Control
and Cybernetics
VOL. 10 (1981) No. 1-2

Some contribution to the method of minimal sets for large graphs

by

JANUSZ KACPRZYK
Polish Academy of Sciences
Systems Research Institute
Warszawa, Poland

WIESŁAW STAŃCZAK
Polish Academy of Sciences
Institute of Computer Science
Warszawa, Poland

The paper concerns the method of minimal sets (minimally interconnected subnetworks) for partitioning a given graph of similarity. The case of large scale graphs is considered. Some tools for improving the efficiency of the algorithm given in [4] are proposed. First, some new properties of minimal sets are formulated and proved. Then, a notion of concentrate, being some subset of vertices with appropriate edge weights, is introduced. Some properties of concentrates are derived. A numerical comparison of the algorithm without and with the mechanisms developed is shown.

1. Introduction

An important part of many systems analytic approaches, both in theory and practice, is the analysis of relations, connections, similarities, etc. between elements of systems under consideration. A convenient tool for representing such dependences is a weighted graph, whose vertices represent the system's elements and edges — the dependences between these elements. Then, many problems of analysis and synthesis may be formulated as some partitioning of that graph, called in the sequel the graph of similarity.

Among various methods of graph partitioning, a revelant role is played by the methods of minimally interconnected subnetworks [called also minimally interconnected groups, minimally interconnected (sub) sets, or — briefly — minimal groups or minimal (sub) sets]. Further on, we will use the term minimal set, for brevity.

Roughly speaking, a minimal set of a weighted graph is a collection of vertices, such that the sum of weights between them is greater than the sum of weights between them and other vertices. Thus, the vertices in a minimal set are connected stronger with themselves than with the "outer world". The notion of minimal set and basic properties were formulated first by Luccio and Sami [6]. Then, they were considerably developed by Kacprzyk and Stańczak [2, 4], Nieminen [7], Nowicki and Stańczak [8] and Stańczak [10]. The method was successfully applied for, e.g. structuring a set of enterprises [1], designing a computer network [3, 10], structuring a data base [9], and designing a telephone netweork [3].

In general, the method of minimal sets performs the graph decomposition in a quite efficient way. However, as all the combinaterial procedures, it looses its performance as the size of the problems becomes large. Moreover, its efficiency for large scale problems depends to a considerable extent on how officiently we may determine the first minimal set. This is particularly true for practical problems in which there are many edges with more or less equal weights, large differences in weights, etc.

The above difficulties have motivated this work. Its general goal is to propose some approximate approach incorporating an interaction with the designer and to provide the efficient algorithm proposed in [4, 8] with some mechanisms for alleviating the mentioned specifics of large practical problems.

We begin with a brief recalling of some factors previously given which are relevant for our considerations. Then, we formulate and prove some new properties. In the paper's main part, the notion of condensate is introduced. It is, roughly speaking, a product of restructuring the given weighted graph, mainly by merging appropriate vertices and redefining appropriate weights. Important properties of condensates are formulated and proved.

It should be stressed that the determination of condensates is a somewhat subjective matter which is closely related to the specific problem under consideration. Thus, the "principles", or -better to say- "rules of thumb" may be quite different for, e.g. computer networks, telephone networks, etc. The key factor is here the designer's experience and knowledge of problem's specific features. The approach proposed should, therefore, be meant as the one which incorporates both the "science" and the "art". At the expense of loosing some "formalization" or "strict optimality", we gain, however, a considerable efficiency increase, which — for large practical problems- may even be by two orders of magnitude.

As an example, the derivation of a condensate for some given weighted graph is shown. The efficiency of the tool proposed is discussed.

2. Preliminaries

We consider a given complete undirected graph $G=(X, E)$ without leeps and multiple edges, where X is its vertex set, and $E=\{\{x, y\}: x, y \in X, x \neq y\}$ is its edge set, i.e. we assume an edge to be an unordered pair of vertices. We assign a non-negative weight $w(x, y)$ to each edge $\{x, y\}$. Then we obtain an ordered pair $\langle G, w\rangle$ ·

$\cdot\langle G, w\rangle$ is said to be the graph of similarity. Let A and B be nonempty and disjoint subsets of X. We denote

$$f(A,B)=\sum_{x\in A}\sum_{y\in B} w(x,y) \tag{1}$$

We assume that, by definition, $f(A,\emptyset)=0$ for each $A\subset X$.

