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Controllability of linear discrete-time systems with constrained controls in Banach spaces

by

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For a given discrete-time system $x_{k+1}=Ax_k+Bu_k$, where $x_k \in X$, $u_k \in U$, X and U are Banach spaces and A, B are linear bounded operators, some controllability problems are considered. Throughout the paper the assumption $u_k \in \Omega \subset U$, where 0 is not necessarily an interior point of Ω, is made. Necessary and sufficient conditions for local and global controllability are given. They are illustrated by some examples. For finite-dimensional systems corollaries to the main theorem include some well-known results of Evans and Murthy. The proof of the main result is based on the prominent Krein-Rutman theorem concerning properties of a convex cone invariant with respect to a linear bounded operator in a Banach space.

1. Introduction

In this paper we shall be concerned with controllability of linear time-invariant systems whose input state dynamics are described by difference equations in (infinite-dimensional) Banach spaces.

The study of controllability for discrete-time systems has received considerable attention during the last twenty years. One of the first works devoted to this problem is the paper by Kalman and others [1], in which the authors introduced a concept of complete controllability and derived algebraic testes for this property for linear discrete-time systems of the form $x_{k+1}=Ax_k+Bu_k$ in the finite-dimensional space R_n. Controllability for various types of discrete-time systems with delays was studied by Gabasov and others [2]. In a series of papers (see, e.g. [3], [4]) Fuhrmann undertook the study of controllability and some other related properties such as realizability and observability for linear discrete-time systems which have a Hilbert space as their states space.

We note that all the above references are concerned only with systems with unconstrained controls, i.e. the case where input u_k takes values from a linear space of controls. For systems in R_n with constraints on controls of the form $u_k \in \Omega$, where Ω is a closed convex subset of R_m with the origin in its interior, a necessary and sufficient condition for controllability was presented in [5]. Recently, some easily checkable criteria for complete controllability have been derived in [6] for single input discrete-time systems with positive controls, i.e. for the system $x_{k+1}= =Ax_k+bu_k,\ x_k \in R_n,\ u_k \geqslant 0$.

In this paper we consider controllability of linear discrete-time systems in Banach space with control constraints of the general form $u_k \in \Omega$ where the origin need not be an interior point of Ω. From the results obtained in this paper we derive, as a corollary, a criterion for local controllability of linear discrete-time systems in R_n with constrained controls, which can be considered as an extention to discrete systems of the well-known result due to Brammer [7] for continuous finite-dimensional linear system $\dot{x}=Ax+Bu,\ u \in \Omega$. Our method is based on some fundamental propositions of analysis, among which the theorem of Krein and Rutman (concerning with the properties of convex cones in Banach spaces invariant under a certain family of commutable linear bounded operators) plays a crucial role. This approach enables us to obtain more general results in a unified and much simpler way than techniques of other authors. It is worth to note that the methodology used in [6], [7], clearly, does not apply to the case of infinite-dimensional systems. An analogous technique has been used in [11] for continuous-time systems in R_n and in [12] for ones in Banach spaces.

For the aim of our paper, the mentioned above Krein-Rutman's theorem can be stated as follows.

THEOREM A (Krein M.G. and Rutman M.A., [8]). *Let C be a convex cone with a nonempty interior in a Banach space X and C not be dense in X. Let A be a linear bounded operator mapping the cone C into itself, i.e. $AC \subset C$. Then there exists a bounded positive linear functional $x_0^* \in C^* \subset X^*$, which is an eigenvector of the dual operator A^* corresponding to a nonnegative eigenvalue λ: $A^* x_0^* = \lambda x_0^*$.*

2. Preliminaries

Consider the linear discrete-time system

$$x_{k+1}=Ax_k+Bu_k,\ x_k \in X,\ u_k \in \Omega \subset U, \tag{1}$$

where X and U are real Banach spaces of states and controls, respectively; $A: X \to X$, $B: U \to X$ are bounded linear operators; Ω is a given nonempty convex subset of the control space U, which satisfies, in general, only one requirement:

$$\exists u_0 \in \Omega: \quad Bu_0=0. \tag{2}$$

The solution of system (1) after k steps corresponding to a sequence of inputs $u_j \in \Omega$ for $j=1, 2, \ldots, k$ and an initial state $x=x_1$ is given by

$$x_{k+1}=A^k x_1 + A^{k-1}Bu_1 + \ldots + ABu_{k-1} + Bu_k.$$

For each integer $k \geqslant 1$ we denote by U^k the Banach space of all vectors of the form $u^k \triangleq (u_1, u_2, \ldots, u_k)$, $u_j \in U$ for $j=1, 2, \ldots, k$ with algebraic operations defined in the natural way and with the norm $\|u^k\|=\|u_1\|+\|u_2\|+\ldots+\|u_k\|$.

