y_{h,\tau} \in E_\tau(0, T+\tau; V_h)$ be the solution of* (3.9). *Then*

$$\|\xi_\tau\|_{L^\infty(H^0)} \leqslant C(\tau+h+\|f-f_\tau\|_{L^2(H^0)}). \tag{4.3}$$

Proof. Averaging (2.1) by means of the operation (3.4) we obtain

$$\left(\partial_\tau\left(\overline{\frac{dy}{dt}}\right)_\tau(t), v\right)+a(\bar{y}_{\tau+1/2}(t), v)=(\bar{f}_\tau(t), v)+$$

$$+a(\bar{y}_{\tau+1/2}(t)-\bar{y}_\tau(t), v) \quad \forall v \in V, \ t \in (0, T], \tag{4.4}$$

since it can be easily checked that

$$\partial_\tau\left(\overline{\frac{dy}{dt}}\right)_\tau(n\tau)=\left(\overline{\frac{d^2y}{dt^2}}\right)_\tau(n\tau).$$

Substracting (4.4) from (3.9) we get by Lemma 3.1 and (4.1)

$$(\partial_\tau \vartheta_\tau(t), v_h)+a(\xi_{\tau+1/2}(t), v_h)=\left(\partial_\tau\left(\overline{\frac{d\eta}{dt}}\right)_\tau(t)+r_\tau(t), v_h\right)$$

$$\forall v_h \in V_h, \quad t\in(0, T], \tag{4.5}$$

where for notation simplicity it has been put

$$\begin{aligned} r_\tau(t)&=A\Delta y_\tau(t)+\Delta f_\tau(t), \\ \Delta y_\tau(t)&=-\bar{y}_{\tau+1/2}(t)+\tilde{y}_\tau(t), \\ \Delta f_\tau(t)&=f_\tau(t)-\tilde{f}_\tau(t). \end{aligned} \tag{4.5a}$$

Now, by the same techniques as those developed in [1], we can transform (4.5) to the equation

$$(\partial_\tau \xi_\tau(n\tau), v_h)+a(\varphi_{\tau+1/2}(n\tau), v_h)=(\varepsilon_\tau(n\tau), v_h)+\tau a(\rho_\tau(n\tau), v_h)$$

$$\forall v_h\in V_h, \quad 0\leqslant n\leqslant N-1, \tag{4.6}$$

where the new functions φ_τ, ε_τ and ρ_τ are defined as follows

$$\varphi_\tau(0)=0,$$

$$\varphi_\tau(n\tau)=\tau\sum_{k=0}^{n=1}\xi_{\tau+1/2}(k\tau) \quad \text{for } 1\leqslant n\leqslant N, \tag{4.6a}$$

$$\varepsilon_\tau(0)=\partial_\tau\eta_\tau(0)-\sigma_\tau(0)+\frac{\tau}{2}\Delta f_\tau(0),$$

$$\varepsilon_\tau(n\tau)=\partial_\tau\eta_\tau(n\tau)-\sigma_\tau(n\tau)+\frac{\tau}{2}\Delta f_\tau(0)+\tau\sum_{k=0}^{n-1}\Delta f_{\tau+1/2}(k\tau) \tag{4.6b}$$

$$\text{for } 1\leqslant n\leqslant N-1,$$

$$\rho_\tau(0)=\frac{1}{2}\Delta y_\tau(0), \tag{4.6c}$$

$$\rho_\tau(n\tau)=\frac{1}{2}\sum_{k=0}^{n}\Delta y_\tau(k\tau)+\frac{1}{2}\sum_{k=0}^{n-1}\Delta y_\tau(k\tau) \quad \text{for } 1\leqslant n\leqslant N-1, \tag{4.6d}$$

$$\sigma_\tau(t)=\partial_\tau\bar{y}_\tau(t)-\left(\overline{\frac{dy}{dt}}\right)_{\tau+1/2}(t).$$

Putting in (4.6) the test function of the form

$$v_h=\partial_\tau\varphi_\tau(t)=\xi_{\tau+1/2}(t) \quad \forall t\in[0, T]$$

we obtain

$$\frac{1}{2}\|\xi_\tau((n+1)\tau)\|_0^2-\frac{1}{2}\|\xi_\tau(n\tau)\|_0^2+\frac{1}{2}a(\varphi_\tau((n+1)\tau), \varphi_\tau((n+1)\tau))+$$

$$-\frac{1}{2}a(\varphi_\tau(n\tau), \varphi_\tau(n\tau))=\tau(\varepsilon_\tau(n\tau), \xi_{\tau+1/2}(n\tau))+\tau^2 a(\rho_\tau(n\tau), \xi_{\tau+1/2}(n\tau))$$

$$\text{for } 0\leqslant n\leqslant N-1. \tag{4.7}$$

Summing in (4.7) from $n=0$ to $n=l-1$ for any $1\leqslant l\leqslant N$, and using (2.4), (2.6a) as well as the known inequality

$$a\cdot b\leqslant \varepsilon a^2+\frac{1}{4\varepsilon}b^2 \quad \forall a,b\in R;\ \varepsilon>0, \tag{4.8}$$

we get

$$\|\xi_\tau(l\tau)\|_0^2+\alpha_0\left\|\tau\sum_{n=0}^{l-1}\xi_{\tau+1/2}(n\tau)\right\|_1^2\leqslant$$

$$\leqslant\|\xi_\tau(0)\|_0^2+2_\tau\sum_{n=0}^{l-1}(\varepsilon_\tau(n\tau),\ \xi_{\tau+1/2}(n\tau))+2\tau^2\sum_{n=0}^{l-1}a(\rho_\tau(n\tau),\ \xi_{\tau+1/2}(n\tau))\leqslant$$

$$\leqslant\|\xi_\tau(0)\|_0^2+8T\tau\sum_{n=0}^{l-1}\|\varepsilon_\tau(n\tau)\|_0^2+$$

$$+\frac{\tau}{8T}\sum_{n=0}^{l-1}\|\xi_{\tau+1/2}(n\tau)\|_0^2+2\tau^2\sum_{n=0}^{l-1}a(\rho_\tau(n\tau),\ \xi_{\tau+1/2}(n\tau))\leqslant$$

$$\leqslant\|\xi_\tau(0)\|_0^2+8T\|\varepsilon_\tau\|_{L^2(H^0)}^2+\frac{1}{4}\max_{0\leqslant n\leqslant N}\|\xi_\tau(n\tau)\|_0^2+$$

$$+2\tau^2\sum_{n=0}^{l-1}a(\rho_\tau(n\tau),\ \xi_{\tau+1/2}(n\tau)). \tag{4.9}$$

