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# A constructive approach to the deterministic stopping time problem

by

ITALO CAPUZZO DOLCETTA
MICHELE MATZEU

Istituto Matematico G. Castelnuovo
Università di Roma
Roma, Italy

We propose here an approximation scheme to the optimal stopping time problem for a deterministic system, which has been recently considered by J.L. Menaldi [1], [2] and J.L. Menaldi, E. Rofman [3].

## 1. The problem.

Let us describe by a vector $y_x(t) \in R^N$ ($t \geqslant 0$, $x \in R^N$) the state of a deterministic dynamical system governed by the Cauchy problem:

$$\begin{cases} y_x'(t) = g(y_x(t)), & t>0 \\ y_x(0) = x, \end{cases} \tag{1}$$

where $g$ is a Lipschitz continuous function, with constant $c_g$.

The *optimal stopping* time problem for (1) is to find (if there exists) a $t_x^* \geqslant 0$ minimizing the function

$$J_x(t) = \int_0^t f(y_x(s)) \cdot e^{-\alpha s}\, ds + \psi(y_x(t)) \cdot e^{-\alpha t}, \quad t \geqslant 0, \tag{2}$$

where $f$ and $\psi$ are Borel measurable and bounded and $\alpha$ a positive constant.

It is known (see [1]) that the *Hamilton-Jacobi function*

$$u(x) = \operatorname*{Inf}_{t \geqslant 0} J_x(t) \tag{3}$$

is the *maximal solution* of the following system:

$$\begin{cases} v(x) \leqslant \psi(x) \\ v(x) \leqslant \int_0^t (\Phi(s) f)(x)\, ds + (\Phi(t) v)(x), \ \forall t>0, \ \forall x \in R^N, \end{cases} \tag{4}$$

where

$$(\Phi(s)\, v)(x) = v\,(y_x(s)) \cdot e^{-\alpha s},$$

and that, for continuous $\psi$ and $u$, the time $t_x^*$ defined by

$$t_x^* = \operatorname{Inf}\{t \geqslant 0;\ u\,(y_x(t)) = \psi\,(y_x(t))\} \tag{5}$$

is optimal if finite (if this is not the case, an optimal time doesn't exist).

For recent references to the semigroup approach to optimal control problems see, for example, A. Bensoussan, M. Robin [4] and J. Zabczyk [5].

## 2. Discretization.

Let us consider the problem of finding, for $\Delta t > 0$, the maximal solution of the system (see also [6] for the stochastic case):

$$\begin{cases} v\,(x) \leqslant \psi\,(x) \\ v\,(x) \leqslant \Delta t \displaystyle\sum_{k=0}^{n-1} (\Phi^{\Delta t}(k) f)(x) + (\Phi^{\Delta t}(n)\, v)(x), \ \forall n \in N,\ x \in R^N\,(°) \end{cases} \tag{6}$$

where

$$(\Phi^{\Delta t}(k)\, v)(x) = v\,(y_x^{\Delta t}(k\Delta t)) \cdot (1 - \alpha \Delta t)^k, \ \forall k \in N$$

and $y_x^{\Delta t}(k\Delta t)$ is recursively defined as

$$\begin{cases} y_x^{\Delta t}(k\Delta t) = y_x^{\Delta t}((k-1)\,\Delta t) + g\,(y_x^{\Delta t}(k-1)\,\Delta t) \cdot \Delta t, \ k \geqslant 1 \\ y_x^{\Delta t}(0) = x \end{cases} \tag{7}$$

THEOREM 1. *System* (6) *has a maximal solution* $u^{\Delta t}$ *given by*

$$u^{\Delta t}(x) = \operatorname*{Inf}_{n \geqslant 0} J_x^{\Delta t}(n) = \operatorname*{Inf}_{n \geqslant 0} \left\{ \Delta t \sum_{k=0}^{n-1} (\Phi^{\Delta t}(k) f)(x) + (\Phi^{\Delta t}(n)\, \psi)(x) \right\}. \tag{8}$$

*Moreover* $\{u^{\Delta t}\}$ *is uniformly bounded with respect to* $\Delta t$ *and, if* $f$ *and* $\psi$ *are locally* $\gamma$*-Hölder continuous, with* $\gamma > 0$, *then (for sufficiently small* $\Delta t$*),* $\alpha > c_g \gamma$ *implies* $u^{\Delta t}$ *locally* $\gamma$*-Hölder continuous,* $\alpha \leqslant c_g \gamma$ *implies* $u^{\Delta t}$ *locally* $\beta$*-Hölder continuous for all* $0 < \beta < \alpha / c_g$, *uniformly with respect to* $\Delta t$.