Let the symbol Ω^k stand for the set of all vectors $u^k=(u_1, u_2, ..., u_k) \in U^k$ such that $u_j \in \Omega$ for $j=1, 2, ..., k$.

Consider, for every fixed $k\geqslant 1$, the controllability operator of system (1) F_k: : $U^k \to X$ which is defined as follows

$$F_k u^k = A^{k-1} Bu_1 + A^{k-2} Bu_2 + ... + ABu_{k-1} + Bu_k. \tag{3}$$

Obviously, F_k is a bounded linear operator.

The set $S_k = F_k(\Omega^k)$ will be called the reachable set of system (1) after k steps and the set $S = \bigcup_{k=1}^{\infty} S_k$ will be simply called the reachable set of system (1).

According to Fuhrmann (see. e.g. [3]) system (1) with $\Omega=U$ is said to be weakly controllable (weakly controllable after k steps) if the reachable set S (respectively, the reachable set S_k after k steps) is dense in X and is said to be strongly controllable (strongly controllable after k steps) if the reachable set S (respectively, the reachable set S_k after k steps) coincides with the whole space X.

The above mentions motivate the following definition of local and global controllability for system (1) with constrained controls.

Definition 1. The system (1) is said to be locally controllable (locally controllable after k steps) if the reachable set S (respectively, the reachable set S_k after k steps) contains the origin in its interior, i.e. $0 \in \text{int } S$ (respectively, $0 \in \text{int } S_k$) and is said to be locally ε-controllable (locally ε-controllable after k steps) if S (respectively, S_k) is dense in some neighborhood of the origin, i.e. $0 \in \text{int } \bar{S}$ (respectively, $0 \in \text{int } \bar{S}_k$).

In what follows we denote sometimes, for the sake of convenience, the above four types of local controllability briefly by (LC), $(LC)_k$ $(\varepsilon\text{-}LC)$, $(\varepsilon\text{-}LC)_k$.

If the reachable set S of the system (1) coincides with the whole space X we will talk about global controllability. The concepts of global controllability after k steps, global ε-controllability and global ε-controllability after k steps are defined in a similar fashion.

In Section 3 we illustrate, with the aid of some examples, the difference between the various types of controllability introduced above and establish their relations. The main purpose of Section 4 is to derive necessary and sufficient conditions for local and global controllability of the system (1) with the convex control set Ω satisfying the additional condition (2).

3. Relations Between the Various Types of controllability

Firstly, it directly follows from Definition 1 that $(LC)_k$ implies (LC) and $(\varepsilon\text{-}LC)_k$ implies $(\varepsilon\text{-}LC)$. Further, suppose that the control set Ω is convex and satisfies the condition (2). If the state space X is finite-dimensional, then, by virtue of convexity of the reachable set S, (LC) and $(\varepsilon\text{-}LC)$ are equivalent. Moreover, since $S_k \subset S_{k+1}$ by (2), we may observe that (LC) implies $(LC)_k$ for some finite k. However, the last two properties, in general, do not hold for infinite-dimensional systems.

Example 1. Consider the system $x_{k+1}=Ax_k+u_k$, $u_k\in\Omega$ defined on the space C [0, 1] of all continuous functions with usual uniform norm $\|f\|=\max\{|f(t)|, 0\leqslant t\leqslant 1\}$, while the operator A is given by $Af(t)=tf(t)$ and the control set is $\Omega=\{f(t):f(t)\equiv\text{any real number}\}$. Clearly the reachable set S of this system is equal to the set of all polynomials. Therefore, this system is (ε-*LC*) but not (*LC*).

Example 2. Consider the following system defined in $X=l_1: x_{k+1}=Ax_k+bu_k$, $u_k\in\Omega\subset l_1$, while A is the right shift operator: $A(\xi_1, \xi_2, \ldots)=(0, \xi_1, \xi_2, \ldots)$, $b=(1, 0, 0, \ldots)$ and $\Omega=[-1, 1]$. The reachable set of this system after k steps is equal to $S_k=\{x=(\xi_1, \ldots, \xi_k, 0, 0, \ldots): |\xi_j|\leqslant 1\}$. Thus, this system is not $(\varepsilon\text{-}LC)_k$ for any finite k. On the other hand, for arbitrary point $x_0=(\xi_1^0, \xi_2^0, \ldots)$ in the unit ball of l_1 and $\varepsilon>0$ we can find an integer N such that $\sum_{N+1}^{\infty}|\xi_j^0|<\varepsilon$. Putting $x=(\xi_1^0, \xi_2^0, \ldots, \xi_N^0, 0, 0, \ldots)$ we have $\|x_0-x\|<\varepsilon$ and $x\in S_N\subset S$. Since $\varepsilon>0$ is arbitrary we conclude that the system is (ε-*LC*).

