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Integer Programming Problems with Inexact Objective Function

by

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In the paper an integer linear programming problem is considered in which coefficients of the objective function are given with uncertainty. Several levels of this uncertainty are described and possible methods of dealing with inexact parameters of the problem are mentioned.

The case when the coefficients of the objective function are known only to belong to a given convex set is discussed in details. Two extreme strategies consisting in seeking the solutions of the problem under the assumptions that the objective function is the worst and the best possible are considered. Corresponding max-min and max-max integer programming problems are discussed and three algorithms for solving max-min problem are proposed. Small examples illustrating the algorithms are given.

1. Introduction

Almost all parameters of practical mathematical programming problems are estimated with errors. On the other hand, available methods for solving these problems assume that the parameters of models are exact. To deal with the uncertatinty of data description several approaches are possible. Some of them, in the context of integer programming and inexact objective function, are mentioned in Section 2. The case, when the objective function coefficients of integer linear programming problem are known only to lie in a given convex set is discussed in Section 3. Small numerical examples illustrating the algorithms proposed in Section 3 are given in the Appendix.

2. Uncertainty Levels

Consider an integer linear programming problem (P)

$$(\mathrm{P})\qquad \begin{aligned} &\max c^T x \\ &Ax \leqslant b \\ &x \geqslant 0 \text{ and integer} \end{aligned}$$

Let us assume that the parameters of the constraint set (i.e. matrix A and vector b) are exactly known and the uncertainty concerns only the coefficients of the objective function. The situation when the objective function of the problem is not known exactly occurs frequently in practice. The coefficients of the objective function represent often prices or profits which are estimated with errors and sometime are assumed rather arbitrarily. Another reason, which motivates the analysis of problems with inexact parameters, is the simplification of mathematical programming model with respect to real-life situation. For example, some nonlinearities of the objective function and the dependence of prices on time are frequently neglected.

One can distinguish several levels of uncertainty concerning the objective function and in each case different approaches are applicable. The most frequent case is when

(i) *A particular objective vector $\bar{c}$ is given and no additional information about possible changes of objective coefficients is available.*

In this case the problem (P) with $c=\bar{c}$ can be solved and the optimal solution x^0 or, sometime, several optimal solutions can be obtained.

One can hope that the possible changes of the objective coefficients are "small" enough not to destroy the validity of computed solutions. But even under this assumption several questions immediately arrise. For example, one wants to know for which changes of c solution x^0 remains still optimal (or ε-optimal) or which solution obtained allows maximal changes of c. More formally: it is interesting to find for a given optimal vector x^0 the set $C(x^0)$ of all vectors $\tilde{c}$ such that $\tilde{c} \in C(x^0)$ if and only if x^0 is an optimal (ε-optimal) solution of the problem

$$\max \{\tilde{c}x \mid Ax \leqslant b,\ x \geqslant 0 \text{ integer}\}.$$

Another related question is, how the optimal value of problem (P) varies with small changes of c. All of these questions belong to the sensitivity analysis which has just begun to be developed for integer programming [6, 8].

The approach presented above can be regarded as a passive one. In fact we accept the solution x^0 while the sensitivity analysis is performed mainly for assuming that the choice of x^0 has been good.

Quite different situation occurs, when

(ii) *The set C of all possible vectors c is given.*

One if faced with such a problem when, for example, it is possible to describe the bounds for changes of prices but one can not predict the exact values which will occur. This case contains also the problems for which the objective coefficients are measured with known errors. Sometime one can not describe the exact values of coefficients, but it is possible to establish the functional relations between them. In all these cases instead of one objective vector c we have the well described set C of all possible realizations of c.

Several approaches are possible for such untypically stated mathematical programming problem.

Of course, one can choose an arbitrary vector from C and solve the problem (P) with this vector. But in general there is no rule for such a choice. It is also possible to regard the problem stated as a multiobjective programming problem (in general with infinite number of objectives) and try to find nondominated solutions. There are also possible two extreme strategies: a "pessimistic" and an "optimistic" one which for a given set C yield minimal guaranteed profit and maximal possible profit respectively.

