Control
and Cybernetics

VOL. 9 (1980) No. 3

A certain method of multivariable diagonal controller design

by

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A sequential method of multivariable diagonal controller synthesis with regard to disturbance damping as performance index is presented. The method allows to obtain in one step, not iteratively, a quantitatively stated design objective or to verify that the objective is not attainable in the given class of controllers. A way of implementation is described and numerical example given.

Notations:

$\det(G)_i^j$ ... minor of the matrix G created by deleting i-th row and j-th column.

$$m(f)=\inf_{\omega\in[0,\omega_a]} |f(j\omega)| \quad \text{where } j^2=-1$$

$$M(f)=\sup_{\omega\in[0,\omega_a]} |f(j\omega)|$$

$$\|q(j\omega)\|=M(q_1)+\ldots+M(q_n) \text{ for } q(j\omega)=(q_1(j\omega), \ldots, q_n(j\omega))$$

$$f(q)=\mathrm{o}(q^p) \text{ iff } p=\min\left\{k\text{-integer} \,\Big|\, \lim_{\|q\|\to 0}\frac{|f(q)|}{\|q\|^k}=0\right\}$$

$$\hat{G}=G^{-1}=[\hat{g}_{ij}]_{i,j=1,\ldots n}$$

1. Introduction

One of the main roles in control system design plays the so called British School Its leading representatives are Rosenbrock, MacFarlane, Mayne. The school may be characterised by a tendency to decompose the multivariable controller synthesis into a sequence of scalar problems. The inverse Nyquist array method (Rosenbrock) or the return differences method (Mayne) are typical examples. Recently the paper [6] inspired from the same philosophy has appeared. In all these methods the final step is a synthesis of several scalar close-loop control systems with the requirement that each loop should be "tight". Thus the stability is established, but the performance of the resultant multivariable control system with respect to disturbance damping or steady-state error or another criterion can not be predicted exactly. The only possible predictions are of qualitative nature.

The aim of this paper is to present a method which makes it possible to choose diagonal controller in a given class (for example PID) in such a way, that certain quantitative performance requirements are satisfied.

It should be stressed, that quantitative formulation of performance index is a feature which distinguishes the proposed method from all mentioned above.

2. Statement of the problem

Let us consider linear, stationary, n-input, n-output plant described with the transfer function matrix

$$G(s) = [g_{ij}(s)]_{i,j=1,\cdots n} \tag{1}$$

From here on we shall use s instead of $j\omega$ or omit argument in order to shorten notation.

Let the diagonal controller

$$R(s) = \operatorname{diag}\{r_1(s), \dots, r_n(s)\} \tag{2}$$

be applied and disturbances act additively on plant's outputs. As a result we obtain standard control system shown on Fig. 1. $U(s)$ and $E(s)$ are input vector function and error vector function transforms respectively.

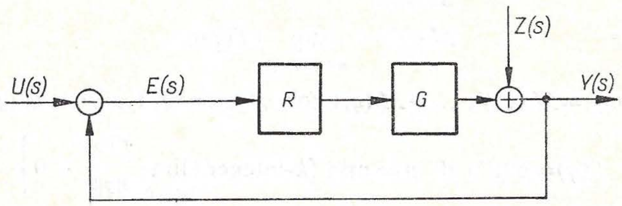

Fig. 1. The control system configuration.

If the system is stable, then its performance can be represented in the frequency domain by transfer function matrix

$$Q(j\omega) = [q_{ij}(j\omega)]_{i,j=1,\cdots n} = (I+GR)^{-1} \tag{3}$$

which may be interpreted as error transfer function matrix or disturbance dampings matrix.

As a goal of controller synthesis we shall consider the obtaining of stable system, in which the elements of matrix Q, namely $q_{ij}(j\omega)$, will have in a given frequency range $[0, \omega_a]$ small gains.

Before we formulate the synthesis goal in a precise quantitative manner, some basic relationships must be given.

3. Basic relationships

At the begining we shall introduse notions of autonomous and diagonal dampings. Assume, that for the given plant all interactions are neglected, $g_{ij}(s)=0$ for $i \neq j$. Then, after applying diagonal controller, the system will split into n independent loops. Disturbance damping in each loop will be represented by function

$$q_i(s) = \frac{1}{1+g_{ii}(s)\, r_i(s)}, \quad i=1, \dots, n. \tag{4}$$

These functions will form a vector

$$q(s)=(q_1(s), \dots, q_n(s))^T \tag{5}$$

of autonomous dampings.

If interactions are taken into account, the matrix Q will not be diagonal. In this case its diagonal elements will be called diagonal dampings.