Example 3. As a state space X we take L_1 [0, 1] (the Banach space of integrable functions). A control set is $\Omega=\{f(t)\in L_1[0, 1]: \int_0^1|f(t)|dt\leqslant 1,\ f(t)\equiv 0$ on some interval $[0, \delta], \delta>0\}$. It is easy to show that Ω is a convex set with empty interior in L_1 [0, 1]. We verify that $0\in\operatorname{int}\Omega$. Let $f(t)$ be an arbitrary element in the unit ball of the space L_1 [0, 1]. By the absolute continuity property of integrable functions, for any ε, $0<\varepsilon<1$ we can find $\delta>0$ such that $\int_0^\delta|f(t)|\,dt<\varepsilon$. If $f_\varepsilon(t)$ is a function which is equal to f on $(\delta, 1)$ and is null otherwise, then we have $f_\varepsilon\in\Omega$ and $\|f_\varepsilon-f\|<\varepsilon$. Therefore, Ω is dense to the unit ball of L_1 [0, 1]. Now, setting $A=0$ and $B=I$ (the identity operator of $X=L_1$ [0, 1]). we obtain the system $x_k=u_k$, $u_k\in\Omega$, which is $(\varepsilon-LC)_1$ but not $(LC)_1$.

If we set

$$Af(t)=\begin{cases}0 & \text{for} \quad t\in[0, 1/2],\\ f(t-1/2) & \text{for} \quad t\in(1/2, 1],\end{cases}$$

and

$$Bf(t)=\begin{cases}f(t) & \text{for} \quad t\in[0, 1/2],\\ 0 & \text{for} \quad t\in(1/2, 1],\end{cases}$$

then it can be shown that the system in $X=L_1[0, 1]: x_{k+1}=Ax_k+Bu_k$, $u_k\in\Omega$, where the control set Ω is defined as above is $(\varepsilon-LC)_2$ but not $(LC)_2$.
The previous examples generate the interest to study the relations between the various types of controllability for infinite-dimensional systems. Some of these relations are presented in the following.

THEOREM 1. *Let the control set* Ω *be convex and satisfy the condition* (2). *We have the following*:

a) *If the system* (1) *is locally controllable then there exists an integer* $k_1\geqslant 1$ *such that the system is locally* ε*-controllable after* k_1 *steps.*

b) If the system (1) *is locally controllable and, in addition, the control set* Ω *has a non-empty interior in U, then there exists an integer* $k_1 \geqslant 1$ *such that the system is locally controllable after* k_1 *steps.*

c) *Suppose that the control set* Ω *is bounded and contains the origin in its interior. If the system* (1) *is locally* ε*-controllable after k steps then it is also locally controllable after k steps.*

The proof of Theorem 1 is based on the lemma below.

LEMMA 1. *Let* M_j $(j=0, 1, \ldots)$ *be a sequence of convex subsets with non-empty interior in a Banach space X such that* $0 \in M_j \subset M_{j+1}$ *and* $0 \in \operatorname{int}\,(\bigcup_{j=0}^{\infty} M_j)$. *Then there exists an integer* j_0 *such that* $0 \in \operatorname{int}\, M_{j_0}$.

Proof. We will prove Lemma 1 by contradiction. Suppose that for every j, $0 \notin \operatorname{int}\, M_j$. Let N_j denote the cone of support functionals to M_j at the origin. i.e.

