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A Schur approach to pole assignment problem

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A new approach to the pole assignment in linear systems is proposed which is based on unitary or orthogonal transformation of the closed loop system matrix to its Schur canonical form. The method has a number of advantages over the other known methods. In particular it does not require the computation of the characteristic polynomial of the open loop system matrix or the transformation to phase-variable canonical form.

There is an analogy between the method proposed and the *QR*-method for finding the Schur canonical form of a matrix. The new approach can be the basis for development of a numerically stable algorithm for pole assignment.

1. Introduction

The problem of pole assignment (PA) in linear time-invariant systems is completely solved from theoretical point of view [1]. However in many cases its numerical solution is unsatisfactory especially for high order systems. This may be a result of the illconditioning of the problem and/or the numerical instability of the methods used for this purpose. Unfortunately there are no investigations up to this moment concerned with the conditioning of the PA problem and the numerical properties of the existing methods have not been considered in detail. Moreover, it can be observed that most of these methods are numerically unstable. For example the methods based on preliminary transformation of the system in Luenberger canonical form are unstable since in general the Frobenius form of a matrix can not be obtained by stable similarity transformations [2].

In the present paper a new approach to PA in linear systems is proposed. It is based on the well known fact that an arbitrary matrix can be reduced to Schur canonical form using unitary or orthogonal similarity transformations [2]. The latter are extremally favorable from computational point of view since they do not increase the norm of the matrix.

According to the new approach the gain matrix is determined so that the Schur form of the closed loop system matrix (CSM) has a desired spectrum. The method for PA is realized by two algorithms. In the first of them unitary transformations are implemented in order to reduce the CSM into complex upper triangular form (complex Schur canonical form) whose diagonal elements are the desired poles. In the second one the CSM is reduced into real upper quasitriangular form (real Schur canonical form) by orthogonal transformations.

Further on the following abbreviations are used: F_n^m — the space of $n\times m$ — matrices over $F\,(F_1^n=F^n, F_n^1=F_n, F_n^1=F)$, where F is the field of real ($F=R$) or complex ($F=C$) numbers; $A^T\,(A^H)$ — the transposed (conjugate transposed) matrix A; $S\,(n)\subset C$ — the set of all collections of n pair-wise conjugate complex numbers $s_1, \ldots, s_n$; $\mathrm{spect}(A)\subset S\,(n)$ — the spectrum of $A\in R_n^n$; $\|A\|$ and $\|A\|_e$ — the spectral and Euclidean norms of A; $GL\,(n, F)\subset F_n^n$, $U\,(n)\subset C_n^n$ and $O\,(n)\subset R_n^n$ — the groups of nonsingular, unitary and orthogonal matrices resp.

2. Statement of the Problem

Consider the completely controllable time-invariant single-input system

$$\dot{x}\,(t)=Ax\,(t)+bu\,(t), \tag{1}$$

where $x\,(t)\in R^n$, $u\,(t)\in R$ and $A\in R_n^n$, $b\in R^n$.

Denote $L=[A, b)\in LS\,(n)$, where $LS\,(n)\subset R_n^n\times R^n$ is the set of matrix pairs $[A, b)$ with A cyclic and b a generator for R^n relative to A, and let $s=\{s_1, \ldots, s_n\}\in$ $\in S\,(n)$. Then the PA problem for system (1) is defined in the following way: Find a gain matrix $k=k\,(s)\in R_n$ such that the control law $u\,(t)=kx\,(t)$ preassignes the spectrum of CSM, $\mathrm{spect}(A_c)=s$, $A_c=A+bk$. Note that in the single-input case the matrix $k\,(s)$ is uniquely determined for each s.

Denote by $h_s\,(p)=(p-s_1)\ldots(p-s_n)=p^n-(s_1+\ldots+s_n)\,p^{n-1}+\ldots+(-1)^n s_1\ldots s_n=$ $=p^n+d_{n-1}\,(s)\,p^{n-1}+\ldots+d_0\,(s)$ the desired characteristic polynomial of A_c. Then the relation $\det(pI_n-A_c)=h_s\,(p)$ for determining $k=k\,(s)$ is equivalent to the linear algebraic equation

$$kD\,(L)=d\,(s_A)-d\,(s) \tag{2}$$

where $s_A=\mathrm{spect}\,(A)$ and $d\,(s)=[d_0\,(s)\,\vdots\ldots\vdots\,d_{n-1}\,(s)]\in R_n$. The matrix $D\,(L)\in R_n^n$ in (2) is uniquely determined and is nonsingular iff the pair L is completely controllable.

