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Existence theorems for solutions of optimal problems with variable delays

by

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Two existence theorems are proved for optimal problems containing variable delays in phase coordinates as well as in controls.

1. Introduction.

In the present paper the author considers an optimal problem with an integral cost functional and boundary conditions of general form for nonlinear systems with variable delays:

$$\dot{x}(t)=f\big(t, x(t), x(\tau(t)), u(t), u(\theta(t))\big).$$

The delay $\tau(t)\leqslant t$ in phase coordinates is a piecewise continuous function. The delay $\theta(t)<t$ in controls is an absolutely continuous and increasing function.

For the problem stated above, with the help ot the method offered in [1], existence theorems for solutions are proved. These theorems are analogues to the well-known theorems of C. Olech [1].

2. Formulation of the problem.

Existence theorems. Let the motion of an object be described by the system of differential equations

$$\dot{x}(t)=f\big(t, x(t), x(\tau(t)), u(t), u(\theta(t))\big),\ t\in I=[t_0, T]. \tag{1}$$

The right-hand side $f(t, x, y, u, v)$ is an n-dimensional vector function continuous with respect to $(x, y, u, v)\in G^2\times 0^2$ for a fixed $t\in I$, and measurable with respect to $t\in I$ for a fixed $(x, y, u, v)\in G^2\times 0^2$, where G and 0 are open sets from the Euclidean spaces R^n and R^r, respectively. The delays $\tau(t)$ and $\theta(t)$ satisfy the above -mentioned conditions on the interval I.

Let $U: I\times G\to 2^0$ be an upper semicontinuous (u s.c.) multivalued mapping, i.e., the set

$$\{(t, x, y) | y \in U(t, x)\}$$

is closed in R^{1+n+r}. It is not difficult to notice that, if $U(t, x)$ is an u.s.c. mapping, the set $U(t, x)$ is closed; $\varphi(t)\in G$, $\min(\theta_0, \tau_0)=\tau\leqslant t\leqslant t_0$, $\theta_0=\theta(t_0)$, $\tau_0=\min\limits_I \tau(t)$, —a continuous initial function; $g^i: I\times G^2\to R^1$, $i=1, ..., p$, — continuous functions.

Consider an integral functional

$$I(t_1, x(\cdot), u(\cdot))=\int\limits_{t_0}^{t_1} f^0\big(t, x(t), x(\tau(t)), u(t), u(\theta(t))\big)\, dt, \tag{2}$$

where f^0 satisfies the same conditions as f.

An element $(t_1, x(\cdot), u(\cdot))$, $t_1\in I$, will be called admissible if the following conditions are satisfied:

1) $x(t)=\varphi(t)$, $\tau_0\leqslant t<t_0$, on the interval $t_0\leqslant t\leqslant t_1$, $x(t)\in G$ — absolutely continuous, and

$$g^i(t_1, x(t_0), x(t_1))=0,\ i=1, ..., k,\quad g^{k+j}(t_1, x(t_0), x(t_1))\leqslant 0,\ j=1, ..., P-k, \tag{3}$$

2) $u(t)$ — a measurable function on the interval $\theta_0\leqslant t\leqslant t_1$, satisfying the condition $u(t)\in U(t, x(t))$ almost everywhere (a.e.),

3) The pair $(x(\cdot), u(\cdot))$ on the interval $t_0\leqslant t\leqslant t_1$ satisfies system (1) a.e.,

4) $I(t_1, x(\cdot), u(\cdot))<\infty$.

The set of admissible elements will be denoted by Ω.

An element $(\tilde{t}_1, \tilde{x}(\cdot), \tilde{u}(\cdot))$ is called a solution to optimal problem (1)–(3) if

$$I(t_1, \tilde{x}(\cdot), \tilde{u}(\cdot))=\min.$$

Let us decompose the interval $[\theta_0, T]$ into the intervals $E_\alpha=[\xi_\alpha, \xi_{\alpha+1}]$, $\alpha=0, ..., \sigma$,

$$\xi_\alpha=\theta(\xi_{\alpha+1}),\ \alpha=0, ..., \sigma-1,\ \xi_1=t_0,\ \xi_{\sigma+1}=T.$$

In the space R^{m+nm} we shall introduce the ordering defined by the cone C_m:

$$C_m=\{(x^1, ..., x^m, 0) | x^i\geqslant 0,\ i=1, ..., m,\ 0\in R^{nm}\},\ m=1, ..., \sigma.$$

Let us introduce the notations:

$$F_0^i(t, x_i, y_i, u_i, u_{i-1})=\dot{\zeta}^{i-1}(t) f^0\big(\zeta^{i-1}(t), x_i, y_i, u_i, u_{i-1}\big),$$

$$F_1^i(t, x_i, y_i, u_i, u_{i-1})=\dot{\zeta}^{i-1}(t) f\big(\zeta^{i-1}(t), x_i, y_i, u_i, u_{i-1}\big),$$

$i=1, ..., \sigma$, $t\in E_1$, $\zeta(t)$ — the inverse function of $\theta(t)$, $\zeta^i(t)=\zeta(\zeta^{i-1}(t))$, $\zeta^0(t)=t$,

$$\zeta(t)=T,\ \theta(T)\leqslant t\leqslant T;$$

$$F_m=(F_0^1, ..., F_0^m, F_1^1, ..., F_1^m),\ m=1, ..., ;$$

$P_m(t, x_1, ..., x_m, y_1, ..., y_m, z_0, ..., z_m)=\{q=(q^1, ..., q^m, q_1, ..., q_m) | q\geqslant F_m(t, x_1, ..., x_m, y_1, ..., y_m, u_0, ..., u_m), (u_0, ..., u_m)\in U(\theta(t), z_0)\times U(t, z_1)\times ... \times U(\zeta^{m-1}(t), z_m)\}$, $t\in E_1$, $m=1, ..., \sigma$.