The pessimistic strategy consists in seeking x^* which gives maximal profit among all feasible solutions under assumption that c is the worst possible. It means that for any $x \in X = \{y | Ay \leqslant b,\ y \geqslant 0 \text{ integer}\}$ the following inequality holds:

$$\min_{c \in C} c^T x^* \geqslant \min_{c \in C} c^T x \tag{1}$$

The pessimistic strategy gives minimal guaranteed profit $\underline{v}(P, C)$, where

$$\underline{v}(P, C) = \min_{c \in C} c^T x^* \tag{2}$$

The optimistic strategy consists in the choice of feasible vector $x^{**} \in X$ maximizing the profit, under the assumption that the objective vector is the most convenient, i.e., for any $x \in X$ the following relation is fulfilled

$$\max_{c \in C} c^T x^{**} \geqslant \max_{c \in C} c^T x \tag{3}$$

The optimistic strategy gives the upper bound $\bar{v}(P, C)$ for all possible profits, where

$$\bar{v}(P, C) = \max_{c \in C} c^T x^{**} \tag{4}$$

Having found the values $\underline{v}(P, C)$ and $\bar{v}(P, C)$, we may characterize the problem and its sensitivity on data changes. It also provides the decision maker with useful information about conservative solution (x^*) and risky solution x^{**}).

It is interesting to note some connections between case (ii) and (i). Suppose that we a given the set C and for a particular $c \in C$ we have obtained the optimal solution x^0 of problem (P). Then it is easy to see, that

PROPOSITION 1. If $C \in C(x^0)$ then $x^* = x^{**} = x^0$.

This follows immediately from the definition of stability region $C(x^0)$.

Although frequently it is too difficult to find the set $C(x^0)$, sometime we have a subset $\tilde{C}(x^0)$ of $C(x^0)$ obtained as an inexpensive by-product of solving (P) (see [5] and [8]). In this case we can try to use Proposition 1 with $\tilde{C}(x^0)$ instead of $C(x^0)$.

Different approach can be used when it is assumed that c is a stochastic variable and

(iii) *Probability distribution of c is known.*

In this case the decision maker has several strategies characterized by the expected profit and the risk level. Two of them are the most straightforward.

The first one consists in choice of feasible vector $x^E \in X$, which maximizes the expected value of profit. It is easy to see that in this case the following linear integer programming problem has to be solved

$$x^E = \arg\max\{\bar{c}x | Ax \leqslant b,\ x \geqslant 0 \text{ integer}\}$$

where $\bar{c} = E(c)$ ($E(\cdot)$ denotes the expected value).

Another approach consists in calculating feasible solution $x^V \in X$ which minimizes a variance of profit with the assumption that the expected value of profit is not less than a given value q. In this case a nonlinear programming problem

$$x^V = \arg\min\{f(x) | Ax \leqslant b,\ cx \geqslant q,\ x \geqslant 0 \text{ integer}\}$$

has to be solved, where $f(x) = E(c^T x - E(c^T x))^2$ is a quadratic function (see [7] for methods of solving nonlinear integer programming problems).

3. Extreme Strategies in the Case when All Possible Objective Vectors are Given

In this Section two extreme strategies mentioned in Section 2 are analysed. We assume that in the problem (P) constraint matrix $A \in R^{m \times n}$ and right hand side vector $b \in R^m$ are fixed. Thus the set

$$X = \{x | Ax \leqslant b,\ x \geqslant 0 \text{ integer}\}$$

of feasible solution is also fixed. Assume that X is nonempty and finite.

Any realization of objective vector belongs to a given set $C \subseteq R^n$.

The pessimistic strategy as defined in Section 2 leads to the following max-min problem

$$x^* = \arg\max_{x \in X} \min_{c \in C} c^T x \tag{5}$$

The problem (5) can be reformulated by introducing an additional variable $t \in R$ into the following programming problem

$$\begin{aligned} &\max t \\ &\quad t \leqslant c^T x \text{ for any } c \in C \\ &\quad Ax \leqslant b,\ x \geqslant 0 \text{ integer} \\ &\quad t \in R \end{aligned} \tag{6}$$

The problem (6) is a mixed integer programming problem with single continuous variable t. If the set C is infinite (which is almost always the case), it is the problem with infinite number of constraints. Mathematical programming problems of this type have been considered in the context of continuous nonlinear programming (see e.g. [1]). The first two algorithms presented in this Section employ the basic ideas of methods used in infinitely constrained programming problems for discrete case. Both of them construct the sequences of integer vectors $\{x^k\}$ and real vectors $\{c^k\}$, $k = 0, 1, 2, \ldots,$ by solving auxiliary problems (Pk), (Sk) and (Qk).