The first statement can be proved via the Bellman's optimality principle. We have in fact, by the definition of $u^{\Delta t}$, that

$$u^{\Delta t}(x) = \operatorname{Min}\,(J_x^{\Delta t}(0),\ \operatorname*{Inf}_{n \geqslant 1} J_x^{\Delta t}(n)) \tag{9}$$

From (9), the additivity of $J_x^{\Delta t}$ on the trajectories of (7) and the semigroup property of the positive operators $\Phi^{\Delta t}(k)$, it follows that $u^{\Delta t}$ is the maximal solution of (6).

(°) We put $\sum_{k=0}^{n-1} (\Phi^{\Delta t}(k) f)(x)\, \Delta t = 0$

The uniform boundedness is a consequence of the estimate

$$\mathrm{Min}\,(\mathrm{Inf}\,\psi,\ 1/\alpha \cdot \mathrm{Inf}\,f) \leqslant u^{\Delta t}(x) \leqslant \mathrm{Sup}\,\psi.$$

The Hölder-continuity of $u^{\Delta t}$ on each compact $K \subset R^N$ follows, for $\alpha > c_g \gamma$, from the inequalities (valid for all $n \in N$, and $x_1, x_2$ in $R^N$):

$$|J_{x_1}^{\Delta t}(n) - J_{x_2}^{\Delta t}(n)| \leqslant c_f\, \Delta t \sum_{k=0}^{n-1} (1-\alpha\Delta t)^k |y_{x_1}^{\Delta t}(k\Delta t) - y_{x_2}^{\Delta t}(k\Delta t)|^\gamma + \\ + c_\psi (1-\alpha\Delta t)^n |y_{x_1}^{\Delta t}(n\Delta t) - y_{x_2}^{\Delta t}(n\Delta t)|^\gamma \tag{10}$$

$$|y_{x_1}^{\Delta t}(n\Delta t) - y_{x_2}^{\Delta t}(n\Delta t)| \leqslant |x_1 - x_2| \cdot (1 - c_g\,\Delta t)^{-n}, \tag{11}$$

where $c_f$ and $c_\psi$ are the Hölder constants on $K$. Hence,

$$|J_{x_1}^{\Delta t}(n) - J_{x_2}^{\Delta t}(n)| \leqslant c_f\, \Delta t \sum_{k=0}^{\infty} \left( \frac{(1-\alpha\Delta t}{(1-c_g\,\Delta t)^\gamma} \right)^k |x_1 - x_2|^\gamma + \\ + c_\psi \left( \frac{1-\alpha\Delta t}{(1-c_g\,\Delta t)^\gamma} \right)^n |x_1 - x_2|^\gamma \tag{12}$$

and the uniform Hölder continuity of $u^{\Delta t}$ follows from (12) through a minimizing sequences argument.

The case $\alpha \leqslant c_g \gamma$ can be treated in a similar way.

**3. Convergence of the approximated solutions.**

The convergence result is the following:

THEOREM 2. *If $f$ and $\psi$ are locally $\gamma$-Hölder continuous, $\gamma > 0$, then $u^{\Delta t}$ converges, as $\Delta t \to 0^+$, to the function $u$ defined by* (3), *uniformly on compact sets. Moreover, for sufficiently small $\Delta t$, the estimate*

$$\sup_{x \in K} |u(x) - u^{\Delta t}(x)| \leqslant c(t, K)\, \Delta t^\beta + C e^{-\alpha t} \tag{13}$$

*holds for all $t \geqslant \Delta t$ and $0 < \beta < \gamma$ ($\beta \leqslant \gamma$ if $\alpha > c_g \gamma$).*

*Remark* 1. It follows from theorems 1 and 2 that, if $f$ and $\psi$ are locally Lipschitz continuous and $\alpha > c_g \gamma$ then $u$ is locally Lipschitz continuous; in this case $u$ is a local solution of the differential problem:

$$\begin{cases} u \leqslant \psi, \quad -g \cdot \nabla u + \alpha u \leqslant f \\ [u-\psi]\,[-g\nabla u + \alpha u - f] = 0 \quad \text{a.e.} \end{cases}$$

as it can be seen passing to the limit, as $t \to 0^+$, in the inequality

$$\frac{u(x) - (\Phi(t)\,u)(x)}{t} \leqslant \frac{1}{t} \int_0^t (\Phi(s)\,f)(x)\, ds$$

We refer to [2] for results of this type.