THEOREM 1. *An optimal exists if the following conditions are satisfied:*

1) $\Omega \neq \emptyset$,

2) *there exists some* $M>0$ *such that* $|x(t)| \leqslant M$, $t_0 \leqslant t \leqslant t_1$, $\forall$ $(t_1, x(\cdot),$

$$u(\cdot)) \in \Omega,$$

3) *for any* $d \in R^n$, *there exists an integrable function* $\Phi_d(t)$, $t \in I$, *such that*

$$-f^0(t, x, y, u, v) + \langle d, f(t, x, y, u, v)\rangle \leqslant \Phi_d(t)^{1)}, \ (u, v) \in U(t, x) \times U(\theta(t), z),$$

$x \in A_M = \{x \,|\, |x| \leqslant M\}$, $y \in B_M = \{\varphi(t) \,|\, \tau_0 \leqslant t \leqslant t_0\} \bigcup A_M$, $z \in D_M = \{\varphi(t) \,|\, \theta_0 \leqslant t \leqslant t_0\} \bigcup A_M$

4) *the set* $P_m(t, x_1, \ldots, x_m, y_1, \ldots, y_m, z_0, \ldots, z_m)$ *is convex for each fixed* $(t, x_1, \ldots, \ldots, x_m, y_1, \ldots, y_m, z_0, \ldots, z_m) \in E_1 \times G^{(3m+1)n}$ *and* $m=1, \ldots, \sigma$,

5) *the mapping*

$$(x_1, \ldots, x_m, y_1, \ldots, y_m, z_0, \ldots, z_m) \to P_m(t, .)$$

is u.s.c. for each $t \in E_1$ *and* $m=1, \ldots, \sigma$.

Let the functions f and f^0 be defined on $I \times R^{2n} \times U^2$ and satisfy the above-mentioned conditions, and $U(t, x)$ — u.s.c. on $I \times R^n$.

THEOREM 2. *An optimal solution exists if the following conditions are satisfied:*

1) $\Omega \neq \emptyset$,

2) *the set* $P_m(t, x_1, \ldots, x_m, y_1, \ldots, y_m, z_0, \ldots, z_m)$ *is convex for each fixed* $(t, x_1, \ldots, \ldots, x_m, y_1, \ldots, y_m, z_0, \ldots, z_m) \in E_1 \times R^{(3m+1)n}$ *and* $m=1, \ldots, \sigma$,

3) *the mapping*

$$(x_1, \ldots, x_m, y_1, \ldots, y_m, z_0, \ldots, z_m) \to P_m(t, .)$$

is u.s.c. for each $t \in E_1$ *and* $m=1, \ldots, \sigma$,

4) *there exists some* $M>O$ *with a property that, for any element* $(t_1, x(\cdot), u(\cdot)) \in \in \Omega$, *one can find some* $t \in [t_0, t_1]$ *such that* $|x(t)| \leqslant M$,

5) *for any* $d=(-1, d_1) \in R^{1+n}$ *and* $\gamma>0$, *there exists a function* $\Phi_d(t, \gamma)$ *integrable on* I, *such that*

$$-f^0(t, x, y, u, v) + \langle d_1, f(t, x, y, u, v)\rangle \leqslant \Phi_d(t, \gamma),$$
$$(u, v) \in U(t, x) \times U(\theta(t), z), \ (x, y, z) \in A_\gamma \times B_\gamma \times D_\gamma,$$

where $\Phi_d(t, \gamma)$ *possesses the following properties:*

a) *the function* $\Phi_{\bar d}(t, \gamma)$, $\bar d=(-1, 0)$, *does not depend on* γ, *there exists some* $\eta>0$ *such that, for each* $d_1 \in R^n$, $|d_1|=1$, *the function* $\Phi_{d_\eta}(t, \gamma)$ *is linear with respect to* γ, $d_\eta=(\eta-, d_1)$.

b) *the functions* $\Phi_{\bar d}(t, \gamma)$ *and* $\Phi_{d_\eta}(t, \gamma)$ *are linear with respect to* γ *for each* $d_1 \in R^n$, $|d_1|=1$ *and* $\eta \in (0, \eta_0]$, $\eta_0>0$, *if* $\Phi_d(t, \gamma) = \Phi_{d_\eta}(t) + \gamma\psi_{d_\eta}(t)$, *then*

$$\int_I \psi_{d_\eta}(t)\, dt \leqslant L < \infty, \ \forall d_1 \in R^n, \ |d_1|=1, \ \eta \in (0, \eta_0].$$

Below we give an example which shows that, for the existence of solutions, the convexity of the set P_m is essential.

1) $\langle .,. \rangle$ — denotes the sclar product

EXAMPLE. Consider an optimal problem

$$\dot{x}(t)=-x(t-1)+u(t)+u^2(t-1),\ 0\leqslant t\leqslant 2,$$
$$x(t)=0,\ -1\leqslant t\leqslant 0,\ |x(2)|\leqslant 3.$$
$$u(t)=1,\ -1\leqslant t\leqslant 0,\ U=[-1,1],$$
$$\int_0^2 (x(t)-t)^2\,dt=min.$$

For a given $k=1, 2, \dots$, we shall decompose the interval $[0, 1]$ into the intervals I_i, $i=1, \dots, k$, of length $1/k$. Let us define a control $u_k(t)$, $0\leqslant t\leqslant 2$: $u_k(t)=v_k(t)$, $0\leqslant t\leqslant 1$, $u_k(t)=t-1$, $1\leqslant t\leqslant 2$, where $v_k(t)$ is an oscillating control, i.e., $v_k(t)=+1$, $t\in I_1$, $v_k(t)=-1$, $t\in I_2$, etc.