It follows from eqn. (2) that the accuracy of the computation of k is determined by the condition number $c\,(L)=\mathrm{cond}\,D\,(L)=\|D\,(L)\|\,\|D^{-1}\,(L)\|$ of the matrix $D\,(L)$. Further on we shall refer to $c\,(L)$ as the condition number of the PA problem for the concrete realization L of (1).

Note that the matrix $D(L)$ may be very ill-conditioned even if the controllability matrix $C(L)=[b \vdots Ab \vdots \dots \vdots A^{n-1} b]$ is well-conditioned.

The number $c(L)$ is inherent for the problem considered and its value is crucial for the computation only of k in the presence of uncertain data and round-off errors. At the same time spect (A_c) may be very sensitive in variations of k. This imposes even more accuracy requirements on the numerical algorithms for PA.

In each method for solving the PA problem the matrix $D(L)$ is obtained in a specific way which in turn may deteriorate the solution. For example the method based on transformation of L into phase-variable canonical form corresponds to the factorization $D(L)=C(L)\, M(s_A)$ where $M(s_A)=[m_{ij}] \in R_n^n$, $m_{ij}=m_{ji}=$ $=d_{i+j-1}(s_A)$ $(d_n=1, d_i=0$ for $i \geq n+1)$ and $\det M(s_A)=1$ [3]. However it is well known [4] that the computation of $C(L)$ may be associated with great errors which leads to the numerical instability of this method.

Analogous difficulties arise also in the other methods for PA. Thus the development of numerically stable PA methods turns out to be an important and urgent problem in the synthesis of linear systems.

The method proposed in this paper is based on successive determination of the matrices $V \in U(n)$, $F \in C_n^n$ and k satisfying the relation

$$V^H (A+bk)\, V=F, \tag{3}$$

where

$$F=\begin{bmatrix} s_1 & f_2 & & & \\ & s_2 & & & f_n \\ & & \ddots & & \\ & & & \ddots & \\ & 0 & & & s_n \end{bmatrix}$$

and $f_i \in C^{i-1}$ $(i=2, \dots, n)$ are unknown vectors.

A modification of the method using orthogonal transformations is also developed.

A preliminary step in both cases is the transformation of L into the form $\tilde{L}=$ $=[\tilde{A}, \tilde{b})=[P^T AP, P^T b)$, $p \in O(n)$, where $\tilde{b}=[\|b\| \vdots 0 \vdots \dots \vdots 0]^T$. This may be accomplished by one numerically stable Householder transformation.

3. Pole Assignment by Unitary Triangularization

Rewrite equation (3) as

$$(\tilde{A}+\tilde{b}\tilde{k})\, \tilde{V}=\tilde{V}F, \tag{4}$$

where $\tilde{V}=P^T V \in U(n)$, $\tilde{k}=kP \in R_n$, and denote $e_n=[1 \vdots 0 \vdots \dots \vdots 0] \in R_n$, $E_n=[0 \vdots I_{n-1}] \in$ $\in R_n^{n-1}$. Now equation (4) takes the form

$$e_n (\tilde{A}+\tilde{b}\tilde{k})\, \tilde{V}=e_n \tilde{V}F,$$

$$E_n (\tilde{A}+\tilde{b}\tilde{k})\, \tilde{V}=E_n \tilde{V}F.$$

Since $E_n \tilde{b}=0$ one obtains

$$\tilde{k}=(v^1 F\tilde{V}^H-a^1)/\|b\| \tag{5}$$

and

$$E_n \tilde{A}\tilde{V}=E_n \tilde{V}F, \tag{6}$$

where $v^1=e_n \tilde{V}$, $a^1=e_n \tilde{A}$.