We shall estimate the terms on the right hand side of (4.9). Observe first that by (4.5a)

$$\left\|\frac{\tau}{2}\Delta f_\tau(0)-\tau\sum_{k=0}^{n-1}\Delta f_{\tau+1/2}(k\tau)\right\|_{L^2(H^0)}^2\leqslant T^2\|f-f_\tau\|_{L^2(H^0)}^2.$$

Hence, using the definitions (4.6 b, d) we obtain by Lemma 3.1 and Lemma 3.2

$$\|\varepsilon_\tau\|_{L^2(H^0)}^2\leqslant\|\partial_\tau\eta_\tau\|_{L^2(H^0)}^2+\|\sigma_\tau\|_{L^2(H^0)}^2+T^2\|f-f_\tau\|_{L^2(H^0)}^2\leqslant$$

$$\leqslant C\left[h^2\left\|\frac{dy}{dt}\right\|_{L^2(H^1)}^2+\tau^2\left\|\frac{d^2y}{dt^2}\right\|_{L^2(H^0)}^2+\|f-f_\tau\|_{L^2(H^0)}^2\right], \tag{4.10}$$

since

$$(\partial_\tau\eta_\tau)(t)=\left(\overline{\frac{d\eta}{dt}}\right)_\tau(t).$$

The following inequality is derived taking advantage of (2.3), (4.5a), (4.6c) and (4.8).

$$2\tau^2\sum_{n=0}^{l-1}a(\rho_\tau(n\tau),\ \xi_{\tau+1/2}(n\tau))-\tau^2 a(\Delta y_\tau(0),\ \xi_{\tau+1/2}(0))=$$

$$=\tau^2\sum_{n=1}^{l-1}a\left(\sum_{k=0}^{n-1}\Delta y_\tau(k\tau)+\sum_{k=0}^{n}\Delta y_\tau(k\tau),\ \xi_{\tau+1/2}(n\tau)\right)=$$

$$=\tau^2 \sum_{k=0}^{l-2} a(\Delta y_\tau(n\tau), \sum_{k=n+1}^{l-1} \xi_{\tau+1/2}(k\tau)+ \sum_{k=n+2}^{l-1} \xi_{\tau+1/2}(k\tau))=$$

$$=\tau^2 \sum_{n=0}^{l-2} a(\Delta y_\tau(n\tau), 2\sum_{k=0}^{l-1} \xi_{\tau+1/2}(k\tau)- \sum_{n=0}^{n} \xi_{\tau+1/2}(k\tau)- \sum_{k=0}^{n+1} \xi_{\tau+1/2}(k\tau))\leqslant$$

$$\leqslant 4\alpha_1 \tau \sum_{n=0}^{l-2} \|\Delta y_\tau(n\tau)\|_1 \cdot \max_{1\leqslant l\leqslant N} \left\|\tau \sum_{k=0}^{l-1} \xi_{\tau+1/2}(k\tau)\right\|_1 \leqslant$$

$$\leqslant 8\frac{\alpha_1^2}{\alpha_0}\tau \sum_{n=0}^{N} \|\Delta y_\tau(n\tau)\|_1^2 + \frac{1}{2}\alpha_0 \max_{1\leqslant l\leqslant N} \left\|\tau \sum_{k=0}^{l-1} \xi_{\tau+1/2}(k\tau)\right\|_1^2 =$$

$$=8\frac{\alpha_1^2}{\alpha_0} \|\tilde{y}_{\tau+1/2}-\tilde{y}_\tau\|^2_{L^2(H^1)} + \frac{1}{2}\alpha_0 \max_{1\leqslant l\leqslant N} \left\|\tau \sum_{k=0}^{l-1} \xi_{\tau+1/2}(k\tau)\right\|_1^2, \tag{4.11}$$

where the constants α_0 and α_1 are defined by (2.3) and (2.4), respectively.

Note that by (4.2) ξ_τ is bounded in $L^\infty(0, T; H^1(\Omega))$ by a constant C independent on τ and h (compare the proof of Lemma 4.2 below). Hence, by (4.5a), (4.11) and Lemma 3.2, the last term of the right — hand side in (4.9) can be estimated as follows

$$2\tau^2 \sum_{n=0}^{l-1} a(\rho_\tau(n\tau), \xi_{\tau+1/2}(n\tau)) \leqslant C\tau^2 \left(\|y\|_{L^\infty(H^1)} + \left\|\frac{dy}{dt}\right\|^2_{L^2(H^1)}\right) +$$

$$+\frac{1}{2}\alpha_0 \max_{1\leqslant l\leqslant N} \left\|\tau \sum_{n=0}^{l-1} \xi_{\tau+1/2}(n\tau)\right\|_1^2. \tag{4.12}$$

It follows from (3.1), (4.1) and Lemma 3.1 that

$$\|\xi_\tau(0)\|_0 \leqslant \|\eta_\tau(0)\|_0 + \|y^1 - y_{h,\tau}(0)\|_0 \leqslant Ch\|y^1\|_1. \tag{4.13}$$