With k sufficiently large, the element $(x_k(\cdot), u_k(\cdot))$ is admissible. Here

$$x_k(t)=t+\int_0^t u_k(s)\,ds,\ 0\leqslant t\leqslant 1,\ x_k(t)=x_k(1)+$$
$$+\int_1^t (s-1-x_k(s-1))\,ds+t-1,\ 1\leqslant t\leqslant 2.$$

Furthermore, $\lim_{k\to\infty} x_k(t)=t$ uniformly with respect to $t\in[0, 1]$, and the sequence of Dirac measures $\delta_{v_k(t)}$ is weakly convergent to $\frac{1}{2}\delta-1+\frac{1}{2}\delta+1$, [2]. It is easy to observe that the trajectory $\tilde{x}(t)=t$, $0\leqslant t\leqslant 2$, corresponds to the control $\tilde{u}(t)$: : $\tilde{u}(t)=0$, $0\leqslant t\leqslant 1$, $\tilde{u}(t)=t$, $1\leqslant t\leqslant 2$, but $(\tilde{x}(\cdot), \tilde{u}(\cdot))$ is not admissible.

Consequently, in the problem under consideration there is no optimal solution since the set

$$P(t, y_1, y_2)=\left\{\begin{pmatrix}-y_1+u_1+u_0^2\\-y_2+u_2+u_1^2\end{pmatrix}\middle|\ u_i\in U,\ i=0, 1, 2\right\}$$

is not convex.

REMARK 1. It should be noticed that the existence theorem given in [3] is not true; in the example considered, all assumptions of the theorem in question are satisfied and yet, no solution exists.

The lemmas formulated below play an essential part in proving theorem 1. The proofs of these lemmas are given in [1].

3. Fundamental lemmas.

Let P be a subset of the space R^n. We shall define a set

$$K_P=\{\alpha \mid p+\lambda\alpha\in P,\ \forall p\in P,\ \lambda\geqslant 0\}.$$

The set K_P is a cone and possesses the following properties:

$$K_{\mathrm{co}P}=\mathrm{co}\,K_P,\ K_{\mathrm{cl}P}=\mathrm{cl}\,K_P \qquad (4)$$

Let in R^n the ordering be defined by the cone C:

$$C=\{x=(x^1, \ldots, x^s, 0)|x^i \geqslant 0, \; i=1, \ldots, s\}.$$

The set

$$X=C-C=\{X|x=c_1-c_2, \; c_1, c_2 \in C\},$$

is a subspace in R^n. Denote by Y an orthogonal complement of the subspace X, and by C^0 — the polar of the cone C:

$$C^0=\{(x^1, \ldots, x^s, y)|x^i \leqslant 0, \; i=1, \ldots, s, \; y \in R^{n-s}\}$$

LEMMA 1. (C. Olech). *Let $P: I \rightarrow 2^{R^n}$ be a multivalued mapping such that, for every t, the set $P(t)$ is closed and convex. Besides, we shall assume that*

$$K_{P(t)}=C, \; t \in I,$$

and, for each $d \in int\ C^0$, there exists an integrable function $\Phi_d(t)$, $t \in I$, such that

$$\max_{p \in P(t)} \langle d, p \rangle \leqslant \Phi_d(t).$$

If the sequence of absolutely continuous functions $z_k(t)$, $k=1, 2, \ldots$, is uniformly bounded and

$$\dot{z}_k(t) \in P(t) \; a.e. \; on \; I,$$

then there exists a subsequence $z_{k_i}(t)$, $i=1, 2, \ldots$, which is pointwise convergent to a function $z(t)+v(t)$, where

1°. *$z(t)$ is absolutely continuous, and $\dot{z}(t) \in P(t)$ a.e. on I,*

2°. *$\dot{v}(t)=0$ a.e. on I, $v(s) \leqslant v(t)$, $s \leqslant t$, $v(t_0)=0$,*

3°. *$\lim_{i \rightarrow \infty} Pr_y z_{k_i}(t)=Pr_y z(t)$[2] uniformly on I.*

LEMMA 2. (*Q — property of L. Cesari* [4]). *Let a mapping $Q: I \times R^n \rightarrow 2^{R^n}$ be u.s.c. for each fixed $t \in I$, the values of $Q(t, x)$ — be convex sets, and the following conditions are satisfied:*

$$K_{Q(t,x)}=C,$$

for each $d \in int\ C^0$ and $\gamma > 0$, there exists a function $\Phi_d(t, \gamma)$ integrable on I, such that

$$\sup_{|x| \leqslant \gamma} \; \sup_{q \in Q(t,x)} \langle d, q \rangle \leqslant \Phi_d(t, \gamma).$$

Then the mapping Q has the property

$$Q(t, x_0)=\bigcap_{\varepsilon \geqslant 0} \mathrm{cl\ co} \bigcup_{|x-x_0| \leqslant \varepsilon} Q(t, x) \textit{ for each fixed } t \in I$$

(see statement 3 and remark 3, [1]).