Let us introduce the following denotions:

$$A=\frac{\det G}{\prod\limits_{k=1}^{n} g_{kk}} \tag{6}$$

$$A_i=\frac{\det (G)_i^i}{\prod\limits_{\substack{k=1\\k\neq i}}^{n} g_{kk}}, \quad i=1, \dots, n. \tag{7}$$

$$B=\frac{\det (I+GR)}{\prod\limits_{k=1}^{n} (1+g_{kk}\, r_k)} \tag{8}$$

A, A_i and B are functions of frequency. Now we shall formulate the main theorem, proved in [2].

THEOREM 1. *For the given control system the following relationships hold*:

$$q_{ii}=\frac{A_i}{B}\, q_i+o(q), \quad i=1, \dots, n. \tag{9}$$

$$q_{ij}=(-1)^{i+j}\frac{\det (G)_j^i}{\det (G)_i^i}\, q_{ii}+o(q); \quad \underset{i\neq j}{i, j=1, \dots, n.} \tag{10}$$

$$B=A\left(1-\sum_{i=1}^{n} q_i\right)+\sum_{i=1}^{n} A_i\, q_i+o(q) \tag{11}$$

for all $\omega \in [0, \omega_a]$.

By means fo this theorem we shall estimate gains of elements of matrix Q, but to do so, we have to presuppose that autonomous dampings are in a given frequency range small. In a formulated farther algorithm the supposition will allways hold.

4. Quantitative formulation of synthesis problem

The Theorem 1 implies that nondiagonal elements of matrix Q are approximately proportional to diagonal dampings. Let us note, that proportionality factors do not depend on a type of controller used. They are simply ratios of the elements of the matrix inverse to G.

$$|q_{ij}|=\frac{|\hat{g}_{ij}|}{|\hat{g}_{ii}|}\,|q_{ii}| \qquad i,j=1,\ldots,n \tag{12}$$

If we calculate the values of $M(\hat{g}_{ij}/\hat{g}_{ii})$ then we shall easily find for every set of numbers $\delta_{ij}>0$, $i,j=1,\ldots,n$ such $\delta_i>0$, $i=1,\ldots,n$ that

$$\underset{i\leqslant n}{\forall}\; M(q_{ii})\leqslant\delta_i \Rightarrow \underset{i,j\leqslant n}{\forall}\; M(q_{ij})\leqslant\delta_{ij}. \tag{13}$$

It means that in order to achieve small gains of elements of matrix Q it is enough to impose constraints on diagonal dampings only.

Now we are able to give ultimate formulation of design problem: find diagonal controller in such a way that the close-loop system is stable and diagonal dampings fulfil inequalities

$$\sup_{\omega\in[0,\,\omega_a]} |q_{ii}(j\omega)|=M(q_{ii})\leqslant\delta_i \qquad i=1,\ldots,n.$$

where ω_a, $\delta_i>0$ are given numbers.

5. Sufficient conditions

In [2] the following theorem was proved:

THEOREM 2. *Let* $M(q_i)=x_i$, $\quad i=1,\ldots,n$.

If the numbers x_i *satisfy the set of inequalities*

$$M(A_i)\,x_i+\sum_{k=1}^{n}(m(A)\,\delta_i+M(A_k)\,\delta_i)\,x_k\leqslant\delta_i\,m(A), \qquad i=1,\ldots,n. \tag{14}$$

then diagonal dampings are in the given frequency range $[0,\omega_a]$ *bounded by* δ_i

$$M(q_{ii})\leqslant\delta_i \qquad i=1,\ldots,n$$

Let us notice, that (14) implies inequality

$$x_i\leqslant\frac{m(A)\,\delta_i}{M(A_i)(1+\delta_i)+m(A)\,\delta_i} \tag{15}$$

for every $i=1, \ldots, n$. It means, that if upper bounds of diagonal dampings δ_i are small, then upper bounds of autonomous dampings are small as well.

Assume, that we have found set of numbers $x_i^* > 0$, $i=1, \ldots, n$ satisfying constraints (14) and a diagonal controller with such gains that

$$M(q_i) \leqslant x_i^* \quad i=1, \ldots, n \tag{16}$$

Then we can omit $o(q)$ on right hand sides in (9), (10), (11) and use evaluations of the Theorem 1.

6. Stability

In order to achieve stability of the close-loop system the return differences method (Mayne, [3]) was used, proved also in a slighty generalized version in [2].

