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Numerical solution of the c-control problem for nonlinear systems

by

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A successive approximation method for solving the c-control problem of nonlinear systems is proposed.

Some constructive sufficient conditions are presented for the convergence of the iterative process. In particular, there are obtained the estimations for the small parameter μ and radius r^* of the neighborhood of the equilibrium state in state space in which the considered nonlinear system will be c-controllable. This is very interesting in the practical applications and presents a new result, to the best of the knowledge of the author, not obtainable by other approaches to the problem of the controllability for the nonlinear systems.

1. Statement of the c-control problem

Let the motion of an object be described by the nonlinear system of ordinary differential equations in the following matrix form

$$\frac{dx}{dt}=A(t)x+B(t)u(t)+\mu f(x,t),\ \ x(t_0)=a^0, \tag{1.1}$$

where x is an n-dimensional vector representing the state of the considered object at time t, $u(t)$ is an m-dimensional real vector-valued function representing the control input of the system, $f(x, t)$ is an n-dimensional real vector-valued function, nonlinear with respect to x, $A(t)$ is an $(n\times n)$-matrix, $B(t)$ is an $(n\times m)$-matrix, a^0 is an n-dimensional real vector representing the initial state of the object at time t_0 and μ is a small parameter.

The function $f(x, t)$ on the right-hand side of Eq. (1.1) is assumed to be continuous in x and t in some domain of admissible values for x and t. Furthermore, this function must satisfy certain other conditions mentioned later.

All the functions-elements of the matrices $A(t)$ and $B(t)$ are assumed to be $(n-1)$ times continuously differentiable on the finite time interval $t_0\leqslant t\leqslant t_f$.

The c-control problem for nonlinear system (1.1) *can be stated as the mathematical problem of finding the continuous function* $u(t)\in C_{(t_0,t_f)}$ *for which the following condition of „hit" is satisfied*

$$x(t_f, u)=a^f, \tag{1.2}$$

where a^f is an n-diemensional vector preassigned in the state space, the n-dimensional vector-valued function $x(t, u)$ is a solution of the matrix differential equation (1.1) subject to the requisite control $u(t)$ representing the trajectory of the considered system in the state space and t_f is a given fixed final time.

This problem for the corresponding $\mu=0$ linear systems has been stated and solved by N. N. Krasovskii in Ref. [1].

It should be observed that a similar problem has been investigated by Ia N. Roitenberg in Ref. [2].

The Roitenberg's problem of "hit" can be stated as the mathematical problem of finding the discontinuous (namely, piecewise constant) function $u(t)$ for which the condition of „hit" (1.2) is satisfied.

We shall refere to the Roitenberg's problem of „hit" as the d-control problem.

Roitenberg's investigations were further developed by Nguyen in Refs. [3–6], who demonstrated that the classical approximation method of Picard can be directly applied to solve the broad class of the d-control problem for nonlinear systems.

The purpose of this paper is to show that the numerical technique developed by Nguyen in Refs. [3–6] can be extended in an iterative fashion to solve the c-control problem for nonlinear systems.

2. Numerical solution of the *c*-control problem

It is well known that the system of differential equations (1.1) can be written in the following equivalent integral form

$$x(t,u)=X(t)\Big[a^0+\int_{t_0}^{t}X^{-1}(\xi)B(\xi)\,u(\xi)\,d\xi+\mu\int_{t_0}^{t}X^{-1}(\xi)\,f(x(\xi,u),\xi)\,d\xi\Big] \tag{2.1}$$

Here $X(t)$ is a normalized fundamental $(n\times n)$-dimensional matrix for the corresponding $\mu=0$ linear system, i.e.

$$\frac{dX(t)}{dt}=A(t)\,X(t),\quad X(t_0)=I$$

where I is an identity $(n\times n)$-dimensional matrix.

From (2.1) it is easy to see that the condition of „hit" (1.2) will be fulfilled, if the required control function $u(t)$ satisfies the following equation

$$\int_{t_0}^{t_f}X^{-1}(t)\,B(t)\,u(t)\,dt=\lambda^0-\mu\int_{t_0}^{t_f}X^{-1}(t)\,f(x(t,u),t)\,dt, \tag{2.2}$$

where $x(t, u)$ is a solution of the matrix integral equation (2.1) and λ^0 is the following known constant vector

$$\lambda^0=X^{-1}(t_f)\,a^f-a^0 \tag{2.3}$$

Thus, for determining the required control function $u(t)$ we have to solve equation (2.2) associated with the original nonlinear matrix integral equation (2.1).

Since it is actually impossible to obtain an analytic solution (in the closed form) of these equations, they must be solved by means of various numerical method.