Let us begin with the formal description of Algorithm 1.

Algorithm 1

1°. Choose $c^0 \in C$
$k:=1$
Go to 2°

2°. Solve the problem (Pk)

$$\text{(Pk)} \qquad \begin{aligned} &\max t \\ &t \leqslant (c^j)^T x, \; j=0, 1, \ldots, k-1 \\ &x \in X, \quad t \in R \end{aligned}$$

Let x^k, t^k denote an optimal solution of (Pk).
Go to 3°

3°. Solve the problem (Qk)

$$\text{(Qk)} \quad \min_{c \in C} c^T x^k$$

Let c^k denote an optimal solution of (Qk).
If $(c^k)^T x^k \geqslant t^k$ then STOP: x^k is an optimal solution of the problem (5) and

$$\underline{v}(P, C) = t^k$$

else
$k:=k+1$
Go to 2°.

By the step of the above algorithm a solution of pair of problems (Pk), (Qk) is meant. If X is finite, then

PROPOSITION 2. The Algorithm 1 solves the problem (5) in a finite number of steps.

Proof. To prove the finiteness of Algorithm 1 it is enough to see that no vector x^k can be repeated before the optimality because if $x^k = x^l$, $l > k$, then $(c^k)^T x^k = (c^l)^T x^l = t^l$ and the stop criterion is fulfilled. Assume that algorithm terminates after l steps.

For the proof of validity of Algorithm 1 observe that for any k the problem (Pk) is a relaxation of problem (5). So it is enough to prove that if the pair (t^l, x^l) fulfils the stop criterion (i.e.,

$$t^l \leqslant (c^l)^T x^l$$

where c^l is a solution of problem (Ql)) then (t^l, x^l) is feasible to problem (5). To do this we must check that

$$t^l \leqslant c^T x^l \text{ for any } c \in C \tag{6'}$$

But this is immediate from the definition of problem (Ql) and the fact that the stop criterion holds. We have

$$t^l \leqslant (c^l)^T x^l = \min_{c \in C} c^T x^l$$

which is equivalent to (6′).

In the Algorithm 1 for any k the problem (Pk) is a mixed integer (with single continuous variable) programming problem. So this step of algorithm may be very time consuming. It is possible to modify this step by replacing the problem (Pk) by practically easier problem (Sk) consisting in finding a feasible integer solution of a system of linear constraints. The formal description of this modification is the following:

Algorithm 2

1° Choose a feasible solution $x^1 \in X$
$k := 1$, $t := -\infty$, $\bar{x} := x^1$

2° Solve the problem (Qk)
(Qk) $\min_{c \in C} c^T x^k$
Let c^k denote an optimal solution of (Qk), $t^k := (c^k)^T x^k$
If $t^k > t$, then $t := t^k$, $\bar{x} := x^k$
Go to 3°

3° Find a solution of the following system of constraints

$$
\text{(Sk)} \quad \begin{aligned} & t < (c^j)^T x \qquad \text{for } j = 1, \ldots, k \\ & Ax \leqslant b \\ & x \geqslant 0 \text{ integer} \end{aligned}
$$

If (Sk) is inconsistent then STOP: $\bar{x}$ is an optimal solution of (5) and $\underline{v}(P, C) = t$
else:
Let x^{k-1} denote the solution of (Sk)
$k := k+1$
Go to 2°

The proof of validity and finiteness of Algorithm 2 is similar as in the case of Algorithm 1. Instead of solving the problem (Pk) to the optimality, it is enough to find in Step 3° a feasible solution decreasing the actual value t. This step can be realized by any enumerative scheme. A similar structure of algorithm appears when Benders' decomposition method to mixed integer programming is used. Results presented in [3] suggest that in Step 3° the branch and bound scheme used in pseudoboolean programming could be suitable.

One can expect that in Algorithm 2 the problem (Qk) is solved more times than in Algorithm 1. Due to this fact Algorithm 2 seems to be applicable to cases when (Qk) can be solved efficiently. Such a situation potentially occurs when (Qk) can be solved analytically (for example if C is a ball in R^n) or when it is easy to reoptimize (Qk) for successive x^k.