*Sketch of the proof of THEOREM* 2.—The pointwise convergence of a sequence $\{u^{\Delta t}\}$ to some function $u \leqslant \psi$ is an obvious consequence of the uniform boundedness of $\{u^{\Delta t}\}$. Next one shows that $\tilde{u}$ satisfies the $2^{nd}$ inequality in (4); the main tool in this respect are the estimates:

$$\sup_{x \in K} \left| \sum_{k=0}^{[t/\Delta t_j]-1} \int_{k\Delta t_j}^{(k+1)\Delta t_j} (\Phi^{\Delta t_j}(k) f)(x)\, ds - \int_0^t (\Phi(s) f)(x)\, ds \right| \leqslant c_1(t, K)(\Delta t_j)^\beta \tag{14}$$

$$\sup_{x \in K} |(\Phi^{\Delta t_j}([t/\Delta t_j])\, u^{\Delta t_j})(x) - (\Phi(t)\tilde{u})(x)| \leqslant c_2(t, K) \cdot (\Delta t_j)^\beta + \\ + e^{-\Delta t} \sup_{x \in K} |u^{\alpha t_j}(y_x(t)) - \tilde{u}(y_x(t))|, \tag{15}$$

holding for all $t \geqslant \Delta t_j$ and sufficiently small $\Delta t_j$.

The estimate (14) follows from well-known results on the Euler method for ordinary differential equations, namely

$$|y_x^{\Delta t}(k\Delta t) - y_x(s)| \leqslant c_t\, \Delta t, \quad s \in \big(k\Delta t, (k+1)\Delta t\big), \quad k = 0, \ldots, [t/\Delta t] - 1$$

(see [7]).

To prove (15) one makes use of the uniform Hölder continuity of $u^{\Delta t}$ and the contraction property of $\Phi(t)$.

The maximality of $u^{\Delta t}$ implies the maximality of $\tilde{u}$, as it can be seen by contradiction; this gives

$$\tilde{u}(x) = u(x)$$

and, therefore,

$$(u(x) - \psi(x)) \cdot \left(u(x) - \int_0^t (\Phi(s) f)(x)\, ds - (\Phi(t) u)(x)\right) = 0 \quad \forall t > 0 \tag{16}$$

From (16) one finally obtains the estimate (13); to this purpose one considers the sets $K_1 = \left\{x \in K:\ u(x) < \psi(x),\ u(x) = \int_0^t (\Phi(s) f)(x)\, ds + (\Phi(t) u)(x)\right\}$, $K_2 = \left\{x \in K: u(x) = \psi(x), u(x) < \int_0^t (\Phi(s) f)(x)\, ds + (\Phi(t) u)(x)\right\}$ and $K_3 = \left\{x \in K: u(x) = \psi(x), u(x) = \int_0^t (\Phi(s) f)(x)\, ds + (\Phi(t) u)(x)\right\}$.

## 4. Convergence of the optimal stopping times.

Let us define for all $j \in N$ the sets

$$T_j = \{k \in N: u^{\Delta t_j}(y_x^{\Delta t_j}(k\Delta t_j)) > \psi(y_x^{\Delta t_j}(k\Delta t_j)) - 1/j\}$$

Under the same general assumptions of Theorem 2, the following holds:

THEOREM 3. *If there exists* $n_j \in N$ *and* $\Delta t_j > 0$ *such that*:

$\Delta t_j \to 0^+$ *as* $j \to +\infty$, $\{n_j \Delta t_j\}$ *is bounded,* $u^{\Delta t_j}(y_x^{\Delta t_j}(n_j \Delta t_j)) = \psi(y_x^{\Delta t_j}(n_j \Delta t_j))$, *then* $t_x^*$ *defined in* (5) *is finite and therefore optimal. On the contrary, if* $t_x^*$ *is finite, then there exists a sequence* $\Delta t_j \to 0^+$ *such that* $T_j \neq \emptyset$ ($j \in N$), *and the sequence*

$$t_j^* = \Delta t_j \cdot \mathrm{Min}\,\{k \in N: k \in T_j\}$$

*converges to* $t_x^*$ *as* $j \to +\infty$.

*Final comments.* For the details of the proofs the reader is referred to [8]. In that paper the same discretization scheme is applied to the study of the $1^{st}$ order variational inequality arising in connection with the optimal stopping time problem for a deterministic system constrained by an open bounded subset of $R^N$.

## References.


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## Konstruktywne podejście do deterministycznego zagadnienia stopu

Proponuje się tutaj formułę aproksymacyjną dla zagadnienia optymalnego stopu dla układu deterministycznego. Układ taki był ostatnio rozpatrzony przez J. L. Menaldi [1], [2] i J. L. Menaldi, E. Rofman [3].

## Конструктивный подход к вопросу определения оптимального момента задержки

Предлагается аппроксимационная схема для задачи определения оптимального момента задержки детерминистической системы, рассматриваемой Д. Л. Менальди [1], [2] и Д.Л. Менальди и Э. Ройзманом [3].