Let the matrix $\tilde{V}$ be partitioned as $\tilde{V}=[v_1 \vdots \dots \vdots v_n]$, $v_i \in C^n$ and set $V_i=[v_1 \vdots \dots \vdots v_{i-1}] \in C^n_{i-1}$. Then (6) is equivalent to the system of n vector equations

$$E_n \tilde{A}v_1=E_n [v_1 \vdots \dots \vdots v_n]\begin{bmatrix} s_1 \\ \hdashline 0 \end{bmatrix}=s_1 E_n v_1 , \tag{7}$$

$$E_n \tilde{A}v_i=E_n [V_i \vdots v_i \vdots \dots \vdots v_n]\begin{bmatrix} f_i \\ \hdashline s_i \\ \hdashline 0 \end{bmatrix}=E_n V_i f_i+s_i E_n v_i , \tag{8}$$

$i=2, \dots, n$.

It follows from (7), (8) that

$$A_1 v_1=0, \tag{9}$$

$$A_i v_i=W_i f_i,\ i=2, \dots, n \tag{10}$$

where $A_i=E_n (\tilde{A}-s_i I_n)$, $W_i=E_n V_i$.

Now the vectors v_i, f_i and the matrix k can be determined successively by the following algorithm called *UT* (Unitary Triangularization):

Step 1. Since rank $G_1 \leqslant n-1$, $G_1=A_1$, the general solution of (9) is $z_1=v_1=pc_1$, where $0 \neq c_1 \in C^n$ is any non-zero solution of (9) and $p \in C$ is an arbitrary parameter. The condition $v_1^H v_1=1$ yields $|p|^2 c_1^H c_1=1$, i.e. $p=e^{j_0 p_0} (c_1^H c_1)^{-1/2}$, $j_0^2=-1$. Choose for simplicity $p_0=0$. Then

$$v_1=(c_1^H c_1)^{-1/2} c_1 . \tag{11}$$

Step i ($2 \leqslant i \leqslant n$). Let the matrix V_i be determined as a result of steps $1, \dots, i-1$. Denote $z_i=[v_i^T \vdots f_i^T]^T \in C^{n+i-1}$. Then equation (10) together with the orthogonality condition $V_i^H v_i=0$ is equivalent to

$$G_i z_i=0 \tag{12}$$

where

$$G_i=\begin{bmatrix} A_i & -W_i \\ \hdashline V_i^H & 0 \end{bmatrix}.$$

Since rank $G_i \leqslant n+i-2$ the linear equation (12) has a solution $z_i=pc_i$, $p \in C$, where $0 \neq c_i=[v_{i0}^T \vdots f_{i0}^T]^T \in C^{n+i-1}$ is any fixed solution of (12). Moreover, it follows from the complete controllability of L that $v_{i0} \neq 0$ (see the Appendix). Now the condition $v_i^H v_i=1$ yields $|p|^2 v_{i_0}^H v_{i0}=1$, i.e. $p=e^{j_0 p_0} (v_{i0}^H v_{i0})^{-1/2}$.

Choosing $p_0=0$ one obtains

$$v_i=(v_{i0}^H v_{i0})^{-1/2} v_{i0}, \ f_i=(v_{i0}^H v_{i0})^{-1/2} f_{i0}. \tag{13}$$

Final step. After the determination of $\tilde{V}$ and F from (11), (13) the matrix $\tilde{k}$ is given by (5). Thus the solution of the PA problem is

$$k=\tilde{k}P, \ V=P\tilde{V}. \tag{14}$$

Note that the solution of the homogeneous equations $G_j z_j=0$, $j=1, \ldots, n$, can be accomplished reliably by singular value decomposition (SVD) [5]. Let $G_j=L_j [D_j \vdots 0] R_j^H$ ($L_j \in U(n+j-2)$, $R_j \in U(n+j-1)$) be the SVD of G_j, where $D_j=\operatorname{diag}(d_{j1}, \ldots, d_{j,n+j-2})$ and $d_{j1} \geqslant \ldots \geqslant d_{j,n+j-2} \geqslant 0$ are the singular values of G_j. Then a particular non-zero solution z_{j0} is the last column of R_j.

It can be shown that if $s_i \in \boldsymbol{R}$ then the vector v_i can be chosen also real.

4. Pole Assignment by Orthogonal Triangularization

An analogous algorithm can be derived using orthogonal quasitriangularization of A_c.

Note first that if the desired poles $s_1, \ldots, s_n$ are real then the unitary triangularization in algorithm UT can be replaced directly by an orthogonal one choosing $V \in \boldsymbol{O}(n)$, $F \in \boldsymbol{R}_n^n$ and substituting V^T instead of V^H, etc.