Combine (4.10), (4.12), (4.13) with (4.9) making a proper choice of l on the left in (4.9). This yields

$$\|\xi_\tau\|^2_{L^\infty(H^0)} + \alpha_0 \max_{1\leqslant l\leqslant N} \left\|\tau \sum_{n=0}^{l-1} \xi_{\tau+1/2}(n\tau)\right\|_1^2 \leqslant C\left[\tau^2\left(\|y\|_{L^\infty(H^1)} +\right.\right.$$

$$\left.\left.+\left\|\frac{dy}{dt}\right\|^2_{L^2(H^1)} + \left\|\frac{d^2y}{dt^2}\right\|^2_{L^2(H^0)}\right) + h^2\left(\|y^1\|_1^2 + \left\|\frac{dy}{dt}\right\|^2_{L^2(H^1)}\right) + \|f-f_\tau\|^2_{L^2(H^0)}\right] +$$

$$+\frac{1}{2}\|\xi_\tau\|^2_{L^\infty(H^0)} + \alpha_0 \max_{1\leqslant l\leqslant N} \left\|\tau \sum_{n=0}^{l-1} \xi_{\tau+1/1}(n\tau)\right\|_1^2.$$

Thus, making use of ıegularity assumptions (2.1a) and (4.2) we arrive at (4.3). □

In the case of more regular solutions we can establish the following

LEMMA 4.2. *Let*

$$y \in H^{3,3}(Q) \tag{4.14}$$

be the solution of (2.1) *and* $y_{h,\tau} \in E_\tau(0, T+\tau; V_h)$ *be the solution of* (3.9). *Then*

$$\|\vartheta_\tau\|_{L^\infty(H^0)} + \|\xi_\tau\|_{L^\infty(H^1)} \leqslant C(\tau + h + \|f - f_\tau\|_{L^2(H^0)}). \tag{4.15}$$

Proof. Let us substitute in (4.5) as a test function

$$v_h = \partial_\tau \xi_\tau + R_\tau \omega_h = \vartheta_{\tau+1/2},$$

where

$$R_\tau y = \partial_\tau \bar{y}_\tau - \left(\overline{\frac{dy}{dt}}\right)_{\tau+1/2} \qquad \forall y, \frac{dy}{dt} \in C([0, T]; V). \tag{4.16}$$

Then, by some elementary transfoımations we obtain

$$\frac{1}{2}[\|\vartheta_\tau((n+1)\tau)\|_0^2 - \|\vartheta_\tau(n\tau)\|_0^2 + a(\xi_\tau((n+1)\tau), \xi_\tau((n+1)\tau)) +$$

$$-a(\xi_\tau(n\tau), \xi_\tau(n\tau))] =$$

$$= \left(\partial_\tau\left(\overline{\frac{d\eta}{dt}}\right)_\tau(n\tau) + r_\tau(n\tau), \vartheta_{\tau+1/2}(n\tau)\right) - a(\xi_{\tau+1/2}(n\tau), (R_\tau\omega_h)(n\tau)) =$$

$$= \left(\partial_\tau\left(\overline{\frac{d\eta}{dt}}\right)_\tau(n\tau) + r_\tau(n\tau), \vartheta_{\tau+1/2}(n\tau)\right) - a(\xi_{\tau+1/2}(n\tau), (R_\tau y)(n\tau)). \tag{4.17}$$

Applying to (4.17) a standard procedure we obtain a supremum-norm inequality, that by (4.8) can be modified as follows:

$$\|v_\tau\|_{L^\infty(H^0)}^2 + \|\xi_\tau\|_{L^\infty(H^1)}^2 \leqslant \|\vartheta_\tau(0)\|_0^2 + \|\xi_\tau(0)\|_1^2 +$$

$$+C\left[\left(\left\|\partial_\tau\left(\overline{\frac{d\eta}{dt}}\right)_\tau\right\|_{L^2(H^0)} + \|r_\tau\|_{L^2(H^0)}\right) \cdot \|\vartheta_\tau\|_{L^\infty(H^0)} + \|R_\tau y\|_{L^2(H^1)} \cdot \|\xi_\tau\|_{L^\infty(H^1)}\right] \leqslant$$

$$\leqslant \|\vartheta_\tau(0)\|_0^2 + \|\xi_\tau(0)\|_1^2 +$$

$$+\frac{1}{2}C^2\left(\left\|\partial_\tau\left(\overline{\frac{d\eta}{dt}}\right)_\tau\right\|_{L^2(H^0)}^2 + \|r_\tau\|_{L^2(H^0)}^2 + \|R_\tau y\|_{L^2(H^1)}^2\right) +$$

$$+\frac{1}{2}\|\vartheta_\tau\|_{L^\infty(H^0)}^2 + \frac{1}{2}\|\xi_\tau\|_{L^\infty(H^1)}^2. \tag{4.18}$$

Observe that by (3.1), (4.5a), (4.16), Lemma 3.1 and Lemma 3.2 we have respectively

$$\|\vartheta_\tau(0)\|_0 \leqslant \left\|\frac{d\eta}{dt}(0)\right\|_0 + \|y^2 - z_{h,\tau}(0)\|_0 \leqslant Ch\|y^2\|_1,$$

$$\|\xi_\tau(0)\|_1 \leqslant \|\eta(0)\|_1 + \|y^1 - y_{h,\tau}(0)\|_1 \leqslant Ch\|y^1\|_2,$$

$$\left\|\partial_\tau\left(\overline{\frac{d\eta}{dt}}\right)_\tau\right\|_{L^2(H^0)} \leqslant \left\|\frac{d^2\eta}{dt^2}\right\|_{L^2(H^0)} \leqslant Ch\left\|\frac{d^2y}{dt^2}\right\|_{L^2(H^1)},$$

$$\|r_\tau\|_{L^2(H^0)} \leq C\|\Delta y_\tau\|_{L^2(H^2)} + \|\Delta f_\tau\|_{L^2(H^0)} \leq C\left(\tau\left\|\frac{dy}{dt}\right\|_{L^2(H^2)} + \|f-f_\tau\|_{L^2(H^0)}\right),$$

$$\|R_\tau y\|_{L^2(H^1)} \leq \left\|\partial_\tau \bar{y}_\tau - \left(\frac{\overline{dy}}{dt}\right)_{\tau+1/2}\right\|_{L^2(H^1)} \leq C\tau\left\|\frac{d^2y}{dt^2}\right\|_{L^2(H^1)}.$$