$$N_j = \{x^* \in X^*;\ x^*(x) \geqslant 0 \text{ for all } x \in M_j\}.$$

Since $0 \notin \operatorname{int} M_j$, the cone N_j contains nonzero element by the Hahn-Banach theorem. It is evident from the properties of the sequence M_j that the cones N_j are convex and decreasing, i.e. $N_j \supset N_{j+1}$ for $j=0, 1, 2, \ldots$ Besides, it is not difficult to show that N_j are closed in the weak* topology of X^*. Let x_0 be an arbitrary element interior to M_0. Choose a number $\alpha > 0$ such that $x_0 - \alpha e \in M_0$ for all e in the unit ball X^1 of X. Then, for any $x^* \in N_0$ we have

$$x^*(x_0) \geqslant \alpha x^*(e) \quad \text{for all } e \in X^1. \tag{4}$$

Therefore, $x^*(x_0) \geqslant \alpha \|x^*\|$ for every $x^* \in N_0$. Denote by N_j^h the intersection of N_j with the hyperplane $H = \{x^* \in X^*: x^*(x_0) = \alpha\}$. Then, obviously, for every j, N_j^h is non-empty, convex and closed in the weak* topology. Moreover, since N_0^h N_j^h when $j \geqslant 0$, from (4) we get $x^*(x_0) = \alpha \geqslant \alpha \|x^*\|$ for every $x^* \in N_j^h$. It follows that $\|x^*\| \leqslant 1$ for all $x^* \in N_j^h$ and $j \geqslant 0$. On account of Alaoglu's theorem ([9], p. 424) we conclude that N_j^h is a weakly* compact set in X^*. Hence, by the finite intersection property, the set $N_\infty^h = \bigcap_{j=0}^{\infty} N_j^h$ is non-empty. Let x_∞^* be an arbitrary element of N_∞^h. Then, clearly, x_∞^* is a nonzero support functional to the set $\bigcup_{j=0}^{\infty} M_j$ at the origin. This contradicts the assumption $0 \in \operatorname{int}\,(\bigcup_{j=0}^{\infty} M_j)$ and completes the proof.

Proof of Theorem 1.

a) Suppose that system (1) is locally controllable, i.e. $0 \in \operatorname{int} S$. It follows that the reachable set S is absorbing and hence we can write

$$X = \bigcup_{j=1}^{\infty} \bigcup_{K=1}^{\infty} (jS_k).$$

By Baire's theorem, one of the sets, say $j_0 S_{k_0}$, is dense to some ball, i.e. $\operatorname{int}\,(\overline{j_0 S_{k_0}}) \neq \emptyset$. Since the map $x \to j_0 x$ is a homeomorphism of X onto itself, it follows that $\bar{S}_{k_0}$

has non-empty interior in X. By convexity of the control set Ω and the condition (2), $\bar{S}_k$ is convex and $\bar{S}_k \subset \bar{S}_{k+1}$ for all integers k. Now, we put $\bar{S}_{k_0+j} = M_j$ for $j=0, 1, \ldots$ By Lemma 1, there exists an integer j_0 such that $0 \in \text{int}\ (\bar{S}_{k_0+j_0})$. This means that the system (1) is locally ε-controllable after k_1 $(=k_0+j_0)$ steps and concludes the proof of a).

b) Consider the controllability operator F_k of the system (1) defined as (3). If the system is locally controllable, then, by the definition, $0 \in \text{int}\ (\bigcup_{K=1}^{\infty} F_k(\Omega^k))$ and, therefore, $X = \bigcup_{K=1}^{\infty} F_k(U^k)$. By the Baire's theorem, there exists an integer $k_0 \geqslant 1$ such that $F_{k_0}(U^{k_0})$ is of the second category and, therefore, according to the Banach open mapping theorem, we conclude that F_{k_0} is an open mapping from U^{k_0} onto X. On the other hand, by the assumption int $\Omega \neq \emptyset$, the set Ω^{k_0} has also non-empty interior in the Banach space U^{k_0}. Hence, the reachable set S_{k_0} $(=F_{k_0}(\Omega^{k_0}))$ of system (1) has non-empty interior in X. Applying Lemma 1 with $M_j = S_{k_0+j}$ $(j=0, 1, 2, \ldots)$ we obtain that $0 \in \text{int}\ S_{k_1}$ for some finite k_1. This completes the proof of b).

c) Denote by U_1^k the unit ball of the Banach space U^k. Then, by the boundeness of Ω and the assumption $0 \in \text{int}\ \Omega$, we can choose the positive numbers $\varepsilon_1, \varepsilon_2$ $(0 < \varepsilon_1 < \varepsilon_2)$ such that $\varepsilon_1 U_1^k \subset \Omega^k \subset \varepsilon_2 U_1^k$. It follows that

$$\varepsilon_1 F_k(U_1^k) \subset F_k(\Omega^k) \subset \varepsilon_2 F_k(U_1^k). \tag{5}$$

Hence

$$\varepsilon_1 \overline{F_k(U_1^k)} \subset \overline{F_k(\Omega^k)} \subset \varepsilon_2 \overline{F_k(U_1^k)}. \tag{6}$$