Let us now consider the general case when the desired spectrum of A_c is $s=\{a_1-j_0 b_1, a_1+j_0 b_1, \ldots, a_m-j_0 b_m, a_m+j_0 b_m, s_{2m+1}, \ldots, s_n\}$, where $a_1, b_1, \ldots, a_m, \ldots \ldots b_m, s_{2m+1}, \ldots, s_n \in \boldsymbol{R}$; $2m \leqslant n$.

When only $\boldsymbol{O}(n)$ — transformations are used the matrix F in (3) must be in the form

$$F=\left[\begin{array}{cccccccc} S_1 & F_2 & & F_m & f_{2m+1} & & f_n \\ & S_2 & & & & & \\ & & \ddots & & & & \\ & & & S_m & & & \\ & & & & s_{2m+1} & & \\ & 0 & & & & \ddots & \\ & & & & & & s_n \end{array}\right] \in \boldsymbol{R}_n^n, \tag{15}$$

where

$$S_j=\left[\begin{array}{c:c} a_j & -b_j/r_j \\ \hdashline b_j r_j & a_j \end{array}\right] \in \boldsymbol{R}_2^2, \ F_i=[f_{2i-1} \vdots f_{2i}] \in \boldsymbol{R}_2^{2i-2} \tag{16}$$

$f_r \in \boldsymbol{R}^{r-1}$; $i=2, \ldots, m$; $j=1, \ldots, m$; $r=2m+1, \ldots, n$ and $r_j \neq 0$ are real parameters which will be determined later.

REMARK. Using $O(2)$—transformations a real 2×2—matrix with complex spectrum can not be reduced to the form $\left[\begin{array}{c|c} a_j & -b_j \\ \hline b_j & a_j \end{array}\right]$, the latter being attainable in general by $GL(2, \boldsymbol{R})$—transformations.

Let the matrix $\tilde{V}\in \boldsymbol{O}(n)$ be partitioned as

$$\tilde{V}=[\tilde{V}_1 \mid \dots \mid \tilde{V}_m \mid v_{2m+1} \mid \dots \mid v_n],\ \tilde{V}_i=[v_{2i-1} \mid v_{2i}]\in \boldsymbol{R}^n_2,\ v_j\in \boldsymbol{R}^n.$$

It follows from (4), (15), (16) that

$$\tilde{k}=(v^1 F\tilde{V}^T-a^1)/\|b\| \tag{17}$$

and

$$E_n \tilde{A} \tilde{V}_1=E_n \tilde{V}_1 S_1, \tag{18}$$

$$E_n \tilde{A}\tilde{V}_i=E_n V_{2i-1} F_i+E_n \tilde{V}_i S_i,\ i=2, \dots, m, \tag{19}$$

$$E_n \tilde{A}v_r=E_n V_r f_r+s_r E_n v_r,\ r=2m+1, \dots, n, \tag{20}$$

where $v^1=e_n \tilde{V}$, $a^1=e_n \tilde{A}$ and $V_j=[v_1 \dots v_{j-1}]\in \boldsymbol{R}^n_{j-1}$.

Having in mind (16) equations (18)-(20) take the form

$$A_1 v_1=b_1 r_1 E_n v_2,\ A_1 v_2=-(b_1/r_1) E_n v_1, \tag{21}$$

$$\begin{gathered} A_i v_{2i-1}=W_{2i-1} f_{2i-1}+b_i r_i E_n v_{2i}, \\ A_i v_{2i}=W_{2i-1} f_{2i}-(b_i/r_i) E_n v_{2i-1},\ i=2, \dots, m, \end{gathered} \tag{22}$$

$$A_r v_r=W_r f_r,\ r=2m+1, \dots, n, \tag{23}$$

where $A_j=E_n(\tilde{A}-a_j I_n)$, $j=1, \dots, m$; $A_r=E_n(\tilde{A}-s_r I_n)$, $r=2m+1, \dots, n$ and $W_j=$ $=E_n V_j$, $j=1, \dots, n$.