In the later proofs we will often use the following preperties of f:

$$f(A,B)\geqslant 0,$$

$$f(A,B)=f(B,A),$$

if $A, B, C, D\subset X$ are pairwise disjoint, then

$$f(A\cup B, C\cup D)=f(A,C)+f(A,D)+f(B,C)+f(B,D),$$

and, in particular,

$$f(A,C)=f(A\cup B, C)-f(B,C)$$

Definition. Let S be a subset of X. If the following inequality

$$f(R, X-R)>f(S, X-S) \tag{2}$$

holds for each nonempty proper subset R of S, then S is said to be a minimally interconnected set or, simply, a minimal set.

The following properties of minimal sets play an important role in the sequel.

Lemma 1 [4]. *The necessary and sufficient condition for S to be a minimal set is*

$$f(R, S-R)>f(R, X-S) \tag{3}$$

for every nonempty proper subset R of S.

Corollary 1 [2]. *If S is a minimal set, then the inequality*

$$f(R, X-R)>0 \tag{4}$$

is satisfied for each nonempty proper subset R of S.

From Lemma 1 and using the identical proof technique as in the above corollary we obtain the next property.

Corollary 2. *Let R, S be as in Corollary* 1. *Then the relation*

$$f(R, S-R)>0 \tag{5}$$

must hold.

Theorem 1 [4]. *Let I and J be nonempty sets of indices, $\{Z_i\colon i\in I\}$ — a family of pairwise disjoint minimal sets. Let $J\subset I$, $|J|\geqslant 2$. We denote*

$$S_J=\bigcup_{i\in J} Z_i \tag{6}$$

If S_J is not a minimal set for each J, $J \neq I$, then S_I is minimal, if and only if the following condition is satisfied

$$f(S_I, X-S_I) < \min \{f(Z_i, X-Z_i) : i \in I\} \tag{7}$$

The following corollary is an immediate consequence of Theorem 1.

COROLLARY 3. *The set S, $S \subset X$, $|S|>1$, is a minimal set, if and only if S is the smallest set, such that for each $x \in S$ the inequality*

$$f(S, X-S) < f(\{x\}, X-\{x\}) \tag{8}$$

holds.

COROLLARY 4 [2]. *Let $S \subset X$, $|S| \geqslant 3$. If the following equality*

$$w(x, y) = w_0 = \text{const} \tag{9}$$

holds for each pair x, y of distinct elements from S, then every nonempty proper subset R of S, $|R|>1$, is not a minimal set.

For the proofs of the above properties see [2, 4].

3. On some specific sets

The general properties of sets given in [2, 4, 7] are quite sufficient for the construction of an efficient algorithm for determining minimal sets. The computational procedure can be, however, not efficient enough for high dimensional problems. The following new properties may here be important.

PROPOSITION 1. Let $S \subset X$, $|S| \geqslant 2$. The necessary and sufficient condition for S to be a minimal set is that for each $x \in S$:

$$(|S|-1)\, w_0 > f(S-\{x\}, X-S) \tag{10}$$

Proof. According to (9) we obtain

$$f(\{x\}, S-\{x\}) = (|S|-1)\, w_0 = f(\{x\}, X-\{x\}) - f(\{x\}, X-S) \tag{11}$$

Furthermore, we have

$$f(S-\{x\}, X-S) = f(S, X-S) - f(\{x\}, X-S) \tag{12}$$

From (11) and (12) it is evident, that the inequality (10) is equivalent to the condition (8). Moreover, each nonempty proper subset R of S is not minimal in view of Corollary 4. Hence, S is the smallest set, which completes the proof due to Corollary 3. Q.E.D.

As an immediate consequence of the above proposition we obtain the following property given in [2].

COROLLARY 5 [2]. *Let $S \subset X$, $|S| \geqslant 2$ and* (9) *hold). We also assume that for every pair $x, y \in X$, $\{x, y\} \not\subset S$, of distinct vertices we have $w(x, y) \neq w_0$, and $w_0 = \max\{w(x, y): x, y \in X,\ x \neq y\}$. Then, the necessary and sufficient condition for S to be a minimal set is to satisfy the inequality* (10) *for each $x \in S$.*

It is evident that Corollary 5 concerns a particular case of the situation discussed in Proposition 1.