Let us define the sequence of matrices R_k, $k \leqslant n$

$$R_k = \operatorname{diag}\{r_1, \ldots, r_{k-1}, r_k, 0, \ldots, 0\} \tag{17}$$

and T_k, $k \leqslant n$

$$T_k = [t_k(i, j)]_{i,j=1,\ldots n} = (I + GR_k)^{-1} G. \tag{18}$$

We shall also use return differences formed in the following way:

$$f_k = 1 + r_k t_{k-1}(k, k), \quad k=1, \ldots, n. \tag{19}$$

The Mayne's theorem [3] states, that if every return difference f_k, $k=1, \ldots, n$ satisfies stability conditions (e.g. Nyquist's) then the close-loop system is stable.

In order to compute matrices T_k the following construction [2], [3] is useful:

$$T_0 = G$$

$$T_k = T_{k-1} - \frac{r_k}{f_k} t_{k-1}(\cdot, k)\, t_{k-1}(k, \cdot) \tag{20}$$

$k=1, \ldots, n$

where $t_{k-1}(\cdot, k)$ k-th column of matrix T_{k-1}

$t_k = _{-1}(k, \cdot)$ k-th row of matrix T_{k-1}.

7. Algorithm

The complete algorithm for multivariable diagonal controller design consists of several steps:

I) Choose the frequency range $[0, \omega_a]$.

II) Choose the bounds for diagonal dampings δ_i, $i=1, \ldots, n$ taking into consideration relations (12).

III) Find numbers x_i^*, $i=1, \ldots, n$ satisfying the constraints (14).
For $k=1, \ldots, n$ execute steps IV) and V)
IV) Compute the function $t_{k-1}(k, k)$.
V) Find a controller r_k in the given class in such a way, that

$f_k(j\omega)$ fulfils stability conditions, $M(q_k) \leqslant x_k^*$.

Step iii) of the algorithm may be performed only under assumption that $M(A_i)$ $i=1, \ldots, n$ exist and $m(A)>0$. Degenerate cases will be analyzed in a separate paper.

Finding the appropriate controller in step V) may also be for certain k impossible, what means that the design objective is not attainable in the given class of controllers.

8. Implementation

The lower part of frequency range, $[0, \omega_a]$ is crucial for the formulated in §4 performance criterion of a control system, while higher frequences, $[\omega_a, \infty)$, decide on stability. In implementation all functions have to be represented by a table of values, so it is important to choose discrete frequences in a reasonable way. In the presented version the following set was adopted:

$$\omega_1=\omega_a \cdot 10^{-0.95}, \quad \omega_{20}=\omega_a, \quad \omega_{60}=100\omega_a,$$

$$\omega_{i+1}=\omega_i \cdot 10^{0.05}.$$

Because of such a choice logarithmic characteristics are easy to obtain.

Accordingly the plant is represented by the sequence of complex matrices

$$G(j\omega_1), \ldots, \quad G(j\omega_{60}).$$

A first action of the program is to compute matrices inverse to G:

$$\hat{G}(j\omega_1), \ldots, \quad \hat{G}(j\omega_{20})$$

and determinants

$$\det G(j\omega_1), \ldots, \quad \det G(j\omega_{20}).$$

As it is easily seen

$$|A_k|=|\det G|\,|\hat{g}_{kk}|/\prod_{\substack{i=1\\ i\neq k}}^{n} |g_{ii}|.$$

Consequently it was assumed, that

$$M(A_k)=\max_{p=1,..20} |A_k(j\omega_p)|$$

$$m(A)=\min_{p=1,..20} |A(j\omega_p)|$$

These values are substituted into (14). The numbers x_k^*, $k=1, ..., n$ are computed by changing in (14) inequalities into equalities.

One of main parts of the program is a procedure for the single-loop controller design (step V). It decides whether the aim of control is realizable in a given class of controllers. In implementation the controller type PID was adopted:

$$r=K(1+\frac{1}{Ts}+Ds)$$

with coefficients from the set Π

$$\Pi=\{K, T, D \mid -K_{max} \leqslant K \leqslant K_{max}, \; T_{min} \leqslant T \leqslant T_{max}, \; 0 \leqslant D \leqslant D_{max}\}$$

For the purpose of satisfying (16) transfer function of the controller must have such a gain, that

$$|r_k(j\omega_i) \cdot g_{kk}(j\omega_i)| \geqslant \frac{1}{x_k^*}+1, \qquad i=1, ..., n.$$

Taking into consideration stability we set a gain margin a $[dB]$ and phase margin φ and then require, that the graph of the function

$$\psi(j\omega_i)=r_k(j\omega_i)\, t_{k-1}(k, k)(j\omega_i), \qquad i=1, ..., 60$$

in the complex plane must avoid the cone

$$\mathrm{Re}\,\psi(j\omega_i) \leqslant -|\mathrm{tg}\,\varphi\,\mathrm{Im}\,\psi(j\omega_i)|-(1-10^{-a/20}), \qquad i=1, ..., 60.$$

Thus we guarantee unconditional stability for the k-th return difference.