It turns out that the numerical technique developed by Nguyen in Refs. [3–6] for solving d-control problem can be applied in an iterative fashion to solve the above-mentioned equations.

The initial approximation may be defined as

$$\int_{t_0}^{t_f} X^{-1}(t)\, B(t)\, u^0(t)\, dt = \lambda^0 \tag{2.4}$$

$$x^0(t) \triangleq x(t, u^0) = X(t)\Big[a^0 + \int_{t_0}^{t} X^{-1}(\xi)\, B(\xi)\, u^0(\xi)\, d\xi\Big] \tag{2.5}$$

It is very easy to see that control vector-valued function $u^0(t)$ determined from (2.4) represents the solution of the c-control problem for the corresponding $\mu=0$ linear system. For this case the following result holds

THEOREM 1. *If the row vectors of the following $(n\times m)$-dimensional matrix*

$$G(t) = X^{-1}(t)\, B(t)$$

are linearly independent on the interval $t_0 \leqslant t \leqslant t_f$, then c-control problem for the corresponding $\mu=0$ linear system has infinite set of solutions, amongst them the control function defined by the following formula

$$u^0(t) = B'(t)\, [X^{-1}(t)]'\, \eta^0, \tag{2.6}$$

where

$$D\eta^0 = \lambda^0, \qquad D = \int_{t_0}^{t} X^{-1}(t)\, B(t)\, B'(t)\, [X^{-1}(t)]'\, dt \tag{2.7}$$

is the unique solution, which has minimal norm in the space $L_2\,[t_0, t_f]$.

It should be observed that Theorem 1 had been proved by N. N. Krasovskii in Ref. [1] for single input case ($m=1$). The proof of Theorem 1 for the presented above general case is similar and will not be repeated here.

In (2.6) the prime denotes the transpose.

To ensure that iterative procedure is well defined, it is convenient to introduce the following terminology.

DEFINITION 1. *The dynamic system (1.1) is c-controllable on the interval $t_0 \leqslant t \leqslant t_f$, if there exists some continuous control function $u(t)$ which brings a system from the initial state a^0 at t_0 to the preassigned terminal state a^f at the given fixed final time t_f.*

We have seen via Theorem 1 that the corresponding $\mu=0$ linear system is c-controllable on the interval $t_0 \leqslant t \leqslant t_f$, if the row vectors of the matrix $G(t)$ are linearly independent on this interval.

It should be noted that the indicated above test for c-controllability of the corresponding $\mu=0$ linear system depends on knowing the matrix $G(t)$. In many practical cases, however, the problem of the construction of the matrix $G(t)$ is either difficult or computationally time consuming. Thus, it would be a distinct advantage to have a test for c-controllability which does not require a knowledge of the matrix $G(t)$.

It turns out that the following proposition holds (see Ref. [1]).

THEOREM 2. *The row vectors of the matrix $G(t)$ are linearly independent on the interval $t_0 \leqslant t \leqslant t_f$, if*

$$\operatorname{rank} P(t)=n, \quad \forall t \in [t_0, t_f] \tag{2.8}$$

where

$$P(t)=(P_1(t), P_2(t), \ldots, P_n(t))$$

$$P_1(t)=B(t), \; P_{i+1}(t)=\frac{dB_i(t)}{dt}-A(t)\,B_i(t), \qquad i=1, 2, \ldots, n-1$$

It is clear from Theorems 1 and 2 that the corresponding $\mu=0$ linear system is c-controllable on the interval $t_0 \leqslant t \leqslant t_f$, if the matrix $P(t)$ is of rank n for any time t over this interval.

Therefore, in what follows we shall assume that the condition (2.8) is fulfilled for the corresponding linear system.

Then, it is easy to show that $\det D \neq 0$ and the required control vector-valued function $u^0(t)$ is uniquely defined from (2.6) and (2.7).

Now, suppose that $(j-1)$-th approximation is already found, i.e. $u^{j-1}(t)$ and $x^{j-1}(t) \triangleq x(t, u^{j-1})$ are known. Then the j-th approximation is determined by

$$\int_{t_0}^{t_f} X^{-1}(t)\,B(t)\,u^j(t)\,dt=\lambda^j, \quad j=1, 2, \ldots . \tag{2.9}$$

$$x^j(t) \triangleq x(t, u^j)=X(t)\Big[a^0+\int_{t_0}^{t} X^{-1}(\xi)\,B(\xi)\,u^j(\xi)\,d\xi+$$

$$+\mu\int_{t_0}^{t} X^{-1}(\xi)\,f(x^{j-1}(\xi), \xi)\,d\xi\Big]\, j=1, 2, \ldots . \tag{2.10}$$

where

$$\lambda^j=\lambda^0-\mu\int_{t^0}^{t_f} X^{-1}(t)\,f(x^{j-1}(t), t)\,dt, \quad j=1, 2, \ldots , \tag{2.11}$$

Note that Eqs. (2.9), (2.10) and (2.11) remain also valid for $j=0$ if in this case we set $\mu=0$.