PROPOSITION 2. We consider a nonempty set $S \subset X$, $S \neq X$. Let

$$w(x, y) \geqslant (|X| - |S|)\, w(x, z) \tag{13}$$

hold for each $x, y \in S$ and for every z belonging to $X - S$. We assume that the vertices of S can be arranged in a path in G such that each edge of this path has a positive weight. If there exists a pair s, t $s, t \in S$, such that for each $y \in S$ there exists a pair $q, r \in X - S$, where

$$w(s, y) > (|X| - |S|\, w(s, q) \tag{14}$$

$$w(t, y) > (|X| - |S|)\, w(t, r) \tag{15}$$

respectively, then Sisa minimal set.

Proof. Due to the existence of path described before, for each nonempty proper subset R of S the condition (5) holds. It means that S can be a minimal set. It is easy to calculate, that

$$f(R, S-R) - f(R, X-S) = \sum_{x \in R} \Big[\sum_{y \in S-R} w(x, y) - \sum_{z \in X-S} w(x, z) \Big] \geqslant \sum_{x \in R} u(x) \tag{16}$$

where

$$u(x) = \sum_{y \in S-R} w(x, y) - w(x, y_A) \tag{17}$$

and $y_A \in A(x)$. By $A(x)$ we define the following set

$$A(x) = \{y: w(x, y) = \min\{w(x, y): y \in S - \{x\}\}\} \tag{18}$$

It is obvious that $u(x) \geqslant 0$ for each $x \in S$. If $(S-R) \cap A(x) = \emptyset$ for some $x \in R$ or $|S| - |R| > 1$, then $u(x) > 0$ for this x, and then the condition (3) holds due to (16).

Let us now assume that $R = S - \{p\}$, $p \in S$, and for each $x \in R$ we have $A(x) \cap \cap \{p\} = \{p\}$. Let $p \neq s$. Through (16), (17), (18) and the above assumptions one can obtain

$$f(R, S-R) - f(R, X-S) = f(S-\{p\}, \{p\}) - f(S-\{p\}, X-S) \geqslant$$

$$\geqslant \sum_{x \in S - \{p, s\}} [w(x, p) - w(x, p)] + w(s, p) - \sum_{z \in X-S} w(s, z) \tag{19}$$

With regard to (13) and (14) we can write

$$w(s, p) - \sum_{z \in X-S} w(s, z) = \frac{w(s, p)}{|X| - |S|} - w(s, q) > 0 \tag{20}$$

Taking into account (20), the relation (3) results from (19). In a similar way using (15) instead of (14), we proceed in the case $R=S-\{s\}$. Hence, Lemma 1 implies that S is a minimal set. Q.E.D. ■

PROPOSITION 3. We consider a nonempty proper subset S of X. Let

$$w(x, y) \geqslant w(x, z) \tag{21}$$

hold for each $x, y \in S$, $z \in X-S$. We assume that R is a nonempty proper subset of S and $P \subset X-S$, $P \neq \emptyset$. If the condition

$$|S| \geqslant |R|+|P| \tag{22}$$

holds, then $H=R \cup P$ is not a minimal set.

Proof. It is easy to notice that

$$f(H, X-H)=f(R, X-(S \cup P))+f(R, S-R)+f(P, X-P)-f(R, P) \tag{23}$$

Then, we have

$$f(R, S-R)-f(R,P)=\sum_{x \in R}[f(\{x\}, S-R)-f(\{x\}, P)]=$$
$$=\sum_{x \in R}\Big[\sum_{y \in S-R} w(x, y)-\sum_{z \in P} w(x, z)\Big]$$

Due to (21) the right-hand side of the above equality has a lower bound given by the expression $(|S|-|R|-|P|)\sum_{x \in R} w(x, y_A)$, where $y_A \in A(x)$ and $A(x)$ is given by (18). Thus, we obtain

$$f(R, S-R)-f(R, P) \geqslant (|S|-|R|-|P|)\sum_{x \in R} w(x, y_A) \tag{24}$$

Due to (24) and (22) we obtain the following inequality

$$f(R, S-R)-f(R, P) \geqslant 0 \tag{25}$$

In view of (23) and (25) we obtain that $f(H, X-H) \geqslant f(P, X-P)$, because $f(R, X+-(S \cup P)) \geqslant 0$. It ensures, by definition, that H is not a minimal set. Q.E.D.

As a natural consequence of Proposition 3 we notice the following property stated and proved in [2].