Using (17), (21)-(23) the following algorithm (called OT — Orthogonal Triangularization) for computation of V, F and k can be derived:

Step 1. Set $x_1=v_1$, $x_2=r_1 v_2$ and rewrite (21) as $H_1 z_1=0$,

where $z_1=[x_1^T \mid x_2^T]^T\in \boldsymbol{R}^{2n}$ and $H_1=\left[\begin{array}{c|c} A_1 & -b_1 E_n \\ \hline b_1 E_n & A_1 \end{array}\right]\in \boldsymbol{R}^{2n-2}_{2n}$. (24)

The linear equation (24) has at least two linearly independent solutions, say $[c_1^T \mid c_2^T]^T$, $[d_1^T \mid d_2^T]^T$. Hence $x_1=pc_1+gd_1$, $x_2=pc_2+gd_2$, where $p, q\in \boldsymbol{R}$. The conditions $v_1^T v_2=0$ and $v_1^T v_1=1$ are equivalent to

$$p^2 c_1^T c_2+pq(c_1^T d_2+d_1^T c_2)+q^2 d_1^T d_2=0, \tag{25}$$

$$p^2 c_1^T c_1+2pqc_1^T d_1+q^2 d_1^T d_1=1. \tag{26}$$

The solution of (25), (26) is

$$q=\pm(w^2 c_1^T c_1+2wc_1^T d_1+d_1^T d_1)^{-1/2} \tag{27}$$

$$p=wq, \tag{28}$$

where

$$w=(-c_1^T d_2-d_1^T c_2\pm((c_1^T d_2+d_1^T c_2)^2-4c_1^T c_2 d_1^T d_2)^{1/2})/(2c_1^T c_2) \tag{29}$$

The parameter r_1 is determined from $v_2^T v_2=r_1^{-2} x_2^T x_2=1$, i.e.

$$r_1=\pm q\,(w^2 c_2^T c_2+2w\, c_2^T d_2+d_2^T d_2)^{1/2}. \tag{30}$$

Step i $(2\leqslant i\leqslant m)$. Set $x_{2i-1}=v_{2i-1}$, $x_{2i}=r_i v_{2i}$. $y_{2i-1}=f_{2i-1}$, $y_{2i}=r_i f_{2i}$ and ewrite (22) together with the orthogonality conditions $V_{2i-1}^T[v_{2i-1} \vdots v_{2i}]=0$ asr

$$H_i z_i=0, \tag{31}$$

where $z_i=[x_{2i-1}^T \vdots x_{2i}^T \vdots y_{2i-1}^T \vdots y_{2i}^T]^T \in R^{2n+2i-2}$ and

$$H_i=\left[\begin{array}{c:c:c:c} A_i & -b_1 E_n & -W_{2i-1} & 0 \\ \hdashline b_i E_n & A_i & 0 & -W_{2i-1} \\ \hdashline V_{2i-1}^T & 0 & & \\ \hdashline 0 & V_{2i-1}^T & \multicolumn{2}{c}{0} \end{array}\right] \in R^{2n+2i-4}_{2n+2i-2}.$$

The homogeneous equation (31) has at least two linearly independent solutions, say $[c_{2i-1}^T \vdots c_{2i}^T \vdots q_{2i-1}^T \vdots q_{2i}^T]^T$, $[d_{2i-1}^T \vdots d_{2i}^T \vdots h_{2i-1}^T \vdots h_{2i}^T]^T$, where one of the vectors c_{2i-1}, d_{2i-1} and c_{2i}, d_{2i} is non-zero (see the Appendix). Hence $x_{2i-1}=pc_{2i-1}+$ $+qd_{2i-1}$, $x_{2i}=pc_{2i}+qd_{2i}$, $y_{2i-1}=pq_{2i-1}+qh_{2i-1}$, $y_{2i}=pg_{2i}+qh_{2i}$, where the parameters $p, q\in R$ are determined from the conditions $v_{2i-1}^T v_{2i}=0$, $v_{2i-1}^T v_{2i-1}=1$, i.e. one can use (27)-(29) where the indices 1, 2 are replaced by $2i-1$, $2i$. Finally r_i is obtained from $v_{2i}^T v_{2i}=r_i^{-2} x_{2i}^T x_{2i}=1$ via (30) after appropriate index substitution.

Steps $2m+1, \ldots, n$ for determining $v_{2m+1}, \ldots, v_n$, $f_{2m+1}, \ldots, f_n$ are carried out as steps $2m+1, \ldots, n$ of algorithm *UT*. Now the solution of the *PA* problem is given by (14) in view of (17).