Letting the system (1) be locally controllable after k steps we find a positive $\lambda > 0$ such that $\lambda X_1 \subset \bar{S}_k = \overline{F_k(\Omega_k)}$, where X_1 denotes the unit ball of the Banach space X. Therefore, by (6) we have $\frac{\lambda}{\varepsilon_2} X_1 \subset \overline{F_k(U_1^K)}$. Since F_k is a linear bounded operator from the Banach space U^k to the Banach space X, it implies from the last inclution that $\frac{\lambda}{\varepsilon_2} X_1 \subset F_k(U_1^K)$ (see [10], Lemma 4.13, p. 113). Hence, by (5) we obtain

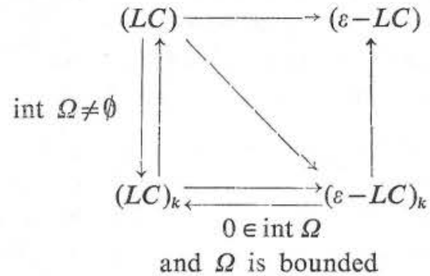
$$\frac{\varepsilon_1 \lambda}{\varepsilon_2} X_1 \subset F_k(\Omega^k) = S_k.$$

This means that $0 \in \text{int}\ S_k$ and so the system (1) is locally controllable after k steps.

The proof of Theorem I is complete.

REMARKS 1. Let u_0 be an arbitrary element of Ω such that $Bu_0 = 0$ and let V be the closed linear span of $\Omega - u_0$: $V = sp\,(\Omega - u_0)$. Denote by $ri\,(\Omega - u_0)$ the relative interior of $\Omega - u_0$, i.e. the interior of $\Omega - u_0$ relative to the inducde norm on V. Then, the parts b) and c) in the theorem 1 remain true if one replace the assumption $(0 \in)$ int $\Omega \neq \emptyset$ by the weaker one, namely by $(0 \in)$ $ri\,(\Omega - u_0) \neq \emptyset$. Indeed, it suffices to replace U^k, U_1^k and Ω^k respectively by V^k, V_1^k and $\Omega^k - u_0^k$, where V^k is defined similarly, V_1^k is the unit ball of V^k and $u_1^k = (u_0, u_0, \ldots, u_0)$, and we may observe that all the argumentation in the above proof remains valid.

The results obtained above are collected in the following diagram, where implications are denoted by arrows; some arrows are drawn together with the additional conditions, under which these implications hold.



To end this section, we note that if the state space X is infinite-dimensional and the operator $B: U \to X$ is compact, the system (1) is not locally controllable. Moreover, if in addition, the control set Ω is bounded, then the system (1) is not even locally ε-controllable after any finite number of steps. This property follows from the fact the controllability operator F_k of the system (1) defined by (3) is compact whenever the operator B is compact.

4. Criteria of Local and Global Controllability

The main result of this paper is the following.

THEOREM 2. *Suppose that the control set Ω is convex, has non-empty interior in U and satisfies the additional condition* (2). *Then a necessary and sufficient condition for the system* (1) *to be locally controllable is that*:

a) *The dual operator A^* has no eigenvector with nonnegative eigenvalue, supporting to the set $B\Omega$ at the origin*;

b) *The corresponding system with unconstrained controls*

$$x_{k+1} = Ax_k + Bu_k, \quad x_k \in X, \quad u_k \in U \tag{7}$$

is strongly (*in Fuhrmann's sense*) *controllable, or, equivalently, for some integer $k \geqslant 1$*

$$\text{sp}\,\{BU, ABU, \ldots, A^{k-1}BU\} = X \tag{8}$$

Proof. Necessity. The necessity of b) evidently follows from the definition. To prove the necessity of a) we assume that there exists a nonzero element $x_0^* \in X^*$ such that $A^* x_0^* = \lambda x_0^*$, $\lambda \geqslant 0$ and x_0^* is a support functional of $B\Omega$ at the origin: $x_0^*(Bu) \geqslant 0$ for all $u \in \Omega$. Then for each $u \in \Omega$ and $j \geqslant 0$ we have $x_0^*(A^j Bu) = (A^{*j} x_0^*) \cdot (Bu) = \lambda^j x_0^*(Bu) \geqslant 0$. Hence, it follows from the definition of the reachable set of system (1) that $x_0^*(x) \geqslant 0$ for all $x \in S$. However, this contradicts the local controllability of the system (1) and so proves the necessity of a).