COROLLARY 6 [2]. *Let S be as described in Corollary 5. If R, P are nonempty proper subsets of S and $X-S$, respectively, and the relation* (22) *is satisfied, then $H=R \cup P$ is not a minimal set.*

The properties formulated and proved in this section concern some specific sets. In them, every two vertices are connected with edges weighted by the same number or weighted by the number with a relatively great value. From the theoretical point of view, it is a rare case. This is not, however, true from the practical point of view [9, 10]. In fact, in applications there often occur situations when we have a great number of approximately equal weights or some group of entities is connected with edges weighted by relatively great numbers. In the former case we have to keep in mind that the parameters mentioned are usually approximated or even estimated. Hence, they can be assumed to be of equal value. With regard to the above remarks, the properties of sets given in this section can be especially useful for the initial estimation of results and, therefore, for the eventual renumbering of vertices in order to reduce the computation time of the algorithm while determining minimal sets.

4. Condensates and their properties

In many real life situations we have to do with large scale problems. In this case, the cardinality of X is a large number and much computation is required for enumerating minimal sets. The natural way in this case is to merge some vertices into one vertex and then to obtain a problem with a smaller dimension. This concept leads to the idea of condensates.

We consider a graph of similarity $\langle G, w\rangle$, $G=(X, E)$. Let W be a nonempty set of indices and $\{S_i : i \in W\}$ — an arbitrary family of pairwise disjoint, nonempty subsets of X. We also assume that $|S_i|>1$ for each $i \in W$. We denote

$$Q=\bigcup_{i \in W} S_i, \tag{26}$$

and $Y=X-Q$. We construct a complete undirected graph $G^*=(Y \cup W, E_{Y \cup W})$ without loops and multiple edges, where $E_{Y \cup W}=\{\{x, y\}: x, y \in Y \cup W, \ x \neq y\}$. We also denote $Y \cup W$ by X^* and $E_{Y \cup W}$ by E^*, for simplicity. Then, we define

$$w^*(x, y)=\begin{cases} w(x, y), & \text{if} \quad x, y \in Y, \\ \sum_{i \in S_x} \sum_{j \in S_y} w(i, j), & \text{if} \quad x, y \in W \\ \sum_{i \in S_x} w(i, y), & \text{if} \quad x \in W, \ y \in Y, \\ \sum_{j \in S_y} w(x, j), & \text{if} \quad y \in W, \ x \in Y \end{cases} \tag{27}$$

It is easy to show that $\langle G^*, w^*\rangle$ is also a graph of similarity. Moreover, we assume that f^* has the same meaning as f, but with respect to $\langle G^*, w^*\rangle$ instead of $\langle G, w\rangle$. The above described $\langle G^*, w^*\rangle$ is called a condensate of $\langle G, w\rangle$.

Hence, we see that the determination of condensates is in fact an arbitrary subjective matter. The best solution would be first to determine some minimal sets, then, to assume them as the condensates. This would, however, be often impossible and, moreover, inefficient. Hence, the best way would be to apply the results of the previous section to determine the merging of which vertices in condensates is expedient. Then, these condensates are handled.

Now, we have the following basic properties.

COROLLARY 7. *The set S, $S \subset Y$ is a minimal set in $\langle G, w \rangle$, if and only if it is a minimal set in $\langle G^*, w^* \rangle$.*

Proof. Let R be a nonempty proper subset of S. Then we can state that

$$f(R, X-R)=f^*(R, X^*-R), \tag{28}$$

and

$$f(S, X-S)=f^*(S, X^*-S) \tag{29}$$

According to (28) and (29) we see that the relation $f^*(R, X^*-R)>f^*(S, X^*-S)$ is equivalent to the formula (2). Q.E.D.

THEOREM 2. *Let a set Z be given, $Z \neq \emptyset$, $Z \subset X$. We assume that Z can be represented in the form*

$$Z=S \cup \bigcup_{i \in I} S_i, \tag{30}$$

where $I \subset W$ and $S \subset Y$. If Z is a minimal set in $\langle G, w \rangle$, then $S \cup I$ is a minimal set in $\langle G^*, w^* \rangle$.