5. Numerical Example

Consider the pair $L=[A,b)\in LS(3)$, $A=\begin{bmatrix}1 & -2 & 2\\ 1 & 0 & 1\\ 0 & 2 & -1\end{bmatrix}$, $b=\begin{bmatrix}1\\0\\0\end{bmatrix}$ and let the desired spectrum of A_c be $s=\{-2, -1-j_0, -1+j_0\}$. Then $D\,(L)=\begin{bmatrix}1 & 1 & -2\\ 0 & 1 & 1\\ 0 & 0 & 2\end{bmatrix}$, the condition of the *PA* problem is $c\,(L)=\|D\,(L)\|\,\|D^{-1}(L)\|=7.0748561$ and the exact solution is $k=[-4 \vdots -3 \vdots -4.5]$. We shall implement *UT* and *OT*—algorithms using machine arithmetics with relative precision eps$=10^{-7}$. Let $\hat{V}, \hat{F}, \hat{k}$ and $\hat{s}$ be the computed values of V, F, k and s. Then the following estimates of the round--off errors can be introduced: $D_s=\|s-\hat{s}\|/\|s\|$ $D_k=\|k-\hat{k}\|/\|k\|$, $D_R=\|R\|_e/\|A_c\|_e$

$(R=(A+bk)\,V-VF)$, $D_A=|\,\|F\|_e/\|A_c\|_e-1|$, $D=\|V^H V-I_3\|_e$, where s stands for $[s_1 \vdots s_2 \vdots s_3]^T \in C^3$. Obviously D_s and D_k are the most important criteria for the accuracy of computations while the other $D-s$ give some information about the computational process.

Using the UT—algorithm the following values have been computed

$$\hat{V}=\left[\begin{array}{c:c:c} -0.7745966 & -0.1299431 & 0.4850712 \\ +0.2581988j_0 & +0.2848747j_0 & \\ \hdashline & 0.8046482 & 0.4850712 \\ -0.2581988j_0 & -0.2249016j_0 & \\ \hdashline 0.5163977 & -0.4498033 & 0.7276068 \\ & -0.0399821j_0 & \end{array}\right]$$

$$\hat{F}=\left[\begin{array}{c:c:c} -1.0 & 3.0389657 & 4.5401295 \\ -j_0 & & +1.7847387j_0 \\ \hdashline 0.0 & -1.0 & 1.6073089 \\ & +j_0 & +1.9061029j_0 \\ \hdashline 0.0 & 0.0 & -2.0 \end{array}\right]$$

$$\hat{k}=\left[\begin{array}{c:c:c} -3.9999995 & -3.0000014 & -4.4999991 \\ -12.7\text{eps}\, j_0 & +6.6\text{ eps}\, j_0 & +4.7\text{ eps}\, j_0 \end{array}\right]$$

The matrix $\hat{k}$ is obtained from k omitting the imaginery terms:

$$\hat{k}=[-3.9999995 \quad -3.0000014 \quad -4.4999991], \text{ and}$$

$$\hat{s}=\{-1.9999984,\ -1.0000005-1.0000004j_0,\ -1.0000005+1.0000004j_0\}$$

The corresponding results obtained via the OT—algorithm are

$$\hat{V}=\begin{bmatrix} -0.8501254 & 0.2049204 & 0.4850708 \\ 0.0665982 & -0.8719350 & 0.4850712 \\ 0.5223515 & 0.4446764 & 0.7276072 \end{bmatrix}$$

$$\hat{F}=\begin{bmatrix} -1.0 & -3.3384992 & 5.0528174 \\ 0.2995357 & -1.0 & -2.1174881 \\ 0.0 & 0.0 & -2.0 \end{bmatrix}$$

$$\hat{k}=[-3.9999965 \quad -2.9999993 \quad -4.5000015]$$

$$\hat{s}=\{-2.0000018,\ -0.9999973-1.0000047j_0,\ -0.9999973+1.0000047j_0\}.$$

The table shows the accuracy estimates for both algorithms:

Algorithm \ D/eps	D_s/eps	D_k/eps	D_R/eps	D_A/eps	D_V/eps
UT	7	3	4	1	12
OT	20	6	6	2	10

6. Conclusions

1. A new method for PA is proposed which is realized by the algorithms UT and OT based on unitary and orthogonal transformation of CSM $A_c=A+bK$ to Schur canonical form.