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[2]) Pr — operator of orthogonal projection

LEMMA 3. *Let a function* $f(t, u): I\times R^r \to R^n$ *be continuous with respect to* $u \in R^r$ *for a fixed* $t \in I$ *and measurable with respect to* t *for a fixed* u.
Let $U(t): I \to 2^{R^r}$ *be a u.s.c. maping. If the measurable function* $y(t)$, $t \in I$, *satisfies the condition*

$$y(t) \in Q(t) = \{z \mid z \geqslant f(t, u),\ u \in U(t)\} \text{ a.e. on } I,$$

then there exists a measurable function $u(t) \in U(t)$ *such that*

$$z(t) \geqslant f(t, u, (t)) \text{ a.e. on } I.$$

Proof of theorem 1. It follows from assumption 3) that the set

$$\{I(t_1, x(\cdot), u(\cdot)) \mid (t_1, x(\cdot), u(\cdot)) \in \Omega\}$$

is bounded from below. Consequently, there exists a sequence $(t_1^{(k)}, x_k(\cdot), u_k(\cdot))$, $k=1, 2, \ldots$, such that

$$\lim_{k\to\infty} I(t_1^{(k)}, x_k(\cdot), u_k(\cdot)) = \inf_{\Omega} I(t_1, x(\cdot), u(\cdot)) = \omega.$$

From the sequence $\{t_1^{(k)}\}$ we choose a subsequence which will again be denoted by $\{t_1^{(k)}\}$, tending to some $\tilde{t}_1 \in I$.

Let $\tilde{t}_1 \in E_l$; we shall introduce the notations:

$$x_k^0(t) = \int_{t_0}^{t} f^0\big(s, x_k(s), x_k(\tau(s)), u_k(s), u_k(\theta(s))\big)\, ds,\ t_0 \leqslant t \leqslant t_1^{(k)},$$

$$z_k(t) = \big(x_k^0(t), \ldots, x_k^0(\zeta^{l-1}(t)), x_k(t), \ldots, x_k(\zeta^{l-1}(t))\big),\ t \in E_1,$$

$$x_k^0(t) = x_k(t_1^{(k)}),\ x_k(t) = x_k(t_1^{(k)}),\ t \geqslant t_1^{(k)},\ P_l\big(t, \chi(\zeta^{l-1}(t)), x_1, \ldots, x_l, y_1, \ldots, y_l, z_0, \ldots, z_l\big) =$$
$$= \{q \in R^{l+nl} \mid q \geqslant \big(F_0^1, \ldots, \chi(\zeta^{l-1}(t)) F_0^l, F_1, \ldots, \chi_k(\zeta^{l-1}(t)) F_0^l\big) \mid (u_0, \ldots, u_l) \in$$
$$\in U(\theta(t), z_0) \times u(t, z_1) \times \ldots \times u(\zeta^{l-1}(t), z_l)\},\ t \in E_1.$$

Here $\chi_k(t)$ — the characteristic function of the interval $t_0 \leqslant t \leqslant t_1^{(k)}$.

It is not hard do see that

$$\dot{z}_k(t) \in P_l\Big(t; \chi_k(\zeta^{l-1}(t)), x_k(t), \ldots, x_k(\zeta^{l-1}(t)), x_k(\tau(t)), \ldots, x_k\big(\tau(\zeta^{l-1}(t))\big),$$
$$x_k(\theta(t)), x_k(t), \ldots, x_k(\zeta^{l-1}(t))\Big),\ t \in E_1.$$

Let us define a set

$$\Pi_j(t) = \bigcup_{k\geqslant j} P_l\Big(t; \chi_k(\zeta^{l-1}(t)), x_k(t), \ldots, x_k(\zeta^{l-1}(t)), x_k(\tau(t)), \ldots, x_k\big(\tau(\zeta^{l-1}(t))\big),$$
$$x_k(\theta(t)), x_k(t), \ldots, x_k(\zeta^{l-1}(t))\Big).$$

We shall show that

$$K_{\Pi_j(t)} = C_l, \quad j = 1, 2, \ldots.$$

Indeed, let $\alpha \in C_l$, then it is evident that

$$q + \alpha\lambda \in \Pi_j(t), \quad \forall q \in \Pi_j(t), \quad \lambda \geqslant 0,$$

i.e., $K_{\Pi_j(t)} \supset C_l$.

Further, let $\alpha \in K_{\Pi_j(t)}$ and $\alpha \notin C_l$, then there exists some $b \in \operatorname{int} C_l^0$ such that $\langle \alpha, b\rangle > 0$.

$$q_0 + \alpha\lambda \in \Pi_j(t), \quad \forall \lambda \geqslant 0,$$

where q_0 — a fixed element from $\Pi_j(t)$.

For any $\lambda \geqslant 0$, there exists some $k_\lambda \in \{1, 2, \ldots\}$ such that $q^{(\lambda)} = q_0 + \alpha\lambda \in P_l(t, \chi_{k_l}(\zeta^{l-1}(t)), x_{k_l}(t), \ldots)$.
We have

$$\langle q^{(\lambda)}, b\rangle = \sum_{i=1}^{l} q_i^{(\lambda)} b_0^i + \sum_{i=1}^{l} q_i^{(\lambda)} b_1^i \leqslant \sum_{i=1}^{q-1} \Big[\dot{\zeta}^{i-1}(t) f^0\Big(\zeta^{i-1}(t), x_{k_\lambda}(\zeta^{i-1}(t)),$$

$$x_{k_\lambda}(\tau(\zeta^{i-1}(t))), u_i^{(\lambda)}, u_{i-1}^{(\lambda)}\Big) b_0^i + \dot{\zeta}^{i-1}(t) f\Big(\zeta^{i-1}(t), x_{k_\lambda}(\zeta^{i-1}(t)), x_{k_\lambda}(\tau(\zeta^{i-1}(t))), u_i^{(\lambda)},$$

$$u_{i-1}^{(\lambda)}\Big) b_1^i\Big] + \dot{\zeta}^{l-1}(t)\, \chi_{k_\lambda}(\zeta^{l-1}(t)) \Big[b_0^l f^0\Big(\zeta^{l-1}(t), x_{k_\lambda}(\zeta^{l-1}(t)), x_{k_\lambda}(\tau(\zeta^{l-1}(t))),$$

$$u_l^{(\lambda)}, u_{l-1}^{(\lambda)}\Big) + b_1^l f\Big(\zeta^{l-1}(t), x_{k_\lambda}(\zeta^{l-1}(t)), x_{k_\lambda}(\tau(\zeta^{l-1}(t))), u_l^{(\lambda)}, u_{l-1}^{(\lambda)}\Big)\Big]. \quad (5)$$