Sufficiency. As in the proof of b) in Theorem 1, it follows from the strong controllability of system (7) that for some finite $k>0$ the controllability operator F_k is open. Therefore, by convexity of Ω and the assumption int $\Omega\neq\emptyset$, the reachable set $S_k\,(=F_k\,(\Omega^k))$ is convex and has non-empty interior in X. Since $S_j\subset S_{j+1}$ by (2), we conclude that the reachable set S of the system is also convex and has non-empty interior in X. It remains to show that $0\in\text{int }S$. Suppose that this is not the case. Consider the cone C generated by the reachable set S, i.e. $C=\bigcup_{\gamma>0}\gamma S$. Clearly C is a convex cone with non-empty interior and C is not dense in X. We show that C is invariant under the operator A, i.e. $AC\subset C$. For this, it suffices to show that $AS\subset S$. Let x be an arbitrary vector in S. Then by definition, there exists a sequence of controls in Ω, say $u_1, u_2, ..., u_m$, such that

$$x=A^m\,Bu_1+A^{m-1}\,Bu_2+...+ABu_{m-1}+Bu_m.$$

Hence, $Ax=A^{m+1}\,Bu_1+A^m\,Bu_2+...+A^2\,Bu_{m-1}+ABu_m$. By (2), putting $u_{m+1}=u_\theta$ we can rewrite the previous relation as follows

$$Ax=A^{m+1}\,Bu_1+A^m\,Bu_2+...+A^2\,Bu_{m-1}+ABu_m+Bu_{m+1}.$$

This means that Ax is reachable from the origin according to the system (1) after $m+1$ steps by virtue of the sequence of controls $u_1, u_2, ..., u_m, u_{m+1}$ taken in the restraint set Ω. It follows that $Ax\in S$ and, on account of arbitrariness of $x\in S$, we get $AS\subset S$. Further, by the Krein-Rutman theorem (see Theorem A in the Introduction) there exists a nonzero positive functional $x_0^*\in C_0^*$ and a nonnegative number λ such that $A^*\,x_0^*=\lambda x_0^*$. Since $x_0^*\in C_0^*$, $x_0^*\,(x)\geqslant 0$ for all $x\in S$. In particular, it follows that $x_0^*\,(Bu)\geqslant 0$ for all $u\in\Omega$, i.e. x_0^* is a support functional of $B\Omega$ at the origin. This contradiction to the condition a) concludes the proof of the sufficiency. The theorem has been completely proved.

We derive some important consequences of Theorem 2.

COROLLARY 1. *Assume that the control set Ω satisfies all conditions of theorem 2 and, besides, the point spectrum of the dual operator A^* is empty. Then the system (1) is locally controllable if and only if the corresponding system with unconstrained controls (7) is strongly controllable.*

Proof is obvious.

As an example, consider a control system described by the following integro-difference equation of Voltera type

$$x_{k+1}\,(t)=\int_0^t x_k\,(s)\,ds+u_k\,(t),\ t\in[0,1],$$

$$x_k\,(\cdot)\in L_2\,[0,1],\ u_k\,(\cdot)\in\Omega\subset L_2\,[0,1].$$

The system is locally controllable if for the control set we take

$$\Omega=\left\{f(\cdot)\in L_2\,[0,1],\quad \int_0^1 |f(s)-f_0\,(s)|^2 ds\leqslant 1\right\},$$

where $f_0\,(\cdot)$ is a function with a L_2-norm equal to 1.

Indeed, the above restraint set Ω satisfies all conditions of Theorem 2. Further, the spectrum of the operator A defined by our example consists only of the points of the continuous spectrum, and, therefore, the point spectrum of A^* is empty. Note that the system with unconctrained controls

$$x_{k+1}(t)=\int_0^t x_k(s)\,ds+u_k(t),\ x_k(\cdot)\in L_2\,[0,1],\ u_k(\cdot)\in L_2\,[0,1]$$

is strongly controllable. Consequently, this system is locally controllable by the previous corollary.

Applying Theorem 2 to finite-dimensional systems, where $X=R_n$, $U=R_m$ and so A and B are real matrices, we firstly note that the condition b) of Theorem 2 can be replaced by the wellknown algebraic test of Kalman for complete controllability

$$\text{rank}\,\{B, AB, A^2B, \ldots, A^{n-1}B\}=n. \tag{9}$$

Therefore, we obtain the following corollary of Theorem 2, which can be considered as a discrete version of Brammer's criterion of controllability for continuous linear systems with constrained controls [7].

