Control
and Cybernetics
VOL. 9 (1980) No. 3

On the determination of a transfer function matrix from the given state equations for linear multivariable time-lags system

by

MIKOŁAJ BUSŁOWICZ

Technical University of Białystok
Institute of Electrical Engineering
Białystok, Poland

The problem of determination of a trnasfer function matrix for systems with delays is considered in [1, 2, 4]. In the papers [1] and [4] the two different methods of a computation of a matrix transfer functions for a system with one delay are given. The method of a computation of a characteristic quasipolynomial from the state equation for a system with delays is given in [2].

In this paper the new method of calculation of the matrix transfer functions for the linear multivariable time — invariant system with delays on the base of description of this system in the state space is given. This method can be used for numerical computation.

1. Statement of the problem

Consider the linear delay — differential system described in the state space by the equations

$$\dot{x}(t)=\sum_{i=0}^{l} A_i\, x(t-ih)+Bu(t),\ t\geqslant 0, \tag{1}$$

$$y(t)=Cx(t) \tag{2}$$

with the initial condition

$$x(t)=\begin{cases} x_0 & \text{for } t=0 \\ \varphi(t) & \text{for } t\in[-lh,\, 0), \end{cases} \tag{3}$$

where $0<h=\text{const.}$, $x(t)\in R^n$, $y(t)\in R^m$, $u(t)\in R^r$ and A_i $(i=0, 1, \ldots, l)$, B, C are matrices of appropriate dimensions with constant elements.

The characterisitc matrix of the system described by the equations (1), (2) has the form

$$\Delta(s, e^{-sh})=sI-M(e^{-sh}), \tag{4}$$

where

$$M(e^{-sh})=\sum_{i=0}^{1} A_i\, e^{-sih} \tag{5}$$

and the characteristic quasipolynomial, determinant of the matrix $\Delta(s, e^{-sh})$, has the following form

$$q(s, e^{-sh})=\sum_{i=0}^{n} \alpha_i(e^{-sh})\cdot s^{n-i}, \tag{6}$$

where

$$\alpha_i(e^{-sh})=\sum_{j=0}^{il} a_{i,j}\, e^{-shj},\quad a_{i,j}\in R,\ a_{0,0}=1. \tag{7}$$

The matrix transfer functions of the system described by the equations (1), (2) (with initial condition equal to zero) has the following form

$$G(s, e^{-sh})=C\Delta^{-1}(s, e^{-sh})B=\frac{1}{q(s, e^{-sh})}\, C\cdot adj\Delta(s, e^{-sh})\, B \tag{8}$$

for all complex s for which the determinant of the matrix $\Delta(s, e^{-sh})$ does not vanish.

Calculation of the matrix transfer functions from the formula (8) required the knowledge of the algebraic adjoint of $\Delta(s, e^{-sh})$ and of the characteristic quasipolynomial $q(s, e^{-sh})$. The classical computation of the matrix adj $\Delta(s, e^{-sh})$ (by calculation of the minors of $\Delta(s, e^{-sh})$) and of the quasipolynomial $q(s, e^{-sh})$ by calculation of the determinant of $\Delta(s, e^{-sh})$ is very complicated for large n and l ($n\geqslant 3$, $l\geqslant 2$).

The purpose of this paper is to give the new method for determining the transfer function matrix $G(s, e^{-sh})$ without classical computation of $\Delta^{-1}(s, e^{-sh})$.

2. Solution of the problem

The elements of the characteristic matrix (4) and characteristic quasipolynomial (6) are the polynomials of two variables s and e^{-sh}. Therefore we will further consider thease polynomials as the polynomials of the new pair of variables s and p, where variable p should be put instead of e^{-sh}.

The matrix adj $\Delta(s, p)$ can be expressed as

$$\text{adj}\,\Delta(s, p)=\sum_{i=0}^{n-1} s^{n-1-i} F_i(p),\quad F_0(p)=I. \tag{9}$$

We can prove (as in [3] for $M(p)=M$ – real matrix) that the matrices $F_i(p)$ and the polynomials $\alpha_i(p)$ (7) can be computed by the following algorithm

$$H_i(p)=M(p)\,F_{i-1}(p),\ F_0(p)=I, \tag{10}$$

$$\alpha_i(p)=-\frac{1}{i}\,\mathrm{tr}\,H_i(p),\ i=1,2,\ldots,n, \tag{11}$$

$$F_i(p)=H_i(p)+I\alpha_i(p). \tag{12}$$

From the above formulas it follows that $F_i(p)$ is a matrix polynomial in p of degree at most il.