As it is mentioned above, the main drawback of Algorithms 1 is the necessity of solving discrete auxiliary subproblems (Pk), which is in general a difficult task. But it worth to mention an important advantage of algorithm of this kind. Namely, in any step of Algorithm 1 the bounds for value of optimal solution are known and one can stop the computation with known approximate solution and its accuracy. Let for any k

$$
z^k = \max \{c^q)^T x^q, \; q = 0, 1, \ldots, k\}
$$

and r be any index q for which this maximum is attained. Then choosing x^r as an approximate solution we have

$$z^k \leqslant \underline{v}(P, C) \leqslant t^k$$

In both of algorithms described above the initial problem is in fact decomposed and optimization problems over X and C were solved separately. The next algorithm uses quite different approach. It applies ordinary branch and bound scheme (see [5]) in which bounds in nodes are computed by solving continuous relaxation of the initial max-min problem.

Algorithm 3

To describe algorithm we specify main elements of the branch and bound scheme, i.e., bounds calculation and separation rule.

The candidate problems have the form

$$(P_l^u) \qquad \max_{x \in X_l^u} \min_{c \in C} c^T x$$

where $X_l^u = \{x \in X \mid l \leqslant x \leqslant u\}$, $u, l \in R_+^n$ and integer.

If in the optimal solution x of relaxation of (P_l^u) the element x_i for some i, $1 \leqslant i \leqslant n$, is noninteger, then two new candidate problems are created by adding to (P_l^u) one of the constraints $x_i \leqslant [x_i]$ or $x_i \geqslant [x_i]+1$ (where $[a]$ denotes integer part of a).

As a relaxation of candidate problem (P_l^u) a continuous relaxation of (P_l^u) is used, i.e., the following continuous max-min problem must be solved for bound calculation

$$(\bar{P}_l^u) \qquad \max_{x} \min_{c \in C} \{c^T x \mid Ax \leqslant b,\ l \leqslant x \leqslant u\}$$

The efficiency of the above algorithm depends mainly on possibility of efficient solution of problem $(\bar{P}_l^u)$. Several papers [2, 9, 11] addresses the methods of solving continuous max-min problem of the form

$$z = \sup_{x} \inf_{c \in C} \{c^T x \mid Ax \leqslant b,\ x \geqslant 0\} \tag{7}$$

In [11] Soyster has formulated a dual problem to (7) under assumption that $C = C + R_+^n$. Observe that this assumption does not change the optimal solution of (7). This dual has the form

$$v = \inf \{b^T y \mid A^T y = c,\ c \in C,\ y \geqslant 0\} \tag{8}$$

where y denotes a dual variable vector, $y \in R^m$.

If C is a closed convex set in R^n, then (8) is a convex programming problem with linear objective function. When some constraint qualification holds (see [9] for different formulations of these conditions) then the following duality theorem is true [11]:

THEOREM 1

One of the three following cases occurs:

(i) *Both of the problems* (7′) *and* (8) *are feasible. Then* x^0 *optimal to* (7) *and* y^0 *optimal to* (8) *exist and there is no duality gap, i.e.,*

$$z = \min_{c \in C} c^T x^0 = b^T y^e = v$$

Moreover, complementary slackness holds

$$(Ax^0 - b)^T y^0 = 0$$

(ii) *One of two problems* (7), (8) *is infeasible and another is unbounded.*

(iii) *Both of problems* (7), (8) *are inconsistent.*

This theorem provides a way of computing the upper bound for subproblem (P_l^u) in the case when C is a convex set by solving the convex programming problem

$$v_l^u = \inf \{ b^T y + u^T \bar{y} - l^T \underline{y} \mid A^T y + \bar{y} - \underline{y} \in C,\ y,\ \bar{y},\ \underline{y} \geqslant 0 \} \tag{9}$$

where $\bar{y}$, $\underline{y} \in R^\alpha$ denote the dual variables corresponding to the constraints $l \leqslant x \leqslant u$.