The left-hand side of inequality (5), with $\lambda \to \infty$ is not bounded, and the right-hand side, by condition 3), is bounded by an integrable function; we get a contradiction

In this way.

$$K_{\Pi_j(t)} = C_l,$$

and taking into account conditions (4), we obtain

$$K_{P_j(t)} = C_l,$$

where

$$P_j(t) = \operatorname{cl} \operatorname{co} \Pi_j(t).$$

For each fixed $j = 1, 2, \ldots$,

$$\dot{z}_{i+j}(t) \in P_j(t), \quad i = 0, 1, \ldots .$$

The sequence $z_{i+j}(t)$, $i = 0, 1, \ldots$, and the mapping $t \to P_j(t)$ satisfy all conditions of Lemma 1, and therefore there exists a subsequence $z_{k_j}(t)$, $j = 1, 2, \ldots$, such that, with each $t \in [t_0, \tilde{t}_1]$,

$$\lim_{j\to\infty} z_{k_j}(t) = z(t) + v(t) = \big(x_l^0(t) + v_1(t), \ldots, x_l^0(t) + v_l(t), x_1(t), \ldots, x_l(t)\big),$$

$$\dot{z}(t) \in P_j(t) \text{ a.e. on } [t_0, \tilde{t}_1],$$

where $v_i(t)$, $i = 1, \ldots, l$, are nondecreasing scalar functions, such that $\dot{v}_i(t) = 0$ a.e, and $\dot{v}_i(t_0) = 0$, and $z(t)$ is an absolutely continuous function.

Let us represent the function $z(t) + v(t)$ in the following form

$$z(t) + v(t) = \big(\tilde{x}_l^0(t) + \tilde{v}_1(t), \ldots, \tilde{x}_l^0(t) + \tilde{v}_l(t), x_1(t), \ldots, x_l(t)\big),$$

where

$$\tilde{x}_l^0(t)=x_l^0(t), \tilde{v}_1(t)=v_1(t), \tilde{x}_i^0(t)=x_i^0(t)-v_{i-1}(\xi_i), \tilde{v}_i(t)=\dot{v}_i(t)+v_{i-1}(\xi_1),$$
$$i=2, \dots, l, \ \tilde{v}_i(t)\geqslant 0, \ \tilde{x}_l(\tilde{t}_1)+\tilde{v}_l(\tilde{t}_1)=\omega.$$

It is not hard to notice that

$$\tilde{x}_1^0(t_0)=0, \ \tilde{x}_i^0(t_0)=\tilde{x}_{i-1}^0(\xi_1), \quad i=2, \dots, l, \tag{6}$$

since

$$\lim_{k\to\infty} x_k^0(\zeta^{i-1}(\xi_1))=\lim_{k\to\infty} x_k(\zeta^i(t_0)), \quad i=1, \dots, l.$$

We shall introduce the notations:

$$\tilde{x}^0(t)=\begin{cases}\tilde{x}_1^0(t), \ t\in E_1,\\ \tilde{x}_2^0(\theta(t)), \ t\in E_2,\\ \dots\dots\dots\dots\dots\dots\\ \tilde{x}_l^0(\theta^{l-1}(t)), \ t\in[\xi_l, \tilde{t}_1],\end{cases} \qquad \tilde{x}(t)=\begin{cases}\varphi(t), \ t\in E_0,\\ x_1(t), \ t\in E_1,\\ x_2(\theta(t)), \ t\in E_2,\\ \dots\dots\dots\dots\dots\dots\\ x_l(\theta^{l-1}(t)), \ t\in[\xi_l, \tilde{t}_1].\end{cases}$$

It is obvious that

$$\dot{\tilde{z}}(t)\in P_j(t) \text{ a.e. on } E_j, \quad j=1, 2, \dots,$$

where

$$\tilde{z}(t)=\left(\tilde{x}^0(t), \dots, \tilde{x}^0(\zeta^{l-1}(t)), \ \tilde{x}(t), \dots, \tilde{x}(\zeta^{l-1}(t))\right).$$

The mapping

$$(t, \chi, x_1, \dots, x_l, y_1, \dots, y_l, z_0, \dots, z_l)\to P_l,$$

$t\in E_1$, $\chi\in\{0, 1\}$, $x_i\in A_M$, $y_i\in B_M$, $i=1, \dots, l$, $z_j\in D_M$, $j=0, \dots, l$, satisfies all requirements of lemma 2, and therefore $\dot{\tilde{z}}(t)=\bigcup_{j\geqslant 1} P_j(t)=P_l\left(t; \chi(\zeta^{l-1}(t)), \ \tilde{x}(t), \dots, \right.$ $\dots, \tilde{x}(\zeta^{l-1}(t)), \tilde{x}(\tau(t)), \dots, \tilde{x}(\tau(\zeta^{l-1}(t)), \tilde{x}(\theta(t)), \tilde{x}(t), \dots, \left.\tilde{x}(\zeta^{l-1}(t))\right)$ a.e. on E_1. Here $\chi(t)$ — the characteristic function of the interval $[t_0, \tilde{t}_1]$.