Proof. Let us assume, that Z is a minimal set in $\langle G, w \rangle$. If $I=\emptyset$, then the theorem follows from Corollary 7. Now, we assume that $I \neq \emptyset$. For simplicity of later notations we introduce the symbol

$$U=\bigcup_{i \in K} S_i, \tag{31}$$

where $K \subset I$. We have

$$f(U, X-U)=\sum_{i \in K}[f(S_i, X-Q)+f(S_i, Q-U)]=$$

$$=f^*(K, Y)+f^*(K, W-K)=f^*(K, X^*-K) \tag{32}$$

In a similar way we derive

$$f(Z, X-Z)=f^*\big(S \cup I, X^*-(S \cup I)\big), \tag{33}$$

and for each U, $\emptyset \neq U \subset Z$, $U \neq Z$, we have

$$f(Z, X-Z)<f(U, X-U)=f^*(K, X^*-K), \tag{34}$$

and for each $R \cup U$, $\emptyset \neq R \cup U \subset Z$, $R \cup U \neq Z$, $R \subset S$, we obtain

$$f(Z, X-Z)<f\big(R \cup U, X-(R \cup U)\big)=f^*\big(R \cup K, X^*-(R \cup K)\big) \tag{35}$$

Due to (34), (35) and (36), the set $S \cup I$ is minimal in $\langle G^*, w^* \rangle$. Q.E.D. ■

It should be noticed that the converse is, in general, not true. A special case in which each S_i, $i \in I$, is a minimal set, is considered in Theorem 1. It is worth noting that the computational algorithm described in [4] is based on that theorem. Moreover, some of further properties derived in [2, 4] concern some specific condensates too.

THEOREM 3. *Let S be a nonempty proper subset of Y. If $S \cup W$ is a minimal set in $\langle G^*, w^* \rangle$, then $Y-R$ is not a minimal set in $\langle G, w \rangle$ for each nonempty proper subset R of S.*

Proof. Due to (35) we have

$$f^*(R \cup W, X^*-(R \cup W))=f(Y-R, X-(Y-R)), \tag{36}$$

$$f^*(S \cup W, X^*-(S \cup W))=f(Y-S, X-(Y-S)) \tag{37}$$

Since $S \cup W$ is a minimal set in $\langle G^*, w^* \rangle$, then $f(Y-R, X-(Y-R))>f(Y-S, X+-(Y-S))$. Since, evidently $Y-R \supset Y-S$, $Y-R \neq Y-S \neq \emptyset$ then the proof is accomplished. Q.E.D.

5. An example

Let us now present an example to show the application of our considerations to derive condensates.

Let there be given a graph of similarity with 41 vertices, i.e. $X=\{1, 2, ..., 41\}$ and edge weights $w(i, j)$ as in Tab. 1 (evidently, only $w(i, j)$'s, $i<j$, are here relevant).

Let us now arbitrary assume the following sets of vertices $S=\{1, 2, ..., 24\}$, $S=\{25, 26, ..., 30\}$, $Q=S_a \cup S_b$. Hence, $W=\{a, b\}$, $Y=\{31, 32, ..., 41\}$, $X^*==W \cup Y$, and the grph $\langle G^*, w^* \rangle$, with w^* as given in Tab. 2, has now only 13 vertices. Its analysis is obviously much easier than of the original graph.

In the first step, in $\langle G^*, w^* \rangle$ we find the following minimal sets: $\{40, 41\}$, $\{33, 34, 35\}$, $\{36, 37, 38, 39\}$. Then, a new graph being a condensate of the above is derived. The obtained minimal sets will become the vertices C, B and A, respectively. In the second step, we find the following minimal sets $\{B, C\}$, $\{8, 31, 32, A\}$. Then, the procedure is interrupted. Thus, we have obtained above. e.g. the following minimal sets consisting of vertices of the original graph: $\{36, 37, ..., 41\}$ and $\{33, 34, 35\}$. They can be assumed to be condensed in the following manner: $S=\{33, 34, 35\}$, $S=\{36, 37, ..., 41\}$. Because we do not know, whether the vertices corresponding to the previous Q form minimal sets then we must split it and perform a test. To increase the numerical efficiency we once more condense the original graph. We see that the set $\{1, 2, ..., 9\}$ satisfies the assumptions of Proposition 1, in approximation. Then we assume $Q_1=\{i: i \in X, i>9\}=S_m$ and we obtain $Y_1=\{1, 2, ..., 9\}$, $W_1=\{m\}$. The test confirms our conjecture that S_m is a minimal set in $\langle G, w \rangle$ (on the base of Corollary 7).