If the value v_l^u is less than or equal to the value of actual incumbent or the problem is inbounded, then the vertex corresponding to the problem (P_l^u) can be fathomed. In the opposite case we must find the solution x of the problem $(\bar{P}_l^u)$ to check it for integrality. Theorem 1 does not give any way to find optimal x. In [2] Falk uses the results of Soyster to propose a method of obtaining an optimal solution of (7) in the case when

$$C = \{ z \in R^n \mid h^i(z) \leqslant 0,\ i = 1, \dots, r \}$$

and $h^i(z)$, $i = 1, \dots, r$, are convex, continuously differentiable functions and the constraint qualification hold. Let y^0 denote an optimal solution of (8) and $c^0 = A^T y^0$. Then [2]

THEOREM 2

The optimal solution x^0 *of* (7) *fulfils the following set of linear constraints*

$$Ax^0 \leqslant b, \quad x^0 \geqslant 0 \tag{10.1}$$

$$(Ax^0 - b)^T y^0 = 0 \tag{10.2}$$

$$x^0 + \sum_{i=1}^{r} \lambda_i \nabla h^i(c^0) = 0 \tag{10.3}$$

$$\lambda_i \geqslant 0, \quad \lambda_i h^i(c^0) = 0 \tag{10.4}$$

where ∇h *denotes the gradient of function* h.

First two conditions (10.1) and (10.2) represent feasibility conditions for x^0 in problem (7) and complementary slackness condition obtained by Soyster. Next two constraints (10.3) and (10.4) are the necessary and sufficient conditions of optimality of c^0 in the inside convex programming problem $\min_{c \in C} c^T x^0$. This theorem gives the possibility of computing the optimal solution of the relaxed problem $(\bar{P}_l^u)$. Thus for any subproblem (P_l^u), i.e., in each vertex in the branch and bound tree, one can distinguish two phases of computations. In the first phase the optimal value $v(\bar{P}_l^u)$ of continuous relaxation of problem (P_l^u) is computed by solving the dual problem (9). If $v(\bar{P}_l^u)=v_l^u$ is not greater than the actual incumbent value or if the problem $(\bar{P}_l^u)$ is unbounded, then the vertex (P_l^u) is fathomed. Else, using Theorem 2, an optimal solution x^0 of $(\bar{P}_l^u)$ is computed. If x^0 is integer, then it becomes to be a new incumbent, if not — two new subproblems are created according to the described above separation rule.

The solution of problems (9) and (10) may be in fact a difficult task and it depends on description of the set C.

The problem is significantly simplified if we assume that C is a polyhedral convex set, i.e.,

$$C=\{c \in R^n \,|\, Kc \geqslant k\} \tag{11}$$

where $K \in R^{r \times n}$ and $k \in R^r$ are given.

One can specify the general considerations for this case and from (9) and (10) obtain formulae for computing the value v_l^u and optimal solution x^0 of $(\bar{P}_l^u)$. The same results in more convenient form can be obtained directly by reformulation of the problem $(\bar{P}_l^u)$. We have

$$v_l^u = \max_{\substack{Ax \leqslant b \\ l \leqslant x \leqslant u}} \min_{Kc \geqslant k} c^T x \tag{12}$$

Replacing the inside problem by its dual we obtain

$$v_l^u = \max_{\substack{Ax \leqslant b \\ l \leqslant x \leqslant u}} \max_{\substack{K^T \lambda = x \\ \lambda \geqslant 0}} k^T \lambda \tag{13}$$

But this problem is equivalent to

$$v_l^u = \max_{\substack{AK^t \lambda \leqslant b \\ l \leqslant K^T \lambda \leqslant u \\ \lambda \geqslant 0}} k^T \lambda \tag{14}$$

The optimal value v_l^u of the problem (12) gives the required upper bound for subproblem (P_l^u) and due to (13) the optimal solution x^0 is immediately given by

$$x^0 = K^T \lambda^0 \tag{15}$$

where λ^0 is an optimal solution of (14). Thus in the case, when C is a polyhedral set, then the relaxation $(\bar{P}_l^u)$ is a linear programming problem and the integrality of obtained solution x^0 can be easily tested because (15) holds. Due to these facts this approach can be computationally efficient and even in the case when C is not polyhedral it may be profitable to approximate C by a polyhedral set. In the Appendix, an example of problem with polyhedral set C is given and all steps of solving it by Algorithm 3 are described.

The "optimistic" strategy as defined in Section 2 leads to the following max-max problem

$$x^{**} = \arg \max_{x \in X} \max_{c \in C} c^T x \tag{16}$$

This problem seems to be more difficult than problem (5) even in the case, when C is a convex polyhedron, $C = \{c \mid Kc \geqslant k\}$. In this case the continuous relaxation of (16) becomes so called bilinear programming problem

$$\max_{\substack{Ax \leqslant b \\ x \geqslant 0}} \max_{Kc \geqslant k} c^T x \tag{17}$$

which is a rather difficult nonconvex problem [4]. So it seems to be not promising to use (17) in branch and bound scheme to calculate the bounds for subproblems of (16).