It follows fiom lemma 3 that there exist measurable functions $\tilde{u}_i(t)$, $t\in E_1$, $i=0, \dots, l$, $\tilde{u}_0(t)\in U(\theta(t), \tilde{x}(\theta(t)))$. $\tilde{u}_i(t)\in U(\zeta^{i-1}(t), \ \tilde{x}(\zeta^{i-1}(t)))$, $i=1, \dots, l$, such that the following conditions are satisfied:

$$\begin{cases}\dfrac{d}{dt}\tilde{x}^0(\zeta^i(t))\geqslant\dot{\zeta}^i(t)f^0\left(\zeta^i(t), \tilde{x}(\zeta^i(t)), \tilde{x}(\tau(\zeta^i(t))), \tilde{u}_{i+1}(t), \tilde{u}_i(t)\right),\\ \qquad\qquad i=0, \dots, l-2,\\ \dfrac{d}{dt}\tilde{x}^0(\zeta^{l-1}(t))\geqslant\dot{\zeta}^{l-1}(t)f^0\left(\zeta^{l-1}(t), \tilde{x}(\zeta^{l-1}(t)), \tilde{x}(\iota(\zeta^{l-1}(t))),\right.\\ \qquad\qquad \left.\tilde{u}_l(t), \tilde{u}_{l-1}(t)\right)\chi(t),\end{cases} \tag{7}$$

$$\begin{cases} \frac{d}{dt}\tilde{x}\left(\zeta^i(t)\right)=\dot{\zeta}^i(t)\, f\Big(\zeta^i(t), \tilde{x}\left(\zeta^i(t)\right), \tilde{x}\left(\tau\left(\zeta^i(t)\right)\right), \tilde{u}_{i+1}(t), \tilde{u}_i(t)\Big), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i=0, \ldots, l-2, \\ \frac{d}{dt}\tilde{x}\left(\zeta^{l-1}(t)\right)=\dot{\zeta}^{l-1}(t)\, f\Big(\zeta^{l-1}(t), \tilde{x}\left(\zeta^{l-1}(t)\right), \tilde{x}\left(\tau\left(\zeta^{l-1}(t)\right)\right), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \tilde{u}_l(t), \tilde{u}_{l-1}(t)\Big)\,\chi(t), \end{cases} \tag{8}$$

a.e. on E_1.

Carrying out elementary transformations and taking into account (6), from (7), (8) we obtain

$$\dot{\tilde{x}}(t)=f\left(t, \tilde{x}(t), \tilde{x}\left(\tau(t)\right), \tilde{u}(t), \tilde{u}\left(\theta(t)\right)\right), \ t_0 \leqslant t \leqslant \tilde{t}_1,$$

$$\tilde{x}(\tilde{t}_1) \geqslant \int_{t_0}^{\tilde{t}_1} f^0\left(t, \tilde{x}(t), \tilde{x}\left(\tau(t)\right), \tilde{u}(t), \tilde{u}\left(\theta(t)\right)\right) dt,$$

where

$$\tilde{u}(t)=\begin{cases} \tilde{u}_0\left(\zeta(t)\right), \ t \in E_0, \\ \tilde{u}_1(t), \ t \in E_1, \\ \tilde{u}_2\left(\theta(t)\right), \ t \in E_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \tilde{u}_l\left(\theta^{l-1}(t)\right), \ t \in [t_l, \tilde{t}_1]. \end{cases}$$

From the definition, $\tilde{x}(\tilde{t}_1)=\tilde{x}_l(\tilde{t}_1) \leqslant \tilde{x}_l(\tilde{t}_1)+\tilde{v}_l(\tilde{t}_1)$ since $\tilde{v}_l(\tilde{t}_1) \geqslant 0$. In this way

$$I\left(\tilde{t}_1, \tilde{x}(\cdot), \tilde{u}(\cdot)\right)=\omega.$$

Of course $\tilde{x}(t)$, $t_0 \leqslant t \leqslant \tilde{t}_1$, satisfies boundary conditions (3). Consequently, the element $\left(\tilde{t}_1, \tilde{x}(\cdot), \tilde{u}(\cdot)\right)$ is the solution. Thus, theorem 1 has been proved. ■

Proof of theorem 2. It follows from assumption 1) that there exists a number L_1 such that

$$\Omega_1=\left\{\left(t_1, x(\cdot), u(\cdot)\right) \in \Omega \,|\, I\left(t_1, x(\cdot), u(\cdot)\right) \leqslant L_1\right\} \neq \emptyset$$

Of course,

$$\inf_{\Omega_1} I=\inf_{\Omega} I.$$

Further, in the same way as in [1], one proves the existence of a number $M \geqslant 0$ such that

$$|x(t)| \leqslant M, \ t_0 \leqslant t \leqslant t_1, \quad \forall \left(t_1, x(\cdot), u(\cdot)\right) \in \Omega_1.$$

In this manner, the proof of theorem 2 is reduced to that of theorem 1. ■

REMARK 2. Theorems 1 and 2 hold true also in the case when the right-hand side of system (1) has the form

$$f\left(t, x\left(\tau_1(t)\right), \ldots, x\left(\tau_s(t)\right), \ u\left(\theta_1(t)\right), \ldots, u\left(\left(\theta_v(t)\right)\right),$$

2. Problem formulation

Consider linear time invariant controllable and observable system described by the equations

$$\dot{x}=Ax+Eu \qquad x(0)=0 \tag{1a}$$

$$y=Cx+Eu, \tag{1b}$$

where $x=x(t)\in R^n$, $u=u(t)\in R^q$, $y=y(t)\in R^p$, $t\geqslant 0$ are the state, input and output vectors respectively.

The transfer function matrix of system (1) is given by the formula

$$H(s)=C(s1_n-A)^{-1}B+E \tag{2}$$

and is a rational proper $p\times q$ matrix.
Assume that a $p\times r$ rational proper matrix T is given.

Synthesis Problem

Find constant matrices $F\in R^{q\times p}$ and $G\in R^{q\times r}$ such that the system (1) under the action of a control law

$$u=Fy+Gv \tag{3}$$

is controllable and observable and its transfer function matrix is T, where $v=v(t)\in R^r$ is a vector of external reference signals.