It is easy to show that the required control function $u^j(t)$ defined by (2.9) has the following form

$$u^j(t)=B'(t)\,[X^{-1}(t)]'\,\eta^j, \quad j=1, 2, \dots . \tag{2.12}$$

where

$$D\eta^j=\lambda^j, \quad j=1, 2, \dots . \tag{2.13}$$

It should be observed that computer implementation of the above steps requires the computation of the inverse matrix $X^{-1}(t)$. In many cases, however, analytical evaluation of this matrix is difficult and, sometimes, impossible, Therefore, it is of interest to find the modified formulation of the presented above iterative procedure which avoids this difficulty.

3. Modified formulation of the iterative procedure

Consider first the determination of the matrix D. It is well known that the matrix D evaluated by formula (2.7) can be represented in the following from (see Ref. [15])

$$D=\int_{t_0}^{t_f} \Phi'(t)\,B(t)\,B'(t)\,\Phi(t)\,dt \tag{3.1}$$

where $\Phi(t)$ is a normalized fundamental matrix of the adjoint differential equation i.e.

$$\frac{d\Phi(t)}{dt}=-A'(t)\,\Phi(t), \quad \Phi(t_0)=I \tag{3.2}$$

Furthermore, the constant vector λ^j evaluated by (2.3) and (2.11) can be represented in the form

$$\lambda^j=\Phi'(t_f)\,[a^f-z^j(t_f)], \quad j=0, 1, 2, \dots \tag{3.3}$$

where $z^j(t)$ is a solution of the following equation

$$\frac{dz^j}{dt}=A(t)\,z^j+\mu_j\,f(x^{j-1}(t), t), \quad j=0, 1, 2, \dots \tag{3.4}$$

subject to the initial condition

$$z^j(t_0)=a^0, \quad j=0, 1, 2, \dots \tag{3.5}$$

Here μ_j, $j=0, 1, 2, \dots$, are defined by

$$\mu_j=\begin{cases}0 & \text{if } j=0\\ \mu & \text{if } j\neq 0\end{cases} \tag{3.6}$$

The required control function $u^j(t)$ in j-th approximation can be evaluated by the following formula

$$u^j(t)=B'(t)\,\psi^j(t), \quad j=0, 1, 2, \dots \tag{3.7}$$

where $\psi^j(t)$ is a solution of the adjoint differential equation

$$\frac{d\psi^j(t)}{dt}=-A'(t)\,\psi^j(t), \quad j=0, 1, 2, \dots \tag{3.8}$$

subject to

$$\psi^j(t_0)=\eta^j, \quad j=0, 1, 2, \dots \tag{3.9}$$

Finally, the trajectory $x^j(t)$ corresponding to $u^j(t)$ can be obtained by solving the following differential equation

$$\frac{dx^j}{dt}=A(t)\,x^j+B(t)\,u^j(t)+\mu_j f(x^{j-1}(t), t), \quad j=0, 1, 2, \dots \tag{3.10}$$

subject to

$$x^j(t_0)=a^0, \quad j=0, 1, 2, \dots \tag{3.11}$$

where μ_j, $j=0, 1, 2, \dots$, are defined by formula (3.6).

Thus, it is not necessary to compute the inverse matrix $X^{-1}(t)$ in order to determine all the requisite values in each iteration.

By combining the above results with those of the previous section, the modified formulation of the iterative procedure can be obtained. It entails the following steps:

Step 0. Solve equation (3.2) from t_0 to t_f. During this integration calculate matrices D and $\Phi(t_f)$.

Step 1. Set $j=0$, $\mu_j=0$ and solve equation (3.4) subject to the initial condition (3.5) from t_0 to t_f and at the end of this integration determine vector $z^j(t_f)$. Next, evaluate vector λ^j by formula (3.3).

Step 2. Solve algebraic equation

$$D\eta^j=\lambda^j$$

in order to find constant vector η^j.

Step 3. Solve equation (3.8) subject to (3.9) from t_0 to t_f. During this integration evaluate function $u^j(t)$ by formula (3.7).

Step 4. Solve equation (3.10) subject to (3.11) from t_0 to t_f. During this integration obtain vector $x^j(t)$. Store $x^j(t)$, $t_0 \leq t \leq t_f$.