From (4), (5) and the properties of determinant it follows that the matrix $\mathrm{adj}\,\Delta(s,p)$ can be written in the following form

$$adj\Delta(s,p)=K_{n-1,0}(s)+\sum_{i=1}^{n-1}\ \sum_{j=(i-1)l+1}^{il} K_{n-1-i,j}(s)\,p^j, \tag{13}$$

where $K_{n-1-i,j}(s)$ denotes an $n\times n$ matrix polynomial in s of degree at most $n-1-i$.

Let (as in [4]) D_p^k denote

$$\frac{1}{k!}\,\frac{\mathrm{d}^k}{\mathrm{d}p^k}\bigg|_{p=0}$$

and let

$$D_p^k F_i(p)=R_i^k, \tag{14}$$

where R_i^k denotes an $n\times n$ real matrix.
From the relations (9), (13) and (14) we have

$$K_{n-1,0}(s)=D_p^0\left[\sum_{i=0}^{n-1}s^{n-1-i}F_i(p)\right]=\sum_{i=0}^{n-1}s^{n-1-i}R_i^0,$$

$$K_{n-2,1}(s)=D_p^1\left[\sum_{i=0}^{n-1}s^{n-1-i}F_i(p)\right]=\sum_{i=1}^{n-1}s^{n-1-i}R_i^1,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$K_{n-2,l}(s)=D_p^l\left[\sum_{i=0}^{n-1}s^{n-1-i}F_i(p)\right]=\sum_{i=1}^{n-1}s^{n-1-i}R_i^l,$$

$$K_{n-3,l+1}(s)=D_p^{l+1}\left[\sum_{i=0}^{n-1}s^{n-1-i}F_i(p)\right]=\sum_{i=2}^{n-1}s^{n-1-i}R_i^{l+1},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$K_{0,(n-1)l}(s)=D_p^{(n-1)l}\left[\sum_{i=0}^{n-1}s^{n-1-i}F_i(p)\right]=R_{n-1}^{(n-1)l}$$

and hence

$$K_{n-1,0}(s)=\sum_{i=0}^{n-1}s^{n-1-i}R_i^0, \tag{15}$$

$$K_{n-1-i,j}(s)=\sum_{k=i}^{n-1} s^{n-1-k} R_{,k}^{j} \tag{16}$$

for $i=1, 2, \ldots, n-1$; $j=(i-1)\,l+1, (i-1)\,l+2, \ldots, il$.
Let us denote

$$\tilde{R}_i^k=D_p^k H_i(p). \tag{17}$$

Lemma 1. *The matrices R_i^k can be computed in the following way:*

$$R_i^k=\tilde{R}_i^k+Ia_{i,k}, \tag{18}$$

where the coefficients $a_{i,k}$ of the polynomial $\alpha_i(p)$ (7) are equal to

$$a_{i,k}=-\frac{1}{i}\,tr\,\tilde{R}_i^k, \tag{19}$$

and the matrices $\tilde{R}_i^k$ are computed by iterative formula

$$\tilde{R}_i^k=A_0 R_{i-1}^k+A_1 R_{i-1}^{k-1}+\ldots+A_l R_{i-1}^{k-l}, \tag{20}$$

where

$$R_0^k=\begin{cases} I \text{ for } k=0, \\ 0 \text{ for } k\neq 0 \end{cases} \tag{21}$$

and

$$R_j^k=0 \text{ for } j<0 \text{ or } k<0. \tag{22}$$

Proof. Taking into account (14), (17) and

$$D_p^k \alpha_i(p)=a_{i,k} \tag{23}$$

from the formula (12) we obtain the relation (18), and from (11)—the relation (19).