The stated above problem has not been solved satisfactorily. The most difficult step in solving it is to find a solution of the nonlinear matrix equation

$$T=H(1_q-FH)^{-1}G \tag{4}$$

with respect to matrices F and G. The attempts to overcome it were based either on very strong assumptions e.g. invertibility of matrix H or required solving a large system of linear equations with no insight in inner properties of the systems.

3. Solution to the problem

Consider S_H^0 — minimal observability matrix for the system with the transfer function matrix H. S_H^0 has following properties (Forney [3]):

(a) $S_H^0=[P\ R]$. $P\in R[s]^{p\times p}$, $R\in R[s]^{p\times q}$ — P and R are polynomial matrices.
(b) P is row proper.
(c) P and R are relatively left prime.
(d) $H=P^{-1}R$

det P is the characteristic polynomial of the system described by the H, deg det P—degree of det P—is the dimension of minimal state space realisation of the H.

Notice that these are the properties of polynomial matrix quotient representation of transfer function matrix H introduced by Wolovich (Wolovich [33]).

Let $S_T^0=[U L]$ be minimal observability matrix for the system with transfer function matrix T. Assume that both matrices (S_H^0 and S_T^0) have their rows ordered in such a way that row indexes i.e. highest degrees of polynomial elements in rows form nondecreasing sequences.

One can easy verify that (4) is equivalent to

$$T=(1_p-HF)^{-1}\, HG. \tag{5}$$

Substitute H and T expressed as quotients of polynomial matrices as in property (d) of minimal observability matrices. We obtain

$$U^{-1}\, L=(P-RF)^{-1}\, RG$$

This is equivalent to

$$DU=P-RF \tag{6a}$$

$$DL=RG, \tag{6b}$$

where D is some $p\times p$ nonsingular polynomial matrix. System (6) can be written in more concise form

$$DS_T^0=S_H^0\begin{bmatrix} 1_p & 0 \\ -F & G \end{bmatrix}$$

Since (1) and closed loop system are assumed to be controllable and observable, the state spaces of both systems should have the same dimensions. Thus det $D \in R$—{0} because

$$\deg(\det U)=\deg(\det P)+\deg(\det D)$$

and deg (det U) as well as deg (det P) are dimensions of minimal realisations of T and H respectively. Let denote

$$\begin{bmatrix} 1_p & 0 \\ -F & G \end{bmatrix}=M \tag{7}$$

We get

$$DS_T^0=S_H^0 M, \tag{8}$$

where M is constant matrix of the form (7) and D is unimodular polynomial matrix. We see that equation (8), is linear with respect to elements of matrices F and G and the left side of it is parametrised by coefficients of elements of D. (8) is much more attractive for computional purpose than (4) but so far one might doubt in advantages of it since elements of the unimodular matrix remain unspecified and can be of arbitrary high degree.

It was shown however ([15]) that if there exists a solution to (8) satisfying the assumptions of the problem statement then the matrix D should have the following form

$$D=\begin{bmatrix} D_1 & 0 & \dots & 0 \\ U_{21} & D_2 & \dots & 0 \\ \dots & \dots & \dots & . \\ U_{t1} & U_{t2} & \dots & D_t \end{bmatrix} \tag{9}$$

Step 5. Check if condition (10) is satisfied. If not then there is no solution to the problem. Otherwise choose maximal number of independent rows of matrix

$$\begin{bmatrix} R_1 \\ E \end{bmatrix}$$

take corresponding rows of matrix

$$\begin{bmatrix} U_1 - P_1 X & P_1 Y - L_1 \\ 1_p - X & Y \end{bmatrix}$$

and solve resulting linear noncontradictory equation with respect to matrices F and G.

4. Example

Let

$$H(s) = g_1^{-1} \begin{bmatrix} s^4 + 2s^2 - 3s & 0 & s^4 + s^3 + 2s^2 - s - 3 \\ s^4 - s^3 + 2s^2 - 5s + 3 & s^3 + 2s - 3 & 0 \\ s^2 - 2s + 1 & 0 & s^4 - s^3 - s^2 + s \end{bmatrix}$$

$$T(s) = g_2^{-1} \begin{bmatrix} (s+3)(-2.5s^3 - 3.5s^2 + 1.5s - 0.5) & (s+3)(5.5s^3 + 6.5s^2 + 7.5s + 0.5) \\ 1.5s^4 + 22.5s^3 + 30.5s^2 + 14s - 3.5 & 7.5s^4 + 25.5s^3 + 37.5s^2 + 19.5s \\ (s+3)(-4s^3 - 3s^2 + 5s + 2) & (s+3)(-2s^3 - 3s^2 + 4s + 1) \end{bmatrix}$$

where $g_1 = (s-1)^2 (s^2 + s + 3)$ and $g_2 = (s+3)(9s^3 + 10s^2 + 5.5s + 0.5)$. Compute minimal observability matrices

$$S_H^0 = \begin{bmatrix} s-1 & 0 & 0 & s & 0 & s+1 \\ 0 & s-1 & 0 & s-1 & 1 & 0 \\ 0 & 0 & s^2 + s + 3 & 1 & 0 & s^2 + s \end{bmatrix}$$

$$S_T^0 = \begin{bmatrix} 4s+1 & 0 & -2.5s & -1 & 3s+1 \\ s-3 & s+3 & -2.5s + 1.5 & s+2 & 2s \\ 2s^2 + 2s + 1 & 0 & s^2 + s + 0.5 & -s^2 - s + 1 & s^2 + s + 2 \end{bmatrix}$$

We see that row indexes of both polynomial matrices constitute the same set $\{1, 1, 2\}$. Polynomial unimodular matrix D has the form

$$D = \begin{bmatrix} d_{11} & d_{12} & 0 \\ d_{21} & d_{22} & 0 \\ a_1 s + b_1 & a_2 s + b_2 & d \end{bmatrix}$$

where $d_{11}, d_{12}, d_{21}, d_{22}, a_1, b_1, a_2, b_2, d$ are real numbers such that $(d_{11} d_{22} + - d_{21} d_{12})\, d \neq 0$.