A selection of the controller is performed according to the following algorithm:

i) Choose the paii T_k, D_k from Π.

ii) Find such a gain, K_k, that

$$|K_k\, g_{kk}(j\omega_i)(1+\frac{1}{T_k j\omega_i}+D_k j\omega_i)| \geqslant \frac{1}{x_k^*}+1, \quad . \; i=1, ..., 20.$$

If $t_{k-1}(k, k)(j0)<0$, then take $K_k<0$.

If $|K_k|>K_{max}$, return to i).

iii) Check the stability conditions.

If they are not fulfilled return to i).

The set Π is explored in a systematic way and structures of increasing complexity P, PI, PID are succesively tried.

It may also be exhausted without finding the right triple K_k, T_k, D_k. That means that a PID controller with parameters from Π can not ıealize unconditionally stable system with prescribed disturbance damping and gain-phase margins for certain $t_{k-1}(k, k)$.

The implemented procedure also gives gain-phase logarithimic characteristics of return differences at every step of synthesis, so it is easily to check validity of controller design. After completing all steps, it produces logarithmic gain characteristics of q_{ii}, $i=1,\ldots, n$, in order to visualize the results.

9. Example

The method performance is illustrated on the example of transfer function matrix

$$G(s)=\begin{bmatrix} \frac{\exp(-0.5s)}{1+s}, & \frac{-\exp(-0.5s)}{1+s}, & \frac{0.5}{1+s} \\ \frac{\exp(-0.5s)}{1+s}, & \frac{1}{1+s}, & \frac{1}{2+s} \\ \frac{-0.5}{1+s}, & \frac{1}{2+s}, & \frac{-1}{1+s} \end{bmatrix}$$

Requirements:

$$\omega_a=0.3,\ \delta_1=\delta_2=\delta_3=0.1$$

Constraints:

$$a=5dB,\ \varphi=20°,\ K_{\max}=50,\ T_{\min}=0.1,\ T_{\max}=10,\ D_{\max}=10.$$

At the beginning the algorithm computes the bounds for autonomous dampings.

$$x_1^*=0.286,\ x_2^*=0.2,\ x_3^*=0.125.$$

The transfer function seen by the first controller, r_1, has a well known shape ($t_0(1, 1)=g_{11}$). The settings of the controller parameters are:

$$K_1=0.157,\ T_1=0.126\ \text{(type PI)}.$$

The gain-phase characteristics of $t_1(2, 2)$ are shonw on Fig. 2 A type PID was chosen for r_2.

$$K_2=0.701,\ T_2=0.398,\ D_2=2.239$$

Fig. 3 shows $t_2(3, 3)$. The proportional controller is sufficient, with gain $K_3=9.304$.

On Fig. 4 the resultant q_{11}, q_{22}, q_{33} are displayed. It is worth noting, that its graphs omit the forbidden area $\omega\leqslant\omega_a$, $|q|\geqslant -20dB$ with the accuracy of 1–2 dB, what confirms the approximations formerly made.