Substituting the above inequalities to (4.18) we obtain

$$\|\vartheta_\tau\|^2_{L^\infty(H^0)} + \|\xi_\tau\|^2_{L^\infty(H^1)} \leq C\left[h^2\left(\|y^1\|_2^2 + \|y^2\|_1^2 + \left\|\frac{d^2y}{dt^2}\right\|^2_{L^2(H^1)}\right) + \right.$$

$$\left. + \tau^2\left(\left\|\frac{dy}{dt}\right\|^2_{L^2(H^2)} + \left\|\frac{d^2y}{dt^2}\right\|^2_{L^2(H^1)}\right) + \|f-f_\tau\|^2_{L^2(H^0)}\right]. \tag{4.19}$$

Applying to (4.19) the regularity assumptions (2.1a) and (4.14), we get (4.15). □

Note that the analogous result formulated in [2] for a similar three-level Galerkin scheme provides H^1 — estimates in terms of $\xi_{\tau+1/2}$ at some additional requirements concerning the initial values in the discrete scheme. Thus, the result of Lemma 4.2 is slightly stronger and more convenient from the standpoint of applications.

5. Convergence of the optimal controls

In this section we estimate the rate of convergence of the discrete optimal control $u^0_{k,\tau}$ to the continuous one u^0 — the solutions of the problems (D) and (P), respectively. We shall apply the known Lagrange formalism technique [7, 8] based on comparing of the saddle point conditions for these two problems.

For this purpose we define a discrete Lagrange functional

$$L_d\colon E_\tau(0, T; W_k) \times E_\tau(0, T+\tau; V_h) \times E_\tau(0, T; Z_k) \times \hat{E}_\tau(-\tau, T; V_h) \to R^1$$

putting

$$L_d(u_{k,\tau}, y_{h,\tau}, \lambda_{k,\tau}, p_{h,\tau}) \stackrel{\text{df}}{=} J(u_{k,\tau}, y_{h,\tau}) + ((\partial_\tau z_{h,\tau}, p_{h,\tau})) +$$

$$+ a(y_{h,\tau+1/2}, p_{h,\tau}) - ((Bu_{k,\tau}, p_{h,\tau})) + ((\psi u_{k,\tau}, \lambda_{k,\tau})), \tag{5.1}$$

where

$$z_{h,\tau+1/2} = \partial_\tau y_{h,\tau}, \tag{5.1a}$$

provided that $z_{h,\tau}(0)$ — given

Note that by (2.14) Lagrangian (5.1) represents a finite dimensional convex programming problem. Hence, by regularity of the constraints there exist [3] uniquely defined Lagrange multipliers $p^0_{h,\tau}$, $\lambda^0_{k,\tau}$ such that the following Kuhn–Tucker conditions hold:

$$((\delta_y L_d(u^0_{k,\tau}, y^0_{h,\tau}, \lambda^0_{k,\tau}, p^0_{h,\tau}), y_{h,\tau} - y^0_{h,\tau})) = 0 \tag{5.2a}$$

$$\forall y_{h,\tau} \in E_\tau(0, T+\tau; V_h), \quad y_{h,\tau}(0) = y^1_{h,\tau}, \quad z_{h,\tau}(0) = z^2_{h,\tau},$$

$$((\delta_u L_d(u^0_{k,\tau}, y^0_{h,\tau}, \lambda^0_{k,\tau}, p^0_{h,\tau}), u_{k\tau} - u^0_{k,\tau})) = 0 \quad \forall u_{k,\tau} \in E_\tau(0, T; W_k), \tag{5.2b}$$

$$((\psi u^0_{k,\tau}, \lambda^0_{k,\tau}))=0, \quad \lambda^0_{k,\tau}\geqslant 0, \tag{5.2c}$$

where $u^0_{k,\tau}$, is a solution of (D) and $y^0_{h,\tau}=y_{h,\tau}(u^0_{k,\tau})$.

By definition (5.1), condition (5.2a) can be expressed in the form

$$((\partial_\tau z_{h,\tau}, p^0_{h,\tau}))+a(y_{h,\tau+1/2}, p^0_{h,\tau})=-((\delta_y J(u^0_{k,\tau}, y^0_{h,\tau}), y_{h,\tau})),$$

$$z_{h,\tau+1/2}=\partial_\tau y_{h,\tau} \tag{5.3}$$

$$\forall y_{h,\tau}\in(E_\tau 0, T+\tau; V_h), \quad y_{h,\tau}(0)=0, \quad z_{h,\tau}(0)=0. \tag{5.3a}$$