Since $[P]_h = 1_3$ we have $X=[DU]_h$ and $Y=[DL]_h$.

$$R_1=\begin{bmatrix}1&0&2\\0&1&0\\1&0&-3\\0&0&0\end{bmatrix},\qquad E=\begin{bmatrix}1&0&1\\1&0&0\\0&0&1\end{bmatrix},\qquad P_1=\begin{bmatrix}1&0&0\\0&1&0\\0&0&-3\\0&0&-1\end{bmatrix}$$

$$X=\begin{bmatrix}4d_{11}+d_{12} & d_{12} & -2.5d_{11}-2.5d_{12}\\ 4d_{21}+d_{22} & d_{22} & -2.5d_{21}-2.5d_{22}\\ 4a_1+a_2+2d & a_2 & -2.5a_1-2.5a_2+d\end{bmatrix},\qquad Y=\begin{bmatrix}d_{12} & 3d_{11}+2d_{12}\\ d_{22} & 3d_{21}+2d_{22}\\ a_2-d & 3a_1+2a_2+d\end{bmatrix}$$

$$U_1=\begin{bmatrix}3d_{12}-d_{11} & -3d_{12} & -1.5d_{12}\\ 3d_{22}-d_{21} & -3d_{22} & -1.5d_{22}\\ 3b_2-b_1-d & -3b_2 & -1.5b_2-0.5d\\ 3a_2-4b_1-a_1-b_2-2d & -b_2-3a_2 & 2.5b_1+2.5b_2-1.5a_2-d\end{bmatrix}$$

$$L_1=\begin{bmatrix}d_{11}-2d_{12} & -d_{11}\\ d_{21}-2d_{22} & -d_{21}\\ b_1-2b_2-d & -2d-b_1\\ a_1+d-b_2-2a_2 & -3b_1-a_1-2b_2-d\end{bmatrix}$$

It is easy to check that

$$\perp\begin{bmatrix}R_1\\E\end{bmatrix}=\begin{bmatrix}0&0&0&1&0&0&0\\0&0&0&0&1&-1&-1\\0&0&1&0&-1&0&4\\1&0&1&0&0&-2&1\end{bmatrix}$$

We compute now the matrices U_1-P_1X and P_1Y-L_1

$$U_1-P_1X=\begin{bmatrix}-5d_{11}+2d_{12} & -4d_{12} & 2.5d_{11}+d_{12}\\ -5d_{21}+2d_{22} & -4d_{22} & 2.5d_{21}+d_{22}\\ 3b_2+12a_1+3a_2+5d-b_1 & 3a_2-3b_2 & -7.5a_1-7.5a_2-1.5b_2+2.5d\\ 3a_1+4a_2-4b_1-b_2 & -2a_2-b_2 & -2.5a_1-4a_2+2.5b_1+2.5b_2\end{bmatrix}$$

$$P_1Y-L_1=\begin{bmatrix}3d_{12}-d_{11} & 4d_{11}+2d_{12}\\ 3d_{22}-d_{21} & 4d_{21}+2d_{22}\\ 4d-3a_2-b_1-2b_2 & -9a_1-6a_2+b_1-d\\ a_2+b_2-a_1 & 3b_1+2b_2-2a_1-2a_2\end{bmatrix}$$

The unique solution to (10) is:

$$a_1=a_2=b_1=b_2=d_{12}=d_{21}=0,\qquad d_{11}=d_{22}=d=1.$$

In the last step we compute a solution to equation

$$\begin{bmatrix}1&0&2\\0&1&0\\1&0&-3\end{bmatrix}[F\ G]=\begin{bmatrix}-5&0&2.5&-1&4\\2&-4&1&3&2\\5&0&2.5&4&-1\end{bmatrix}$$

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Ogólny algorytm syntezy układu o zadanej macierzy transmitancji za pomocą proporcjonalnego sprzężenia zwrotnego od wyjścia

W pracy przedstawiona została metoda sprowadzenia nieliniowego równania macierzowego $T=H(1_q-FH)^{-1}G$ (wiążącego macierze transmitancji operatorowych H i T oraz stałe macierze: sprzężenia zwrotnego od wyjścia F i transformacji wektora wejścia G, występującego w problemie syntezy układu o zadanej macierzy transmitancji) do liniowego układu równań o stałych współczynnikach. Przedstawiony algorytm pozwala znaleźć rozwiązanie problemu w przypadku właściwych macierzy transmitancji operatorowych o dowolnej liczbie kolumn. W przypadku, gdy to możliwe naturalną konsekwencją przekształceń jest parametryzacja otrzymanej rodziny rozwiązań.

Общий алгоритм синтеза системы с заданной матрицей передаточной функции с помощью пропорциональной обратной связи с выхода

В работе представлен метод сведения нелинейного матричного уравнения $T=H(1_q-F\,H)^{-1}\,G$ (связывающего матрицы операторных передаточных функций H и T а также постоянные матрицы: обратной связи с выхода F и преобразования вектора входа G, выступающего в задаче синтеза системы с заданной матрицей передаточной функции) к линейной системе уравнений с постоянными коэффициентами. Представленный алгоритм позволяет найти решение задачи в случае правильных матриц операторных передаточных функций с произвольным числом столбцов. В случае, когда это возможно, естественным следствием преобразований является параметризация полученного семейства решений.