Using the Leibniz formula and (10), (17) we obtain the relation (20):

$$\tilde{R}_i^k=D_p^k[M(p)\,F_{i-1}(p)]=\sum_{j=0}^{k}(D_p^j M(p))\,(D_p^{k-j} F_{i-1}(p))=$$
$$=A_0 R_{i-1}^k+A_1 R_{i-1}^{k-1}+\ldots+A_l R_{i-1}^{k-l},$$

because

$$D_p^j M(p)=A_j, \quad j=0, 1, \ldots, l. \tag{24}$$

The conditions (21), (22) follows from the definition of D_p^k and formulas (10), (11), (12) and (14). This completes the proof. ∎

Lemma 2. *For $i>jl$*

$$\tilde{R}_j^i=0 \text{ and } R_j^i=0. \tag{25}$$

Proof (by induction).

For $j=1$ from (20) we have

$$\tilde{R}_1^i = A_0 R_0^i + A_1 R_0^{i-1} + \ldots + A_l R_0^{i-l}$$

and by (21) $\tilde{R}_1^i=0$ for $i>l$ and hence by (19), (18) $R_1^i=0$ for $i>l$. Assume that the formulas (25) are true for $j-1$

$$\tilde{R}_{j-1}^i=0 \text{ and } R_{j-1}^i=0 \text{ for } i>(j-1)\,l \tag{26}$$

and compute $\tilde{R}_j^i$ from (20). We obtain

$$\tilde{R}_j^i = A_0 R_{j-1}^i + A_1 R_{j-1}^{i-1} + \ldots + A_l R_{j-1}^{i-l}, \tag{27}$$

and from (26), (27) $\tilde{R}_j^i=0$ for $i>jl$. By (19), (18) we have $R_j^i=0$ for $i>jl$. ■

THEOREM 1. *The transfer function matrix for a system with delays described by the equations* (1), (2) *can be written in the following form*

$$G(s, e^{-sh}) = \frac{1}{q(s, e^{-sh})}\left[P_{n-1,0}(s) + \sum_{i=1}^{n-1} \sum_{j=(i-1)l+1}^{il} P_{n-1-i,j}(s)\cdot e^{-sjh}\right], \tag{28}$$

where

$$P_{n-1,0}(s) = \sum_{i=0}^{n-1} s^{n-1-i} C R_i^0 B, \tag{29}$$

$$P_{n-1-i,j}(s) = \sum_{k=i}^{n-1} s^{n-1-k} C R_k^j B, \tag{30}$$

for $i=1, 2, \ldots, n-1$; $j=(i-1)\,l+1, (i-1)\,l+2, \ldots, il$, *where the matrices* R_j^i *are computed from the formulas given in the Lemmas 1 and 2.*

Proof follows from the relations (8), (13) for $p=e^{-sh}$. ■

THEOREM 2. *The transfer function matrix* $G(s, e^{-sh})$ *for systems with delays can be written in the form*

$$G(s, e^{-sh}) = \frac{1}{q(s, e^{-sh})} V(e^{-sh})\, P(s), \tag{31}$$

where the matrices $V(e^{-sh})$ *and* $P(s)$ *have the forms* (32) *and* (33), *respectively.*

Proof. Defining the $m \times [((n-1)\,l+1)\,m]$ polynomial matrix

$$V(e^{-sh}) = [I, Ie^{-sh}, \ldots, Ie^{-sh(n-1)l}] \tag{32}$$

and the $[((n-1)\,l+1)\,m] \times r$ polynomial matrix

$$P(s) = \begin{bmatrix} P_{n-1,0}(s) \\ P_{n-2,1}(s) \\ \cdots\cdots \\ P_{n-2,l}(s) \\ P_{n-3,l+1}(s) \\ \cdots\cdots \\ P_{0,(n-1)l}(s) \end{bmatrix} \tag{33}$$

from the formula (28) we get (31). This completes the proof. ■

From the above considerations the following algorithm of computation of the transfer function matrix for system with delay from the state equations follows:

1. Using the formulas (20), (21), (19), (18) and the conditions (22), (25) calculate the matrices $\tilde{R}_i^k$ and the coefficients $a_{i,k}$ of quasipolynomial $q(s, e^{-sh})$ for $i=1, 2, \ldots, n$; $k=0, 1, \ldots, il$ and the matrices R_i^k for $i=1, 2, \ldots, n-1$; $k=0, 1, \ldots \ldots, il$.
2. From the formulas (6) and (7) write characteristic quasipolynomial $q(s, e^{-sh})$.
3. From the formulas (29) and (30) compute the matrices $P_{n-1,0}(s)$, $P_{n-1-i,j}(s)$ for $i=1, 2, \ldots, n-1$; $j=(i-1)l+1, (i-1)l+2, \ldots, il$.
4. Claculate the matrix $G(s, e^{-sh})$ from the formula (28) or (31).