Return to step 1 with j replaced by $j+1$ and set $\mu_j=\mu$. The process ends when the condition of „hit" (1.2) is satisfied with the requisite accuracy.

Thus, the modified procedure requires that only the systems of differential equations with the known initial conditions and the system of algebraic equations are solved in each iteration, so it seems to be well adapted to computations by digital or hybrid computers.

4. Some constructive sufficient conditions for the convergence of the iterative process

In the previous section we have employed the iterative technique developed by Nguyen in Refs. [3–6] to solve the c-control problem for nonlinear systems and found the modified formulation of the iterative procedure which avoids some difficulties in computer implementation.

Now, it remains for us to establish some sufficient conditions for the convergence of the iterative process.

THEOREM 3. *Assume* 1) *The corresponding* $\mu=0$ *linear system is c-controllable on the finite time interval* $t_0 \leqslant t \leqslant t_f$.

2) *The function* $f(x, t)$ *is continuous in all arguments in some closed domain of the space* (x, t), *determined by the expression*

$$D(\Delta, \tau)=\{(x, t): |x| \leqslant \Delta,\ t_0 \leqslant t \leqslant t_f,\ \tau=t_f-t_0\}$$

where Δ *is some positive number. The norm of the matrix* x *is denoted by* $|x|$.

3) *In the domain* $D(\Delta, \tau)$ *the function* $f(x, t)$ *is Lipschitzian with respect to* x *with Lipschitz's constant* L. *This means that for two arbitrary points* (x^1, t), (x^2, t) *of the domain* $D(\Delta, \tau)$ *the following condition will be fulfilled*

$$|f(x^2, t)-f(x^1, t)| \leqslant L\,|x^2-x^1| \tag{4.1}$$

4) *The parameter* μ *satisfies the condition*

$$0<\mu<\mu^*=\frac{1}{(1+H)\,h^+ h^- \tau}\left[\min\left(\frac{1}{L}, \frac{\Delta-p}{f}\right)\right], \tag{4.2}$$

where

$$h^+=\max_{t_0 \leqslant t \leqslant t_f} |X(t)|,\ h^-=\max_{t_0 \leqslant t \leqslant t_f} |X^{-1}(t)|,\ f=\max_{(x,t) \in D(\Delta,\tau)} |f(x, t)|$$

$$H=(h^-)^2 b^2 d^- \tau,\ d^-=|D^{-1}|,\ b=\max_{t_0 \leqslant t \leqslant t_f} |B(t)|$$

5) *The number* p *connected with* $|a^0|$ *and* $|a^f|$ *by formula*

$$p=(1+H)\,h^+ |a^0|+h^+ h^- H\,|a^f| \tag{4.3}$$

satisfies the condition

$$p \leqslant p^*=\Delta-\mu(1+H)\,h^+ h^- f\tau \tag{4.4}$$

Then the proposed above iterative process for solving the c-control problem is convergent and nonlinear system (1.1) *is c-controllable on the interval* $t_0 \leqslant t \leqslant t_f$.

Proof. We show first that all the approximations determined by (2.5) and (2.10) at any time, t, $t_0 \leqslant t \leqslant t_f$, entirely belong to the domain $D(\Delta, \tau)$.

It is quite easy to show that, if the condition (4.4) is fulfilled, then $x^0(t)$ determined by (2.5) belongs to the domain $D(\Delta, \tau)$. Suppose that $x^j(t) \in D(\Delta, \tau)$ *then* we will prove that $x^{j+1}(t) \in D$.

From (2.10) it follows that

$$\|x^{j+1}(t)\| \leqslant h^{+}\left(|a^{0}|+h^{-} b\tau \|u^{j+1}(t)\|+\mu h^{-} f\tau\right) \tag{4.5}$$

where

$$\|x^{j+1}(t)\| = \max_{t_0 \leqslant t \leqslant t_f} |x^{j+1}(t)|, \quad \|u^{j+1}(t)\| = \max_{t_0 \leqslant t \leqslant t_f} |u^{j+1}(t)|$$

Using the equations (2.12), (2.13) and (2.11) we obtain

$$\|u^{j+1}(t)\| \leqslant bh^{-} d^{-}\left(h^{-} |a^{f}|+|a^{0}|+\mu h^{-} f\tau\right) \tag{4.6}$$

With this inequality for $\|u^{j+1}(t)\|$ substituted into (4.5) we have

$$\|x^{j+1}(t)\| \leqslant \rho+\mu(1+H) h^{+} h^{-} f\tau \leqslant \Delta, \quad t_0 \leqslant t \leqslant t_f$$

Thus, all the approximations belong to the domain $D(\Delta, \tau)$.