Let us assume

$$q^0_{h,\tau-1/2}=\bar{\partial}_\tau p^0_{h,\tau}, \tag{5.4a}$$

where

$$p^0_{h,\tau}(T)=0, \quad q^0_{h,\tau}(T)=0 \tag{5.4b}$$

It can be easily checked that by (5.3a) and (5.4) the following identities hold

$$((\partial_\tau z_{h,\tau}, p^0_{h,\tau}))=-((z_{h,\tau}, \bar{\partial}_\tau p^0_{h,\tau}))=-((z_{h,\tau}, q^0_{h,\tau-1/2}))=$$

$$=-((z_{h,\tau+1/2}, q^0_{h,\tau}))=-((\partial_\tau y_{h,\tau}, q^0_{h,\tau}))=((y_{h,\tau}, \bar{\partial}_\tau q^0_{h,\tau})), \tag{5.5}$$

$$a(y_{h,\tau+1/2}, p^0_{h,\tau})=a(y_{h,\tau}, p^0_{h,\tau-1/2}). \tag{5.6}$$

Making use of (5.4)–(5.6) in (5.3) we obtain the following discrete adjoint equation

$$(\bar{\partial}_\tau q^0_{h,\tau}(t), v_h)+a(p^0_{h,\tau-1/2}, v_h)=-(\delta_y J(u^0_{k,\tau}, y^0_{h,\tau})(t), v_k)$$

$$\forall v_h\in V_h, \ t\in[0,T); \quad q^0_{h,\tau-1/2}=\bar{\partial}_\tau p^0_{h,\tau}, \tag{5.7}$$

where

$$p^0_{h,\tau}(T)=0, \quad q^0_{h,\tau}(T)=0. \tag{5.7a}$$

The formulated above optimality conditions reduce to the respective saddle point conditions [3] for Lagrangian (5.1). It means that $u^0_{k,\tau}$ and $y^0_{h,\tau}=y_{h,\tau}(u^0_{k,\tau})$ are the solutions of (D) iff there exist Lagrange multipliers $p^0_{h,\tau}$, $\lambda^0_{k,\tau}$ such that

$$L_d(u^0_{k,\tau}, y^0_{h,\tau}, \lambda_{k,\tau}, p_{h,\tau})\leqslant L_d(u^0_{k,\tau}, y^0_{h,\tau}, \lambda^0_{k,\tau}, p^0_{h,\tau})\leqslant$$

$$\leqslant L_d(u_{k,\tau}, y_{h,\tau}, \lambda^0_{k,\tau}, p^0_{h,\tau})$$

$$\forall u_{k,\tau}\in E_\tau(0,T; W_k),$$

$$\forall y_{h,\tau}\in E_\tau(0, T+\tau; V_h), \quad y_{h,\tau}(0)=y^1_{h,\tau}, \quad z_{h,\tau}(0)=z^2_{h,\tau},$$

$$\forall \lambda_{k,\tau}\in E_\tau(0,T; Z_k), \quad \lambda_{k,\tau}\geqslant 0,$$

$$\forall p_{h\tau}\in \hat{E}_\tau(-\tau, T; V_h), \quad p_{h,\tau}(T)=0, \quad q_{h,\tau}(T)=0. \tag{5.8}$$

Combining (5.8) and the optimality conditions for (P) we shall estimate the rate of convergence of the sequence $\{u^0_{k,\tau}\}$ to the optimal solution u^0. According to the notation of Section 3 we denote in the sequel

$$\tilde{u}^0_{k,\tau}=\tilde{P}_\tau \tilde{P}_k u^0 \in E_\tau(0, T; W_k),$$
$$\tilde{\lambda}^0_{k,\tau}=\tilde{P}_\tau \tilde{P}_k \lambda^0 \in E_\tau(0, T; Z_k),$$
$$\tilde{y}^0_{h,\tau}=\tilde{P}_\tau \tilde{P}_h y^0 \in E_\tau(0, T+\tau; V_h), (\text{resp. } p^0_{h,\tau}=\hat{P}_\tau \tilde{P}_h p^0),$$
$$\omega^y_{h,\tau}=\tilde{P}_\tau \omega^y_h \in E_\tau(0, T+\tau; V_h), (\text{resp. } \omega^p_{h,\tau}=\hat{P}_\tau \omega^p_h),$$
$$\xi^y_\tau=y^0_{h,\tau}-\omega^y_{h,\tau}, \quad \xi^p_\tau=p^0_{h,\tau}-\omega^p_{h,\tau},$$
$$\eta^y=y^0-\omega^y_h, \quad \eta^p=p^0-\omega^p_h,$$

where ω^y_h and ω^p_h are the elliptic projections of y^0 and p^0 respectively, defined in Lemma 3.1.

It can be shown [8] by (5.1), (5.8) and the definition of $\tilde{\lambda}^0_{k,\tau}$ that

$$L_d(u^0_{h,\tau}, y^0_{k,\tau}, \lambda^0_{k,\tau}, p^0_{h,\tau}) \geq L_d(u^0_{k,\tau}, y^0_{h,\tau}, \lambda^0, p_{h,\tau}), \tag{5.9}$$

since the domain of $L_d(u^0_{k,\tau}, y^0_{h,\tau}, \cdot, p^0_{h,\tau})$ can be extended in a nautral way to the whole space $L^2(0, T; H^0(\Omega))$.

Expanding $L_d(\cdot, \cdot, \lambda^0, \omega^p_{h,\tau-1/2})$ into Taylor series at $(u^0_{k,\tau}, \omega^y_{h,\tau+1/2})$, using (5.9), (2.14), the left hand side inequality in (5.8) and exploiting affinity of ψ—we obtain

$$\begin{aligned} L_d(u^0_{k,\tau}, y^0_{h,\tau}, \lambda^0_{k,\tau}, p^0_{h,\tau}) &\geq L_d(\tilde{u}^0_{k,\tau}, \omega^y_{h,\tau+1/2}, \lambda^0, \omega^p_{h,\tau-1/2})+ \\ &+((\delta_u L_d(\tilde{u}^0_{k,\tau}, \omega^y_{h,\tau+1/2}, \lambda^0, \omega^p_{h,\tau-1/2}), u^0_{k,\tau}-\tilde{u}^0_{k,\tau}))+ \\ &+((\delta_y L_d(\tilde{u}^0_{k,\tau}, \omega^y_{h,\tau+1/2}, \lambda^0, \omega^p_{h,\tau-1/2}), y^0_{h,\tau}-\omega^y_{h,\tau+1/2}))+ \\ &+\alpha\|u^0_{k,\tau}-\tilde{u}^0_{k,\tau}\|^2_{L^2(H^0)}. \end{aligned} \tag{5.10}$$