COROLLARY 2. *Let the control set Ω be convex and have a non-empty interior in R_m. Then a necessary and sufficient condition for the system*

$$x_{k+1}=Ax_k+Bu_k,\ x_k\in R_n,\ u_k\in\Omega\subset R_m \tag{10}$$

to be locally controllable is that:

a) *The dual matrix A^* has no eigenvector with nonnegative eigenvalue, supporting to the set $B\Omega$ at the origin:*

b) rank $\{B, AB, A^2B, \ldots, A^{n-1}B\}=n$.

COROLLARY 3. *Suppose that the control set Ω is a convex cone with a non-empty interior in R_m. Then a necessary and sufficient condition for the system* (10) *to be globally controllable is that:*

a) *The dual matrix A^* has no eigenvectors with nonnegative eigenvalue, supporting to the set $B\Omega$ at the origin;*

b) *rank* $\{B, AB, A^2B, \ldots, A^{n-1}B\}=n$.

Proof. Obvious.

Let b be a vector in R_n. Then for any vector $x\in R_n$, clearly, either x or $-x$ is support to the ray $\{bu, u\geqslant 0\}$. Hence Corollary 3, in particular, implies the following results of Evans and Murthy [6].

COROLLARY 4. *The system in R_n with single input*

$x_{k+1}=Ax_k+bu_k$

is globally controllable by virtue of nonnegative controls $u_k\geqslant 0$ if and only if

a) *The matrix A has no real eigenvalues $\lambda\geqslant 0$;*

b) *rank* $\{b, Ab, A^2b, \ldots, A^{n-1}b\}=n$

REMARK 2. It is worth to emphasize that the condition of the control set Ω to have nonempty interior in the control space U is essential for Theorem 2. We illustrate this with the following

Example 4. Consider the system

$x_{k+1}=Ax_k+u_k,\ u_k\in\Omega\subset l_2,$

defined on $X=l_2$, $x=(\xi_1, \xi_2, ...,)$, while A is the left shift operator: $A(\xi_1, \xi_2, \xi_3, ...)= =(\xi_2, \xi_3, ...)$ and Ω is the set of all nonnegative vectors in l_2, i.e.

$$\Omega=\{u=(\xi_1, \xi_2, ...): \xi_j\geqslant 0\}.$$

Since the dual operator A^* has no eigenvectors, the condition a) of Theorem 2 is satisfied automatically. Further, it is clear that the system with unconstrained controls $x_{k+1}=Ax_k+u_k$, $u_k\in l_2$ is strongly controllable, i.e. the condition b) of Theorem 2 also holds. But this system is not locally controllable since its reachable set S is equal to Ω, and henceforth, int $S=\emptyset$.

However, Theorem 2 can be actually strengthened, replacing the assumption int $\Omega\neq\emptyset$ by the weaker condition that the relative interior of the set $\Omega-u_0$ for some $u_0\in\Omega$ is non-ampty: $ri(\Omega-u_0)\neq\emptyset$. More precisely, we obtain the following theorem.

THEOREM 3. *Suppose Ω is a convex subset of U satisfying the condition* (2). *If for some $u_0\in\Omega$ with $Bu_0=0$ the relative interior of the set $\Omega-u_0$ is not empty, then a necessary and sufficient condition for the system* (1) *to be locally controllable is that*:

a) *The dual operator A^* has no eigenvectors with nonnegative eigenvalues supporting to the set $B\Omega$ at the origin*:

b) *The system with unconstrained controls*

$$x_{k+1}=Ax_k+\hat{B}v_k,\ x_k\in X,\ v_k\in V \tag{11}$$

is strongly controllable, where $\hat{B}$ denotes the restriction of the operator B to the subspace V and $V\triangleq\overline{\text{sp}}\,(\Omega-u_0)$.

We omit the proof of this theorem since it is not difficult to show, using the Remark 1 of Theorem 1.

The following example shows an application of Theorem 3.

Example 5. Consider the system defined as in Example 4 but instead of the previous control set we take

$$\Omega=\{u=(\xi_1, \xi_2, ...): \|u\|\leqslant 1 \text{ and } \xi_j=0 \text{ for } j=1, 2, ..., k\}.$$

We observe that int $\Omega=\emptyset$, but the interior of Ω relative to the subspace V of all vectors of the form $u=(0, ..., 0, \xi_k, \xi_{k+1}, ...)$ is non-empty, i.e. $ri\Omega\neq\emptyset$. Furthermore, clearly

$$\text{sp}\,\{BV, ABV, ..., A^{k-1}BV\}=X=l_2.$$

Thus, the system $x_{k+1}=Ax_k+Bv_k$, $v_k \in V$ is strongly controllable. Since the condition a) and b) of Theorem 3 are satisfied, we conclude that the system under considation is locally controllable.