Now, we pass to the problem of convergence.

From the equations (2.10) and (2.12) it is easy to derive the following estimations

$$\|x^{j+1}(t)-x^{j}(t)\| \leqslant h^{+} h^{-} b\tau \|u^{j+1}(t)-u^{j}(t)\|+\mu h^{+} h^{-} L\tau \|x^{j}(t)-x^{j-1}(t)\| \tag{4.7}$$

$$\|u^{j+1}(t)-u^{j}(t)\| \leqslant \mu (h^{-})^{2} bd^{-} L\tau \|x^{j}(t)-x^{j-1}(t)\| \tag{4.8}$$

Substituting (4.8) into (4.7) gives

$$\|x^{j+1}(t)-x^{j}(t)\| \leqslant \mu(1+H) h^{+} h^{-} L\tau \|x^{j}(t)-x^{j-1}(t)\| .$$

$$\frac{\|x^{j+1}(t)-x^{j}(t)\|}{\|x^{j}(t)-x^{j-1}(t)\|} \leqslant \mu(1+H) h^{+} h^{-} L\tau<1 \tag{4.9}$$

The inequality (4.9) shows that the majorant series converges (due to d'Alambert's criterion). Hence, the sequence of the approximations (2.10) must converge uniformly to a certain continuous vector-valued function $x^*(t) \in D(\Delta, \tau)$, and, due to (4.8) the sequence $\{u^{j}(t)\}$ converges to a continuous function $u^*(t)$. Further, it is easy to see that $x^*(t)$ satisfies the original integral equation (2.1), and $u^*(t)$ — the equation (2.2), i.e. the condition of „hit" will be fulfilled. Thus the proof of theorem 3 is complete. ∎

Finally, we consider the case in which $a^{f}=0$, i.e. it is required to transfer the nonlinear system from a given initial state a^{0} at t_0 to the equilibrium state in state space at time t_f. This case of the c-control problem is interesting in the practical applications, for which the following conclusion holds.

THEOREM 4. *If the conditions* 1), 2), 3) *and* 4) *in Theorem* 3 *are fulfilled and if* $a^{f}=0$, *then nonlinear system* (1.1) *is c-controllable on the interval* $t_0 \leqslant t \leqslant t_f$ *in the neighborhood of the equilibrium state with radius* r^* *defined by formula*

$$r^*=\Delta/(1+H) h^{+}-\mu h^{-} f\tau \tag{4.10}$$

Proof. This follows directly from Theorem 3, Definition 1 and the fact that in the considered case a number ρ must satisfy the condition

$$\rho=(1+H) h^{+} |a^{0}| \leqslant \Delta-\mu(1+H) h^{+} h^{-} f\tau$$

Consequently, in this case $|a^0|$ must satisfy the inequality

$$|a^0| \leqslant r^* = \Delta/(1+H)\, h^+ - \mu\, h^- f\tau$$

which completes the proof of Theorem 4.

It is clear that Theorem 4 gives estimations for the small parameter μ and radius r^* of the neighborhood of the equilibrium state in state space under which the nonlinear system (1.1) will be c-controllable. This is very important for the practical applications and presents a new result, to the best of the knowledge of the author, not obtainable by other approach to the problem of the controllability for nonlinear systems (see Refs. [7–14]).

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Rozwiązanie numeryczne problemu *c*-sterowania dla układów nieliniowych

Zaproponowaną metodę kolejnych przybliżeń do rozwiązywania problemu c-sterowania układami nieliniowymi. Przedstawiono pewne konstruktywne warunki wystarczające zbieżności procesu iteracyjnego.

W szczególności otrzymano oceny małego parametru μ i promienia r^* otoczenia stanu równowagi w przestrzeni stanów, w którym rozpatrywany układ nieliniowy będzie c-sterowalny. Jest to bardzo interesujące w zastosowaniach praktycznych i stanowi nowy wynik, który — o ile autorowi wiadomo — nie może być otrzymany za pomocą innych podejść do zagadnienia sterowalności układów nieliniowych.

Численное решение задачи c-управления для нелинейных систем

Предлагается метод последовательных приближений для решения задачи c-управления нелинейными системами. Представлены некоторые конструктивные достаточные условия сходимости интерационного процесса.

В частности получены оценки малого параметра u и радиуса r^* окрестности состояния равновесия в пространстве состояний, в которой рассматриваемая нелинейная система будет c-управляема. Это может быть весьма интересным для практических промененний и является новым результатом, который — насколько известно автору — невозможно получить с помощью других подходов к задаче управляемости нелинейных систем.