3. Concluding remarks

In the paper the method of computing the transfer function matrix from the state equations for multivariable system with delays is given. This method can be used for numerical computation.

Using the formula

$$Y(s)=G(s, e^{-sh})\, U(s)$$

we can calculate the Laplace transform $Y(s)$ of the output vector $y(t)$ for the system described by the equations (1), (2), with zero initial condition ($x_0=0$ and $\varphi(t)=0$ for $t \in [-lh, 0)$), where $U(s)$ is the Laplace transform of the input vector $u(t)$. It is easy to check that for initial condition (3) the Laplace transform of the output vector has the form

$$Y(s)=G(s, e^{-sh})\, U(s)+C\Delta^{-1}(s, e^{-sh})\, W(s, e^{-sh}), \tag{34}$$

where

$$W(s, e^{-sh})=x_0+\sum_{j=1}^{l} A_j e^{-sjh} \int_{-lh}^{0} e^{-s\tau}\varphi(\tau)\, d\tau.$$

We can calculate the matrix $G(s, e^{-sh})$ using the Theorem 1 or 2, and we can calculate the matrix $C\Delta^{-1}(s, e^{-sh})$ from the formula

$$C\Delta^{-1}(s, e^{-sh})=\frac{1}{q(s, e^{-sh})}\left[\tilde{P}_{n-1,0}(s)+\sum_{i=1}^{n-1}\sum_{j=(i-1)l+1}^{il}\tilde{P}_{n-1-i,j}(s)\, e^{-sjh}\right],$$

where

$$\tilde{P}_{n-1,0}(s)=\sum_{i=0}^{n-1} s^{n-1-i}\, CR_i^0,$$

$$\tilde{P}_{n-1-i,j}(s)=\sum_{k=i}^{n-1} s^{n-1-k} CR_k^j \qquad i=1, 2, \ldots, n-1.$$

4. Example

Claculate the matrix transfer functions $G(s, e^{-sh})$ of a system described by the equations (1), (2) with the matrices

$$A_0=\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix};\quad A_1=\begin{bmatrix}1&0&0\\2&0&0\\0&-1&0\end{bmatrix};\quad A_2=\begin{bmatrix}1&2&0\\0&0&1\\0&-1&0\end{bmatrix};$$

$$B=\begin{bmatrix}1&0\\0&1\\-1&0\end{bmatrix};\quad C=\begin{bmatrix}1&0&0\\0&1&-2\end{bmatrix}.$$

In this case we have $n=3$, $l=2$, $m=2$, $r=2$.

Using the formulas given in Lemma 1 we calculate the matrices $\tilde{R}_i^k$, R_i^k and the coefficients $a_{i,k}$, so we obtain

$$a_{0,0}=1,$$

$$a_{1,0}=-2,\ a_{1,1}=-1,\ a_{1,2}=-1,$$

$$a_{2,0}=1,\ a_{2,1}=1,\ a_{2,2}=1,\ a_{2,3}=-3,\ a_{2,4}=1,$$

$$a_{3,0}=0,\ a_{3,1}=0,\ a_{3,2}=0,\ a_{3,3}=3,\ a_{3,4}=-2,$$

$$a_{3,5}=-2,\ a_{3,6}=1.$$

And hence from (6), (7) we have

$$q(s, e^{-sh})=s^3+s^2(-2-e^{-sh}-e^{-s2h})+s(1+e^{-sh}+e^{-s2h}-3e^{-s3h}+ +e^{-s4h})+3e^{-s3h}-2e^{-s4h}-2e^{-s5h}-e^{-s6h}. \tag{35}$$