Combining (5.10) and the estimation of $L_d(u^0_{k,\tau}, y^0_{h,\tau}, \lambda^0_{k,\tau}, p^0_{h,\tau})$ from above, following directly from (5.8), we get

$$\begin{aligned} \alpha\|u^0_{k,\tau}-\tilde{u}^0_{k,\tau}\|^2_{L^2(H^0)} &\leq [L_d(\tilde{u}^0_{k,\tau}, \omega^y_{h,\tau+1/2}, \lambda^0_{k,\tau}, p^0_{h,\tau})+ \\ &-L_d(\tilde{u}^0_{k,\tau}, \omega^y_{h,\tau+1/2}, \lambda^0, \omega^p_{h,\tau-1/2})]+ \\ &+((\delta_u L_d(\tilde{u}^0_{k,\tau}, \omega^y_{h,\tau+1/2}, \lambda^0, \omega^p_{h,\tau-1/2}), \tilde{u}^0_{k,\tau}-u^0_{k,\tau}))+ \\ &+((\delta_y L_d(\tilde{u}^0_{k,\tau}, \omega^y_{h,\tau+1/2}, \lambda^0, \omega^p_{h,\tau-1/2}), \omega^y_{h,\tau+1/2}-y^0_{h,\tau})). \end{aligned} \tag{5.11}$$

Now we must estimate all the components of the right-hand side in (5.11). For notation simplicity let us denote the first term by ΔL.

It follows from (5.1) that

$$\begin{aligned} \Delta L \leq &((\partial^2_\tau \omega^y_{h,\tau}-B\tilde{u}^0_{k,\tau}, p^0_{h,\tau}-\omega^p_{h,\tau-1/2}))+ \\ &+a(\omega^y_{h,\tau/4}, p^0_{h,\tau}-\omega^p_{h,\tau-1/2})+((\psi(\tilde{u}^0_{k,\tau}-u^0), \lambda^0_{k,\tau}-\lambda^0)), \end{aligned} \tag{5.12}$$

since by (2.19) and (5.2c)

$$-((\psi(u^0), \lambda^0_{k,\tau}-\lambda^0)) \geq 0.$$

Note that averaging the equation (2.1) by means of the operation (3.4) we obtain

$$0=\left(\left(\left(\overline{\frac{d^2y^0}{dt^2}}\right)_\tau - B\tilde{u}^0_\tau,\ p^0_{h,\tau}-\omega^p_{h,\tau-1/2}\right)\right)+\alpha\,(\tilde{y}^0_\tau,\ p^0_{h,\tau}-\omega^p_{h,\tau-1/2})\,. \tag{5.13}$$

Substracting now (5.13) from (5.12) and utilizing (4.1), (5.5) and Lemma 3.1 we get

$$\begin{aligned}\Delta L\leqslant&\left(\left(\partial_\tau\left[\left(\overline{\frac{d\omega^y_h}{dt}}\right)_\tau-\left(\overline{\frac{dy^0}{dt}}\right)_\tau\right],\ p^0_{h,\tau}-\omega^p_{h,\tau-1/2}\right)\right)+\\
&-((B(\tilde{u}^0_{k,\tau}-\tilde{u}^0_\tau),\ p^0_{h,\tau}-\omega^p_{h,\tau-1/2}))+\\
&+\alpha\,(\omega^y_{h,\tau/4}-\tilde{y}^0_\tau,\ p^0_{h,\tau}-\omega^p_{h,\tau-1/2})+\\
&+((\psi(u^0-\tilde{u}^0_{k,\tau}),\ \lambda^0-\lambda^0_{k,\tau}))=\\
=&\left(\left(\left(\overline{\frac{dy^0}{dt}}\right)_\tau-\left(\overline{\frac{dy^0}{dt}}\right)_\tau-\left(\overline{\frac{d\eta^y}{dt}}\right)_\tau,\ \bar{\partial}_\tau(p^0_{h,\tau}-\omega^p_{h,\tau-1/2})\right)\right)+\\
&-((B(\tilde{u}^0_{k,\tau}-\tilde{u}^0_\tau),\ p^0_{h,\tau}-\omega^p_{h,\tau-1/2}))+\\
&+\alpha\,(\tilde{y}^0_{\tau/4}-\tilde{y}^0_\tau,\ p^0_{h,\tau}-\omega^p_{h,\tau-1/2})+\\
&+((\psi(u^0-\tilde{u}^0_{k,\tau}),\ \lambda^0-\lambda^0_{k,\tau}))\,.\end{aligned} \tag{5.14}$$

Noting that by Lemma 3.1 and Lemma 3.2

$$\begin{aligned}\|\bar{\partial}_\tau(\omega^p_{h,\tau}-\omega^p_{h,\tau-1/2})\|_{L^2(H^0)}&\leqslant C\tau\left\|\frac{d^2\omega^p_h}{dt^2}\right\|_{L^2(H^0)}\leqslant\\
&\leqslant C\tau\left(h\left\|\frac{d^2p^0}{dt^2}\right\|_{L^2(H^1)}+\left\|\frac{d^2p^0}{dt^2}\right\|_{L^2(H^0)}\right),\\
\|\omega^p_{h,\tau}-\omega^p_{h,\tau-1/2}\|_{L^2(H^s)}&\geqslant C\tau\left\|\frac{d\omega^p_h}{dt}\right\|_{L^2(H^s)}\leqslant\\
&\leqslant C\tau\left(h^{1-s}\left\|\frac{dp^0}{dt}\right\|_{L^2(H^1)}+\left\|\frac{dp^0}{dt}\right\|_{L^2(H^s)}\right)\quad\text{for } s=0,1\end{aligned} \tag{5.15}$$