Applying Theorem 3 to systems in R_n, where, as is well known, any convex subset has a non-empty relative interior, we can state the following result with the assumption int $\Omega \neq \emptyset$ being removed.

COROLLARY 5. *Suppose Ω is a convex subset of R_m satisfying the condition* (2). *Then for the system*

$$x_{k+1}=Ax_k+Bu_k, \ x_k \in R_n, \ u_k \in \Omega \subset R_m \tag{12}$$

to be locally controllable, a necessary and sufficient condition is that:

a) *No eigenvector of the dual matrix A^* with nonnegative eigenvalues supports to the set $B\Omega$ at the origin;*

b) *No eigenvector of the dual matrix A^* is orthogonal to the set $B\Omega$.*

Proof. Evidently, it suffices to show that the condition b) of this corollary is equivalent to the strong controllability of system (11). Indeed, if $h \in R_n$ is an eigenvector of A^* such that $(h, Bu)=0$ for all $u \in \Omega$, then, clearly, $(h, A^j Bv)=0$ for all $v \in V$ and $j \geqslant 0$. Thus, the vector h is orthogonal to the reachable set of the system (11) and so the system (11) is not strongly controllable. Conversely, assume that the condition b) of Theorem 3 fails. Then the reachable set S of the system (11) is a proper subspace of R_n. On the other hand, since S $(=\mathrm{sp}\,\{BV, ABV, \ldots\})$ is invariant under the operator A, the orthogonal complement S^1 is invariant under the dual operator A^*. Let $h \in S^1$ be (in general, a complex) eigenvector of A^*. Then $(h, x)=0$ for all $x \in S$. In particular, we get $(h, Bu)=0$ for all $u \in \Omega$. This contradiction completes the proof.

COROLLARY 6. *Suppose that the control set Ω is a convex cone in R_m. Then the conditions a) and b) of Corollary* 5 *are necessary and sufficient for the global controllability of system* (12).

5. Conclusion

Some criteria for local and global controllability of linear discrete-time systems, which have, in general, the Banach spaces for their states and controls spaces have been derived. From these results we have obtained, as corollaries, some easily verified tests for controllability of finite-dimensional discrete-time systems with constrained controls. They include, in particular, the previous results of Evans and Murthy. Central to our method is the application of the well-known Krein--Rutman theorem to the controllability problem. An analogous technique has been used in [11] and [12] for continous-time systems of the form $\dot{x}=Ax+Bu$, $u \in \Omega$.

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Sterowalność liniowych układów dyskretnych w czasie z ograniczeniami sterowania w przestrzeni Banacha

Rozpatruje się pewne zagadnienia sterowalności dla danego układu z dyskretnym czasem opisanego równaniem $x_{k+1}=Ax_k+Bu_k$, gdzie $x_k \in X$, $u_k \in U$, X i U są przestrzeniami Banacha oraz A i B są ograniczonymi operatorami liniowymi. W pracy zakłada się, że $u_k \in \Omega \subset U$, gdzie O jest niekoniecznie punktem wewnętrznym zbioru Ω. Podaje się warunki konieczne i wystarczające sterowalności lokalnej i globalnej, ilustrując je przykładami. Dla układów skończeniewymiarowych wnioski z głównego twierdzenia zawierają pewne dobrze znane wyniki Evansa i Murthy'ego. Dowód głównego rezultatu jest oparty na słynnym twierdzeniu Kreina-Rutmana o właściwościach stożka wypukłego niezmienniczego względem ograniczonego operatora liniowego w przestrzeni Banacha.

Управляемость линейных дискретных во времени систем с ограничениями по управлению в банаховом пространстве

Рассматриваются некоторые вопросы управляемости для данной системы с дискретным временем, описанной уравнением $x_{k+1}=Ax_k+Bu_k$, где $x_k \in X$, $u_k \in И$, X и $И$ являются пространствами Банаха, A и B являются ограниченными линейными операторами.

В работе предполагается, что $u_k \in \Omega \subset U$, где O является не обязательно внутренней точкой множестра Ω. Даются необходимые и достаточные условия локальной и глобальной управляемости, иллюстрируя их на примерах.

Для конечномерных систем выводы из основной теоремы содержат некоторые хорошо известные результаты Эванса и Марти. Доказательство основного результата основано на известной теореме Крейна-Рутмана о свойствах выпуклого конуса, инвариантного по отношению к ограниченному линейному оператору в банаховом пространстве.