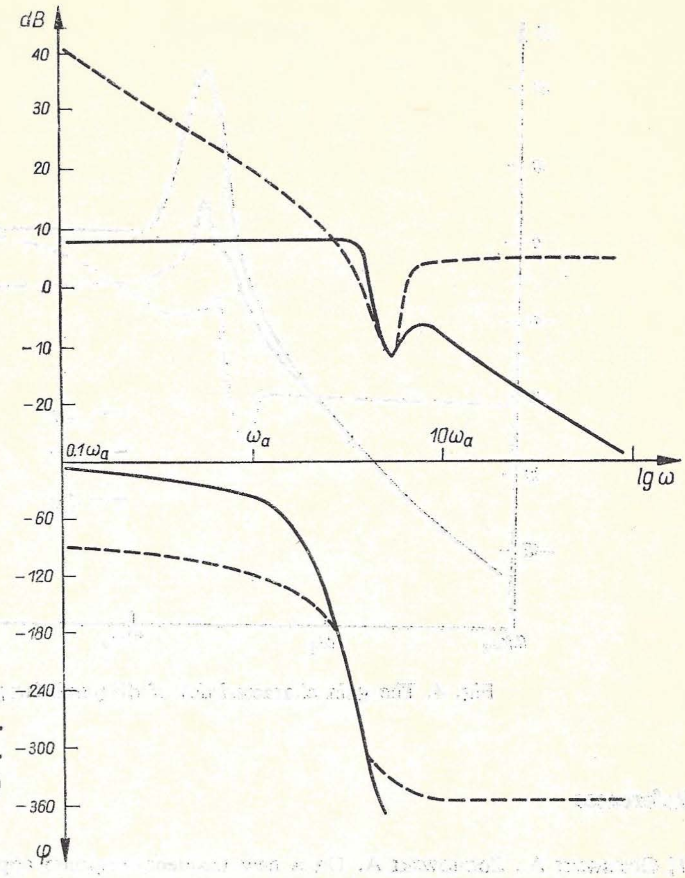

Fig. 2. The gain-phase characteristics of t_1 (2, 2) (continuous line) and $r_2\, t_1$ (2, 2) (dashed line).

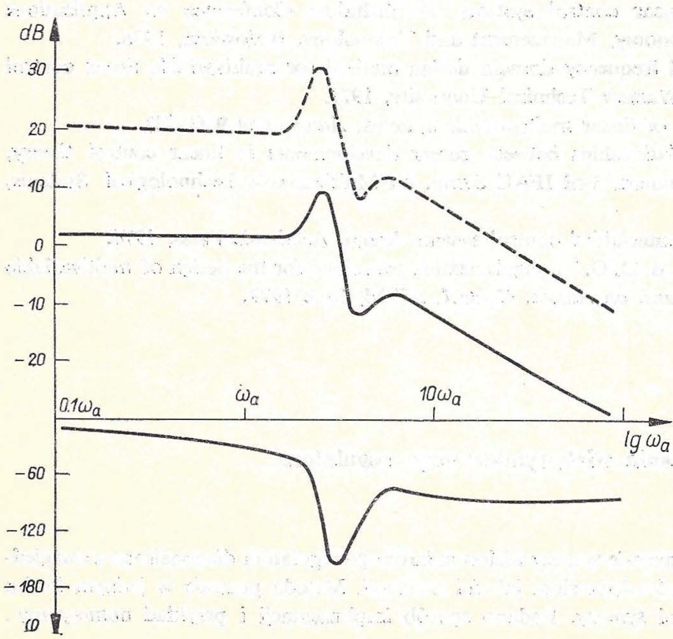

Fig. 3. The gain-phase characteristics of t_2 (3, 3) (continuous line) and $r_3\, t_2$ (3, 3) (dashed line).

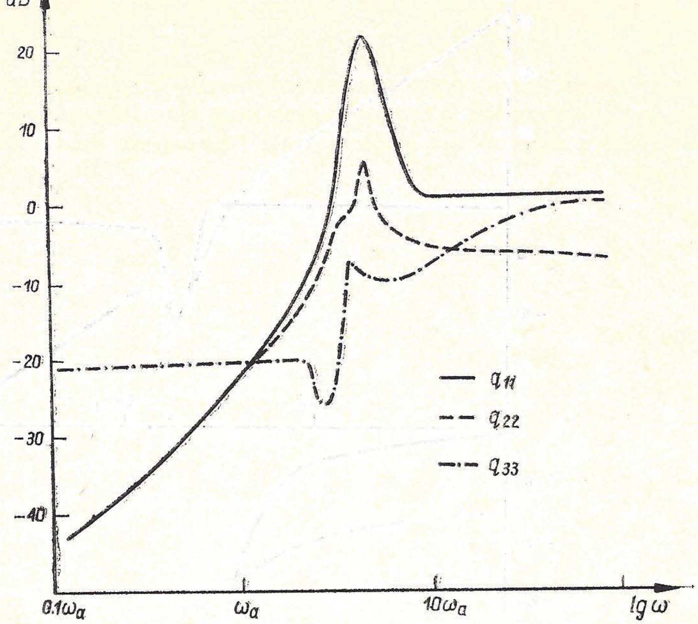

Fig. 4. The gain characteristics of diagonal dampings.

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Received, February 1980.

Pewna metoda projektowania wielowymiarowego regulatora diagonalnego

W pracy zaprezentowano metodę syntezy wielowymiarowego regulatora diagonalnego z uwzględnieniem tłumienia zakłóceń jako kryterium jakości regulacji. Metoda pozwala w jednym kroku osiągnąć zadany ilościowo cel syntezy. Podano sposób implementacji i przykład numeryczny.

Метод проектирования многомерного диагонального регулятора

В работе представлен метод проектирования многомерного диагонального регулятора с учитанием подавления помех как качества регулирования. Метод позгаляет достичь количественно посталенного цели синтеза в одном шаге. Представлен способ осуществления и численный пример.