i \ j	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41
1	22	21	22	21	21	24	23	20	4	1	0	0	2	0	0	0	10	0	0	0	0	0	0	2	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2		25	20	20	24	22	23	21	6	0	0	0	0	0	0	0	0	0	0	10	0	0	0	0	0	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3			20	22	21	22	21	22	3	2	0	0	2	0	0	0	10	0	0	0	0	0	0	1	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4				25	22	24	20	22	3	1	0	0	2	0	0	0	10	0	0	0	0	0	0	1	0	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5					21	21	22	22	10	0	0	0	0	0	0	0	5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6						20	23	22	7	0	0	0	3	0	0	0	5	0	0	0	0	0	0	0	0	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7							21	21	0	2	0	0	3	0	0	0	10	0	0	0	0	0	0	2	0	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8								22	2	2	0	0	3	0	0	0	10	0	0	0	0	0	0	0	0	3	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9									0	2	0	0	5	0	0	0	0	10	0	0	0	0	0	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
10										180	0	0	60	10	0	0	0	20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
11											100	100	0	0	20	10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12												50	70	0	0	0	0	0	0	0	0	10	0	0	0	0	0	0	0	0	0	10	0	0	0	0	0	0	0	0
13													50	30	0	0	0	0	0	0	0	10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14														0	0	0	0	10	0	0	0	10	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
15															70	80	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
16																60	0	0	0	0	0	40	0	0	0	0	0	0	0	0	0	10	5	5	5	0	0	0	2	3
17																	0	0	40	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
18																		70	72	73	70	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19																			70	71	69	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
20																				70	72	20	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
21																					77	0	0	0	0	0	0	0	0	0	0	0	10	10	15	0	0	0	18	7
22																						20	60	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23																							300	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
24																								0	0	0	0	0	0	0	0	0	15	5	0	0	0	0	0	0
25																									90	0	60	0	0	0	20	0	0	20	0	0	0	0	10	0
26																										80	0	0	0	0	20	0	0	0	0	0	0	0	0	0
27																											80	0	0	0	0	0	20	0	0	0	10	0	0	0
28																												0	0	0	30	20	0	0	0	0	0	0	0	0
29																													130	140	120	0	0	0	0	0	0	0	0	0
30																														0	0	20	20	20	0	0	0	0	0	0
31																															0	0	0	0	0	0	10	20	0	20
32																																0	0	0	0	0	0	0	0	0
33																																	70	80	0	0	0	0	0	0
34																																		60	0	0	0	0	0	0
35																																			0	0	0	0	0	0
36																																				30	30	50	5	5
37																																					50	30	10	20
38																																						49	5	15
39																																							10	0
40																																								80

Tab. 1. Edge weights w for the 41-vertex graph considered.

i \ j	b	31	32	33	34	35	36	37	38	39	40	41
a	30	0	0	20	30	20	20	0	0	0	20	10
b		140	190	40	40	40	0	0	10	0	10	0
31			0	0	0	0	0	0	10	20	0	20
32				0	0	0	0	0	0	0	0	0
33					70	80	0	0	0	0	0	0
34						60	0	0	0	0	0	0
35							0	0	0	0	0	0
36								30	30	50	5	5
37									50	30	10	20
38										49	5	15
39											10	0
40												80

Tab. 2. Edge weights w^* for the condensate $\langle G^*, w^* \rangle$.

We notice, as before, that the set {18, 19, 20, 21, 22} fulfills the assumptions of Proposition 1, in approximation. The test confirms again our hypothesis. Then we can define a new condensate. Let S_c, S_d be as above, and $S_g = Y_1$, $S_h = \{18, 19, 20, 21, 22\}$, $Q = S_c \cup S_d \cup S_g \cup S_h$, $W = \{c, d, g, h\}$, $Y = \{10, 11, ..., 17, 23, 24, ..., 32\}$. Hence X^* has 22 vertices. Here we obtain, among the others, the following minimal sets {23, 24}, {15, 16, 17}, {10, 11, ..., 14} and {25, 26, ..., 32,c} i.e. {25, 26, ..., 35} and the calculations are terminated. It is easy to evaluate that the algorithm destribed in [4, 10] needs 210843 tests for obtaining final results. Using properties described in this paper we reduce the number of tests up to 17705 tests.