and applying the norm inequality to (5.14) we obtain

$$\begin{aligned}|\Delta L|\leqslant C&\left[\left\|\left(\overline{\frac{dy^0}{dt}}\right)_\tau-\left(\overline{\frac{dy^0}{dt}}\right)_\tau\right\|_{L^2(H^0)}\cdot\left(\|\bar{\partial}_\tau\xi^p_\tau\|_{L^2(H^0)}+\tau(h+1)\left\|\frac{d^2p^0}{dt^2}\right\|_{L^2(H^1)}\right)+\right.\\
&+\left\|\frac{d\eta^y}{dt}\right\|_{L^2(H^0)}\cdot\left(\|\bar{\partial}_\tau\xi^p_\tau\|_{L^2(H^0)}+\tau(h+1)\left\|\frac{d^2p^0}{dt^2}\right\|_{L^2(H^1)}\right)+\\
&+\|\tilde{u}^0_\tau-\tilde{u}^0_{k,\tau}\|_{L^2(H^0)}\cdot\left(\|\xi^p_\tau\|_{L^2(H^0)}+\tau(h+1)\left\|\frac{dp^0}{dt}\right\|_{L^2(H^1)}\right)+\\
&+\|\tilde{y}^0_\tau-\tilde{y}^0_{\tau/4}\|_{L^2(H^1)}\cdot\left(\|\xi^p_\tau\|_{L^2(H^1)}+\left\|\frac{dp^0}{dt}\right\|_{L^2(H^1)}\right)+\\
&\left.+\|u^0-\tilde{u}^0_{k,\tau}\|_{L^2(H^0)}\cdot\|\lambda^0-\lambda^0_{k,\tau}\|_{L^2(H^0)}\right]\,.\end{aligned} \tag{5.16}$$

Applying (2.18), (4.1) and Lipschitz continuity of $\delta_u J(\cdot, \cdot)$ on bounded sets by the same arguments as those used in [8], we get

$$|((\delta_u L_d(\tilde{u}^0_{k,\tau}, \omega^y_{h,\tau+1/2}, \lambda^0, \omega^p_{h,\tau-1/2}), \tilde{u}^0_{k,\tau}-u^0_{k,\tau}))| \leqslant C\,[\|u^0-\tilde{u}^0_{k,\tau}\|_{L^2(H^0)}+ \\ +\|y^0-\omega^y_{h,\tau+1/2}\|_{L^2(H^0)}+\|p^0-\omega^p_{h,\tau-1/2}\|_{L^2(H^0)}]\cdot\|u^0_{k,\tau}-\tilde{u}^0_{k,\tau}\|_{L^2(H^0)}. \quad (5.17)$$

To handle the last term in (5.11) let us modify that expression utilizing the adjoint equation (2.17). Thus we have

$$((\delta_y L_d(\tilde{u}^0_{k,\tau}, \omega^y_{h,\tau+1/2}, \lambda^0, \omega^p_{h,\tau-1/2}), y^0_{h,\tau}-\omega^y_{h,\tau+1/2}))= \\ =((\delta_y J(\tilde{u}^0_{k,\tau}, \omega^y_{h,\tau+1/2})-\delta_y J(u^0, y^0), y^0_{h,\tau}-\omega^y_{h,\tau+1/2}))+ \\ +\left(\left(\frac{d^2p^0}{dt^2}-\partial^2_\tau\,\omega^p_{h,\tau}, y^0_{h,\tau}-\omega^y_{h,\tau+1/2}\right)\right)+a\,(p^0-\omega^p_{h,\tau/4}, y^0_{h,\tau}-\omega^y_{h,\tau+1/2}).$$

From this identity, regarding Lipschitz continuity of $\delta_y J(\cdot, \cdot)$ on bounded sets of arguments and relations of the form (5.15) applied to $\omega^y_{h,\tau}$, we get

$$|((\delta_y L_d(\tilde{u}^0_{k,\tau}, \omega^y_{h,\tau+1/2}, \lambda^0, \omega^p_{h,\tau-1/2}), y^0_{h,\tau}-\omega^y_{h,\tau+1/2}))| \leqslant \\ \leqslant C\left[\left\|\frac{d^2p^0}{dt^2}-\partial^2_\tau\,\omega^p_{h,\tau}\right\|_{L^2(H^0)}+\|p^0-\hat{p}^0_{\tau/4}\|_{L^2(H^2)}+ \right. \\ \left. +\|u^0-\tilde{u}^0{}_{k,\tau}\|_{L^2(H^0)}+\|y^0-\omega^y_{h,\tau+1/2}\|_{L^2(H^0)}\right]\cdot\|y^0_{h,\tau}-\omega^y_{h,\tau+1/2}\|_{L^2(H^0)} \leqslant \\ \leqslant C\left[\left\|\frac{d^2p^0}{dt^2}-\partial^2_\tau\,p^0_{h,\tau}\right\|_{L^2(H^0)}+\|\partial^2_\tau\,\eta^p_\tau\|_{L^2(H^0)}+ \right. \\ \left. +\|p^0-\hat{p}^0_{\tau/4}\|_{L^2(H^2)}+\|y^0-\omega^y_{h,\tau+1/2}\|_{L^2(H^0)}+\|u^0-\tilde{u}^0_{k,\tau}\|_{L^2(H^0)}\right]\cdot \\ \cdot\left[\|\xi^y_\tau\|_{L^2(H^0)}+\tau\,(h+1)\left\|\frac{dy^0}{dt}\right\|_{L^2(H^1)}\right]. \quad (5.18)$$