Using the formulas (29), (30) we calculate the matrices $P_{n-1,0}(s)$, $P_{n-1-i,j}(s)$ for $i=1, 2$; $j=(i-1)2+1, (i-1)2+2, \ldots, i2$

$$P_{2,0}(s)=C[s^2R_0^0+sR_1^0+R_2^0]B=\begin{bmatrix}s^2-s&0\\2s^2-2s&s^2-2s+1\end{bmatrix}, \tag{36}$$

$$P_{1,1}(s)=C[sR_1^1+R_2^1]B=\begin{bmatrix}0&0\\-2&s-1\end{bmatrix}, \tag{37}$$

$$P_{1,2}(s)=C[sR_1^2+R_2^2]B=\begin{bmatrix}0&2s-2\\-3s+5&s-3\end{bmatrix}, \tag{38}$$

$$P_{0,3}(s)=CR_2^3B=\begin{bmatrix}1&0\\-3&-4\end{bmatrix}, \tag{39}$$

$$P_{0,4}(s)=CR_2^4B=\begin{bmatrix}-1&0\\1&-2\end{bmatrix}. \tag{40}$$

From the formulas (32), (33) it follows that

$$V(e^{-sh})=\begin{bmatrix} 1 & 0 & e^{-sh} & 0 & e^{-s2h} & 0 & e^{-s3h} & 0 & e^{-s4h} & 0 \\ 0 & 1 & 0 & e^{-sh} & 0 & e^{-s2h} & 0 & e^{-s3h} & 0 & e^{-s4h} \end{bmatrix}$$

$$P(s)=\begin{bmatrix} P_{2,0}(s) \\ P_{1,1}(s) \\ P_{1,2}(s) \\ P_{0,3}(s) \\ P_{0,4}(s) \end{bmatrix},$$

where the matrices $P_{2,0}(s)$, $P_{1,1}(s)$, $P_{1,2}(s)$, $P_{0,3}(s)$, $P_{0,4}(s)$ have the forms (36), (37), (38), (39), (40), respectively.

Using the formula (31) we obtain

$$G(s,e^{-sh})=\frac{1}{q(s,e^{-sh})}\begin{bmatrix} s^2-s+e^{-s3h}-e^{-s4h} & e^{-s2h}(2s-2) \\ 2s^2-2s-2e^{-sh}+e^{-s2h}(5-3s)-3e^{-s3h}+e^{-s4h} & s^2-2s+1+e^{-sh}(s-1)+e^{-s2h}(s-3)-4e^{-s3h}-2e^{-s4h} \end{bmatrix},$$

where the characteristic quasipolynomial has the form (35).

References

[1] Busłowicz M. Wyznaczanie transmitancji macierzowej wielowymiarowego układu z opóźnieniem na podstawie równań stanu. *Podstawy Sterowania*, 9, 1 (1979).

[2] Busłowicz M. On the computation of a characteristic quasipolynomial from the state equations for the multidimensional time — lags system. Proceedings of the International Conference „Functional — Differential System and Related Topics", Błażejewko, Poland, May 19-26, 1979.

[3] Gantmacher F. R. Matrix Theory, New York, 1959.

[4] Manitius A., Triggiani R. Sufficient conditions for function space controllability and feedback stabilizability of linear retarded systems. *IEEE Trans. Autom. Contr.* AC-23, 4 (1978).

Received, March 1980.

Wyznaczanie macierzy transmitancji operatorowych liniowego wielowymiarowego układu z opóźnieniami na podstawie równań stanu

W pracy podano metodę obliczania transmitancji macierzowej liniowego wielowymiarowego układu z opóźnieniami nie wymagającą klasycznego odwracania macierzy charakterystycznej $\Delta^{-1}(s, e^{-sh})$ o postaci (4). Proponowana metoda może być wykorzystana do obliczeń numerycznych macierzy transmitancji operatorowych wielowymiarowego układu z opóźnieniami na podstawie jego opisu w przestrzeni stanów.

Определение матрицы передаточных функций многомерной линейной системы с запаздываниями на основе уравнений состояния

В работе дается метод определения матрицы передаточных функций многомерной линейной системы с запаздываниями без классического вычисления обратной матрицы $\Delta^{-1}(s, e^{-sh})$. Предлагаемый метод может быть использован для численного вычисления передаточной матрицы многомерной системы с запаздываниями на основе ее описания в пространстве состояний.