6. Concluding remarks

The aim of the paper was to present some mechanisms for improving the efficiency of the method of minimal sets in the case of large-scale networks. The theoretical analysis of Section 3 and 4 suggests some gain in efficiency. First, the new properties presented provide some additional tool for eliminating more subsets of vertices. Second, the derivation of condensates reduces the dimension of the graph, hence diminishes the time of handling it.

The computations performed for the 41 — vertex graph from Section 5 fully supported the above. As a criterion we used the number of tests during the determination of minimal sets. The use of mechanisms presented in the paper increased the efficiency about 12 times.

Thus, the approach presented is a successful involvement of both the science and the art into some extremely important class of graph partitioning problems. It seems that in general the ability to effectively solve large problems of the type considered, as well as many other ones arising in practice, is closely related to the availability of such procedures which make use of both formal mathematical tools and human experience and knowledge of problem's specifics.

References

[1] Kacprzyk J., Stańczak W. Application of the method of minimally interconnected network for solving the problem of partitioning a group of enterprises into subgroups (in Polish). *Archiwum Automatyki i Telemech.*, **20**, 4, (1975), 513–526.

[2] Kacprzyk J., Stańczak W. On an extension of the method of minimally interconnected subnetworks. *Control and Cybernetics*, **5**, 4, (1976) 61–77.

[3] Kacprzyk J., Stańczak W. Partitioning a computer network into subnetworks and allocation of distributed data bases. Proc. 8th IFIP Conf. Optimization Techniques, Wurzburg, 1976. Springer Verlag, 1977, 464–472.

[4] Kacprzyk J., Stańczak W. On a further extension of the method of minimally interconnected subnetworks. *Control and Cybernetics*, **7**, 2, (1978) 17–31.

[5] Kaliszewski I., Nowicki T., Stańczak W. On the telephone interexchange network structure decomposition using the method of minimally interconnected subgraphs (in Polish). *Rozprawy Elektrotechniczne*, **21**, 2, (1975) 573–580.

[6] Luccio F., Sami M. On the decomposition of networks in minimally interconnected subnetworks. *IEEE Trans. Circuit Theory*, **CT-16**, 2, (1969) 184–188.

[7] Nieminen J. On minimally interconnected subnetworks of a network. *Control and Cybernetics*, **9**, 1–2, (1980) 47–52.

[8] Nowicki T., Stańczak W. On some class of algorithms using for determination of minimal groups in cliques. Proc. International Symp. on Applications of Math. in System Theory, Brasov, Dec. 1978, 275–280 (preprint).

[9] Nowicki T., Stańczak W. Partitioning a set of elements into subsets due to their similarity. Secondes Journées Internationales Analyse des Données et Informatique, Versailles, Oct. 1979. Data Analysis and Informatics. North Holland, 1980, 583–591.

[10] Stańczak W. Application of the method of minimally interconnected subnetworks to computerized designing cf topological structure and configuration for teleprocessing network (in Polish). Ph.D.Thesis, Inst. of Computer Sci., Polish Academy of Sci., Warsaw, 1978.

Received, October 1980.

O metodzie zespołów minimalnych w przypadku grafów o wielkich rozmiarach

Praca dotyczy zastosowania metody zespołów minimalnych do podziału grafu podobieństw (zainteresowań). Rozpatruje się przypadek grafu o dużych rozmiarach. Proponuje się zwiększenie efektywności algorytmu podanego w [4], podając pewną metodę, w której wykorzystuje się interaktywnie wiedzę i doświadczenie projektanta. Na wstępie formułuje się i dowodzi pewnych nowych właściwości zespołów minimalnych. Następnie wprowadza się pojęcie kondensatu, który jest odpowiednikiem podzbioru zbioru wierzchołków i ma odpowiednio określone wagi krawędzi. Wyprowadza się pewne właściwości kondensatów. Analizuje się efektywność zaproponowanego algorytmu, porównując go z algorytmem dotychczas stosowanym.

Метод минимально связанных подграфов в случае великих графов

Статья касается применя метода минимально связанных подграфов для декомпозиции данного графа сходств. Рассмотривается случай великих графов. Предлагается методы повышения эффективности алгоритма представленного в [4]. Вступительно формулируется и доказывается некоторые новые свойства минимально связанных подграфов. После этого вводится идею конденсата, который является подмножеством множества вершин с соответствующими весами на ребрах. Представляется некоторые свойства конденсатов. Потом сравнивается, на примере, эффективность алгоритма, в котором применены выше представлены улучшения и эффективность прежнего вида алгоритма.