Substitute now (5.16)—(5.18) to (5.11) observing that [8]

$$\|\lambda^0-\lambda^0_{k,\tau}\|_{L^2(H^0)} \leqslant C\,[\|u^0-u^0_{k,\tau}\|_{L^2(H^0)}+\|y^0-y^0_{h,\tau}\|_{L^2(H^0)}+\|B^*p^0-B^*p^0_{h,\tau}\|_{L^2(H^0)}]$$

and [2]

$$\|\partial^2_\tau\eta^p_\tau\|_{L^2(H^0)} \leqslant \frac{4}{3}\left\|\frac{d^2\eta^p}{dt^2}\right\|_{L^2(H^0)}.$$

Utilizing the properties of approximation formulated in Lemma 3.1 and Lemma 3.2 as well as regularity of the optimal solutions asserted by Theorem 2.1, we obtain the following estimation:

$$\|u^0-u^0_{k,\tau}\|^2_{L^2(H^0)} \leqslant C\,(\tau+h+k)\,[\|\xi^y_\tau\|_{L^2(H^0)}+\|\xi^p_\tau\|_{L^2(H^1)}+ \\ +\|\bar{\partial}_\tau\,\xi^p_\tau\|_{L^2(H^0)}+\|u^0-u^0_{k,\tau}\|_{L^2(H^0)}+(\tau+h+k)]. \quad (5.19)$$

We are now in a position to estimate the rate of convergence of the finite- dimensional approximation applied to (P). The following theorem provides it in terms of powers of the respective discretization parameters.

THEOREM 5.1. *Let* (u^0, y^0) *and* $(u^0_{k,\tau}, y^0_{h,\tau})$ *are the solutions to (P) and (D), respectively. If* (2.21) *holds then*

$$\|u^0-u^0_{k,\tau}\|_{L^2(H^0)}=0\,(\tau+h+k)\,, \tag{5.20a}$$

$$\|y^0-y^0_{h,\tau}\|_{L^\infty(H^0)}=0\,(\tau+h+k)\,. \tag{5.20b}$$

Proof. Observe that by (4.1), (5.15), Lemma 3.1 and Lemma 3.2 we have

$$\|\bar{\partial}_\tau\,\xi^p_\tau\|_{L^2(H^0)}\leqslant\|\vartheta^p_{\tau-1/2}\|_{L^2(H^0)}+\left\|\bar{\partial}_\tau\,\omega^p_{h,\tau}-\left(\frac{d\omega^p_h}{dt}\right)_{\tau-1/2}\right\|_{L^2(H^0)}\leqslant$$

$$\leqslant C\left[\|\vartheta^p_\tau\|_{L^\infty(H^0)}+\tau\,(h+1)\left\|\frac{d^2p^0}{dt^2}\right\|_{L^2(H^1)}\right]. \tag{5.21}$$

Hence, substituting (5.21) to (5.19) and applying Lemma 4.1, Lemma 4.2 and (2.12) we get (5.20a).

Taking advantage of (5.20a) and (2.12) in Lemma 4.1, we arrive at (5.20b). □

An important feature of the above considerations is that Lagrangian (5.1) is expanded into Taylor series in a neighbourhood of $\omega^y_{h,\tau}$ and $\omega^p_{h,\tau}$, i.e. the elliptic projections of the optimal state variables.

Such a proceeding, along with the use of Lemma 3.1, allows to avoid $L^2(H^1)$— estimates of the state approximation error. Observe in particular [4] that in unconstrained hyperbolic problem, for sufficiently smooth initial data and the more general form of the cost functional, we have $y^0, p^0 \in H^{3,3}(Q)$. Thus, following technique of Section 5 we easily get

COROLLARY 5.1. *Consider (P) for* $U_a=U$ *and for* (2.11c) *neglected. If* (2.21) *holds and moreover*

$$y^1 \in H^{7/2}(\Omega), \qquad y^2 \in H^{5/2}(\Omega),$$

$$B \in \mathscr{L}(H^r(\Omega),\ H^r(\Omega)) \qquad \text{for } r=0,1,2$$

then

$$\|u^0-u^0_{k,\tau}\|_{L^2(H^0)}=0\,(\tau+h+k)\,,$$

$$\|y^0-y^0_{h,\tau}\|_{L^\infty(H^1)}=0\,(\tau+h+k)\,. \qquad \square$$

In comparison with the result of [4], the special choice of starting values in a discrete scheme is not required.

The above remarks have more general sens and apply also to parabolic problems. In particular, it is evident that the final result of [8] is obtainable in this way without using $L^2(H^1)$— estimates of the state approximation error.

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Received, June 1980.

Zbieżność aproksymacji typu Galerkina dla zadań sterowania optymalnego systemami hiperbolicznymi przy ograniczeniach na sterowanie

Badana jest dyskretna względem czasu aproksymacja Galerkina dla klasy zadań sterowania optymalnego układami opisywanymi równaniem hiperbolicznym drugiego rzędu z całkowym wskaźnikiem jakości i przy ograniczeniach amplitudowych na sterowanie. Określono regularność rozwiązań optymalnych, przy której udowodniono liniową zbieżność aproksymacji równania stanu w odpowiednich normach. Wykorzystując uzyskane wyniki oraz stosując formalizm Lagrange'a wykazano liniową zbieżność ciągu sterowań przybliżonych do sterowania optymalnego.

Сходимость аппроксимации типа Галеркина для задач оптимального управления гиперболическими системами при ограничениях на управление

Рассматривается дискретная по времени аппроксимация типа Галеркина для выпуклой задачи оптимального управления системой, описываемой гиперболическим уравнением второго порядка, при локальных ограничениях на амплитуду управления. Получены условия регулярности для оптимальных решений исходной задачи и оценки сходимости аппроксимации уравнения состояния. Используя эти результаты доказана линейна сходимость последовательности приближенных управлений к решению исходной задачи.