2. The UT—algorithm requires usage of complex arithmetics which leads to certain increase of the computation time. On the other hand this increase is compensated by the fact that UT—algorithm involves the solution of linear homogeneous algebraic equations of order up to $2n-1$. Moreover, it treats the complex and real poles of A_c in an unified manner.

3. The OT—algorithm makes possible the implementation of real arithmetics only. However it requires the solution of linear homogeneous equations of order up to $2n+2m-2$. The solution is less accurate in comparison with the UT—algorithm as a result of solving scalar quadratic equations.

4. The method presented has a number of advantages over the other known methods. It requires neither computation of the characteristic polynomial of A nor transformation to phase-variable canonical form — both operations being numerically unstable.

5. There is an interesting analogy between the method proposed and the QR—method [2] for transformation of a non-symmetric matrix into a (quasi) triangular form. In the QR—method a sequence of unitary (orthogonal) similarity transformations is implemented on the original matrix such that the final matrix is (quasi) triangular. In the method presented a somehow opposite problem is solved: Given the diagonal blocks of the (quasi) triangular matrix find a part of A_c and thus the gain matrix k. It is well known that the QR—method is numerically stable and now is the most reliable way to solve the eigenvalue problem for non-symmetric matrices. Although the method proposed may not be characterized in such a definitive manner it is reasonable to think that it can be the basis for development of numerically stable algorithm for PA.

6. The approach presented can be extended directly to linear multi-input systems. In this case the freedom in the gain matrix can be used in order to satisfy additional requirements imposed on the closed loop system.

APPENDIX

PROPOSITION 1. Let $[A, b) \in LS(n)$. Then at each step i of algorithm UT every non-zero solution $z_i=[v_i^T \vdots f_i^T]^T$ of the homogeneous equation $G_i z_i=0$ has the property $v_i \neq 0$.

Proof. Denote $v_i=[x_i \vdots u_i^T]^T$, $x_i \in C$, $u_i \in C^{n-1}$ and $E_n \tilde{A}=[h \vdots H]$; $h \in R^{n-1}$, $H \in R_{n-1}^{n-1}$, where $[H, h) \in LS(n-1)$. Now equations (9), (10) take the form

$$x_1 h+Hu_1=s_1 u_1 \tag{32}$$

$$x_i h+Hu_i=s_i u_i+f_{1i} u_1+\ldots+f_{i-1,i} u_{i-1},\ i=2, \ldots, n, \tag{33}$$

where $f_i=[f_{1i} \vdots \dots \vdots f_{i-1,i}]^T$. We shall prove that

$$\operatorname{rank}\,[u_1 \vdots \dots \vdots u_i]=i \tag{34}$$

for $i=1, \dots, n-1$. In fact, (34) is valid $i=1$ since in the opposite case ($u_1=0$) (32) yields $x_1\, h=0 \Rightarrow x_1=0$ (in view of $h\neq 0$) and $v_1=0$ — a contradiction with $z_1=v_1\neq 0$. Suppose now that $2\leqslant i\leqslant n-1$ is the first integer for which (34) does not hold. Then there exist $i-1$ complex numbers $a_{i1}, \dots, a_{i,\,i-1}$ such that

$$u_i=a_{i1}\, u_1+\dots+a_{i,i-1}\, u_{i-1}\,. \tag{35}$$

Substituting (35) in (33) one gets $b_{i-1,i}\, h+b_{i-1,1}\, u_1+\dots+b_{i-1,i-1}\, u_{i-1}=0$, where

$$\begin{gathered} b_{i-1,i}=x_i-a_{i1}\, x_1-\dots-a_{i,i-1}\, x_{i-1}\,, \\ b_{i-1,j}=a_{ij}\,(s_j-s_i)-f_{j,i-1}+a_{i,j+1}\, f_{j,j+1}+\dots+a_{i,i-1}\, f_{j,i-1}, \qquad j=1, \dots, i-1\,. \end{gathered} \tag{36}$$