In contrast with problem (5) it is possible to prove that when the problem (16) is consistent, then there exists an optimal solution x^{**} which is a vertex of conv (X) (where conv (S) denotes the convex hull of S).

APPENDIX

We give two small examples of solving the problem of the form (5) using described algorithms. Both of the problems have the same set X, but the sets of possible objective vectors are different. In Example 1 the Algorithm 1 is used; in Example 2 the solution by Algorithm 3 is given.

Example 1

Consider the problem

(P) $\max_{x \in X} \min_{c \in C} c^T x$

where $X = \{x \mid 6x_1 + 8x_2 \leqslant 21,\ x_1, x_2 \geqslant 0 \text{ integer}\}$

$C = K\{1/2, 1)^T, 1/2\}$, $K(\rho, r)$ denotes a ball with center in ρ and radius r

We solve this problem using Algorithm 1.

Let $c^0 = (1/2, 1)^T$, $k := 1$.

Problem (P1): $\max \{t \mid t \leqslant (c^0)^T x,\ x \in X,\ t \in R\}$ has the optimal solution $x^1 = (0, 2)^T$ and $t^1 = 2$.

Solving the problem (C1): $\min_{c \in C} c^T x^1$ we obtain:

$c^1 = (1/2, 1/2)^T$ and $(c^1)^T x^1 = 1$. The stop criterion is not fulfilled and we have to increase k ($k: = 2$) and solve the problem (P2).

(P2) $\max t$

$$t \leqslant (c^0)^T x$$
$$t \leqslant (c^1)^T x$$
$$x \in X, \ t \in R$$

We obtain $x^2 = (3, 0)^T$ and $t^2 = 3/2$.

Solving the problem (C2) we have $c^2 = (0, 1)^T$ and $(c^2)^T x^2 = 0$.

The stop criterion is not still fulfilled, so we must solve the problem (Pk) with $k := 3$

(P3) $\max t$

$$t \leqslant (c^i)^T x, \ i = 1, 2, 3$$
$$x \in X, \ t \in R$$

The optimal solution of (P3) is $x^3 = (0, 2)^T$ and $t^3 = 1$.

Solving (C3): $\min\limits_{c \in C} c^T x^3$ we have $c^3 = (1/2, 1/2)^T$ and $(c^3)^T x^3 = 1 = t^3$ which indicates that $x^3 = (0, 2)^T$ is an optimal solution of (P) and $\underline{v}(P, C) = 1$.

Example 2

Consider the problem (see Fig. 1)

(P) $$\max_{x \in X} \min_{c \in C} c^T x$$

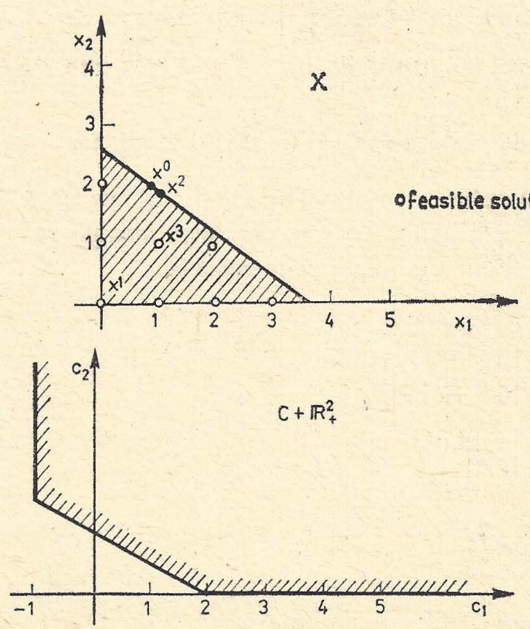

Fig. 1

where $X=\{x|6x_1+8x_2\leqslant 21,\ x_1, x_2\geqslant 0 \text{ integer}\}$

$C=\{c|c_1+2c_2\geqslant 2,\ c_1\geqslant -1,\ c_2\geqslant 0\}$

i.e.,

$$A=[6, 8],\ K=\begin{bmatrix}1 & 2\\ 1 & 0\\ 0 & 1\end{bmatrix},\ k=\begin{bmatrix}2\\ -1\\ 0\end{bmatrix},\ AK^T=[22, 6, 8]$$

We present the solution of this problem using Algorithm 3. The problem ($\bar{P}^0$) is formed by adding obvious bounds for x

$$l^0=(0, 0)^T,\ u^0=(4, 3)^T$$

and relaxing the integrality conditions.

The problem (14) corresponding to (P^0) has the form

$$\begin{aligned}\max\ & 2\lambda_1-\lambda_2\\ & 22\lambda_1+6\lambda_2+\lambda_3\leqslant 21\\ & 0\leqslant \lambda_1+\lambda_2 \leqslant 4\\ & 0\leqslant 2\lambda_1 \quad +\lambda_3\leqslant 3\\ & \lambda_1, \lambda_2, \lambda_3 \geqslant 0\end{aligned}$$

The optimal solution of this problem is $\lambda^0=\left(\frac{21}{22}, 0, 0\right)^T$ and corresponding optimal solution of ($\bar{P}^0$) is $x^0=\left(\frac{21}{22}, \frac{42}{22}\right)^T$, $v(\bar{P}^0)=\frac{42}{22}$. As x^0 is noninteger, two new problems (P^1) with $l^1=(0, 0)^T, u^1=(0, 3)^T$ and (P^2) with $l^2=(1, 0)^T, u^2=(4, 3)^T$ are created.

Solving appropriate problems of the form (14) we obtain

(P^1)

$\lambda^1=(0, 0, 0)^T$

$x^1=(0, 0)^T$

$v(\bar{P}^1)=0$

This solution is integer, so we obtain an incumbent with value 0.

(P^2)

$\lambda^2=\left(\frac{15}{16}, \frac{1}{16}, 0\right)^T$

$x^2=\left(1, \frac{15}{8}\right)^T$

$v(\bar{P}^2)=\frac{29}{16}$

Problem (P^2) separates into two problems: (P^3) with $l^3=(1, 0)^T$, $u^3=(4, 1)^T$ and (P^4) with $l^4=(1, 2)^T$, $u^4=(4, 3)^T$. Problem (14) corresponding to (P^4) is infeasible, so this vertex is fathomed. In the last live vertex (P^3) we obtain the following solutions

$$\lambda^3=(\tfrac{1}{2},\tfrac{1}{2},0)^T$$
$$x^3=(1,1)^T$$
$$v(\bar{P}^3)=\tfrac{1}{2}$$

The solution x^3 is integer and better than the incumbent. There is no live vertex, so algorithm terminates and $x^*=x^3=(1,1)^T$ is an optimal solution of (P) and v(P)$=\frac{1}{2}$. It is interesting to note that x^* is not a vertex of conv (X).

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Problemy programowania całkowitoliczbowego z niedokładnie określoną funkcją celu

W artykule rozpatrywany jest problem programowania całkowitoliczbowego, w którym współczynniki funkcji celu dane są z pewną nieokreślonością. Opisano kilka poziomów nieokreśloności oraz przedstawiono możliwe metody postępowania w przypadku wystąpienia problemów o niedokładnych wartościach parametrów. Szczegółowo omówiono przypadek, w którym wiadomo tylko, że współczynniki funkcji celu należą do danego zbioru wypukłego. Rozpatrzono dwie krańcowe strategie polegające na poszukiwaniu rozwiązań problemu przy założeniach, że funkcja celu jest najgorszą lub najlepszą z możliwych. Omówiono problemy programowania całkowitoliczbowego typu max-min i max-max oraz zaproponowano trzy algorytmy rozwiązywania problemu max-min. Algorytmy zilustrowano niewielkimi przykładami.

Задачи целочисленного программирования с неопределенной функцией цели

В статье рассматривается задача целочисленного программирования, в которой коэффициенты функции цели заданы с некоторой неопределенностью. Описано несколько уровней неопределенности, а также представлены возможные методы решения в случае появления задач с неточными значениями параметров. Подробно рассмотрен случай, в котором известно лишь, что коэффициенты функции цели принадлежат к данному выпуклому множеству. Рассмотрены две крайние стратегии, состоящие в поиске решений задачи при предпосылке, что функция цели является наихудшей либо наилучшей из возможных. Рассмотрены задачи целочисленного программирования типа макс-мин и макс-макс, а также предлагаются три алгоритма решения задачи макс-мин. Алгоритмы иллюстрируются небольшими примерами.