It can be observed that $b_{i-1,i}\neq 0$ since $b_{i-1,i}=0$ together with (36), (35) yields $v_i=a_{i1}\, v_1+\dots+a_{i,i-1}\, v_{i-1}$ which is impossible in view of $v_i^H\, v_j=0$, $j=1, \dots, i-1$. Hence at least one of the coefficients $b_{i-1,j}$ is non-zero; for instance $b_{i-1,i-1}\neq 0$. Therefore

$$u_{i-1}=a_{i-1,i-1}\, h+a_{i-1,1}\, u_1+\dots+a_{i-1,i-2}\, u_{i-2}\,, \tag{37}$$

where $a_{i-1,j}=-b_{i-1,j}/b_{i-1,i-1}$, $j=1, \dots, i-2$; $a_{i-1,i-1}=-b_{i-1,i}/b_{i-1,i-1}$. Seting (37) in (33) and having in mind (35) one obtains

$$\begin{gathered} u_j=a_{jj}\, h+a_{j,j+1}\, Hh+\dots+a_{j,i-1}\, H^{i-j-1}\, h+a_{j1}\, u_1+\dots+a_{j,j-1}\, u_{j-1}\,, \\ u_1=a_{11}\, h+\dots+a_{1,i-2}\, H^{i-2}\, h\,, \end{gathered} \tag{38}$$

where

$$a_{j1}^2+\dots+a_{j,i-1}^2>0, \qquad j=1, \dots, i. \tag{39}$$

Now (38) and (32) yield

$$\begin{aligned} &(x_1-s_1\, a_{11})\, h+(a_{11}-s_1\, a_{12})\, Hh+\dots \\ &\qquad +(a_{1,i-3}-s_1\, a_{1,i-2})\, H^{i-2}\, h+a_{1,i-2}\, H^{i-1}\, h=0\,. \end{aligned} \tag{40}$$

Since $\operatorname{rank}\,[h \vdots \dots \vdots H^{i-1}\, h]=i$ it follows from (40) $a_{1,i-2}=0$, $a_{1,i-3}=s_1\, a_{1,i-2}, \dots, a_{11}=s_1\, a_{12}$, $x_1=s_1\, a_{11}$, i.e. $a_{11}=a_{12}=\dots=a_{1,i-2}=0$ which contradicts (39). Hence (34) is valid for each $i=1, \dots, n-1$, and $v_i\neq 0$, $i=1, \dots, n-1$. Suppose finally that $v_n=0$. Then (33) implies $[u_1 \vdots \dots \vdots u_{n-1}]\, f_n=0$ and $f_n=0$ which is impossible since $z_n=[v_n^T \vdots f_n^T]^T\neq 0$. Proposition 1 is proved.

PROPOSITION 2. Let $[A, b)\in LS\,(n)$. Then at each step i of algorithm OT every non-zero solution $z_i=[x_{2i-1}^T \vdots x_{2i}^T \vdots y_{2i-1}^T \vdots y_{2i}^T]^T$ of the homogeneous equation $H_i\, z_i=0$ has the property $x_{2i-1}\neq 0$, $x_{2i}\neq 0$.

The proof is similar to this of Proposition 1 and is omitted.

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Podejście Schura do zagadnienia rozmieszczenia biegunów

W pracy zaproponowano nowe podejście do zagadnienia rozmieszczenia biegunów w układach liniowych. Jest ono oparte na unitarnej lub ortogonalnej transformacji macierzy układu zamkniętego do postaci kanonicznej Schura. Metoda ma szereg zalet w stosunku do innych znanych metod. W szczególności nie wymaga ona obliczania wielomianu charakterystycznego macierzy układu otwartego lub przekształcenia na postać kanoniczną zmiennych fazowych.

Istnieje pewna analogia między zaproponowaną metodą i metodą *QR* znajdowania postaci kanonicznej Schura macierzy. To nowe podejście może być podstawą opracowania stabilnego numerycznie algorytmu rozmieszczenia biegunów.

Подход Щура к вопросу распределения полюсов

В работе предлагается новый подход к вопросу распределения полюсов в линейных системах. Он основан на унитарном или ортогональном преобразовании матрицы замкнутой системы в канонический вид Щура. Метод обладает рядом преимуществ по сравнению с другими известными методами. В частности он не требует вычисления характеристического многочлена матрицы разомкнутой системы или преобразования в канонический вид фазовых переменных.

Существует некоторая аналогия между предлагаемым методом и методом *QR* нахождения матрицы в каноническом виде Щура. Этот новый подход может быть основой для разработки численно устойчивого алгоритма распределения полюсов.