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A general algorithm of exact model matching by proportional output feedback

by

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In the paper the well known nonlinear equation $T=H(1_q-FH)^{-1}G$ (which describes the relation between transfer function matrices H and T and a pair (F, G) representing output feedback and input transformation in the exact model matching problem) is replaced equivalently by linear (with respect to F and G) equation. Presented algorithm is a general one and gives an answer to the exact model matching question for any pair of proper transfer function matrices with arbitrary number of columns. Parametrisation of solutions (when possible) is obtained.

1. Introduction

The problem of forming dynamical performance of a system represents one of the most important questions of control theory of multivariable linear systems. The main method is based on suitable shifting of poles by feedback techniques. In the case of one dimensional object it is not difficult to express desired dynamical proporties of the closed loop system by means of its poles locations. Closed loop poles of multivariable system do not describe them uniqely. Full characterisation of dynamical performance of linear time invariant controllable and observable system is given by an impuls response matrix or equivalently by transfer function matrix.

The transfer function matrix synthesis problem (called the exact model matching problem) was introduced by W.A. Wolovich in 1971 [29]. The whole bibliography devoted to this problem can be divided in two groups due to aspects they touch. First group is concerned with state feedback techniques. The lack of mesurable state vector is compensated by using a state observer. To this group there belong the following papers: [1, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 16, 18, 20, 21, 22, 23, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35]. The second group is pertained to the situation where the output feedback only is used: [2, 6, 14, 15, 17, 19, 24, 25]. For details see Myszewski [15].

This paper presents an algorithm of the exact model matching with relatively weak assumptions with respect to synthesised system.

2. Problem formulation

Consider linear time invariant controllable and observable system described by the equations

$$\dot{x}=Ax+Eu \qquad x(0)=0 \tag{1a}$$

$$y=Cx+Eu, \tag{1b}$$

where $x=x(t)\in R^n$, $u=u(t)\in R^q$, $y=y(t)\in R^p$, $t\geqslant 0$ are the state, input and output vectors respectively.

The transfer function matrix of system (1) is given by the formula

$$H(s)=C(s1_n-A)^{-1}B+E \tag{2}$$

and is a rational proper $p\times q$ matrix.
Assume that a $p\times r$ rational proper matrix T is given.

Synthesis Problem

Find constant matrices $F\in R^{q\times p}$ and $G\in R^{q\times r}$ such that the system (1) under the action of a control law

$$u=Fy+Gv \tag{3}$$

is controllable and observable and its transfer function matrix is T, where $v=v(t)\in R^r$ is a vector of external reference signals.

The stated above problem has not been solved satisfactorily. The most difficult step in solving it is to find a solution of the nonlinear matrix equation

$$T=H(1_q-FH)^{-1}G \tag{4}$$

with respect to matrices F and G. The attempts to overcome it were based either on very strong assumptions e.g. invertibility of matrix H or required solving a large system of linear equations with no insight in inner properties of the systems.

3. Solution to the problem

Consider S_H^0 — minimal observability matrix for the system with the transfer function matrix H. S_H^0 has following properties (Forney [3]):

(a) $S_H^0=[P\ R]$. $P\in R[s]^{p\times p}$, $R\in R[s]^{p\times q}$ — P and R are polynomial matrices.
(b) P is row proper.
(c) P and R are relatively left prime.
(d) $H=P^{-1}R$

det P is the characteristic polynomial of the system described by the H, deg det P—degree of det P—is the dimension of minimal state space realisation of the H.

Notice that these are the properties of polynomial matrix quotient representation of transfer function matrix H introduced by Wolovich (Wolovich [33]).

Let $S_T^0=[U\,L]$ be minimal observability matrix for the system with transfer function matrix T. Assume that both matrices (S_H^0 and S_T^0) have their rows ordered in such a way that row indexes i.e. highest degrees of polynomial elements in rows form nondecreasing sequences.

One can easy verify that (4) is equivalent to

$$T=(1_p-HF)^{-1}\,HG. \tag{5}$$

Substitute H and T expressed as quotients of polynomial matrices as in property (d) of minimal observability matrices. We obtain

$$U^{-1}\,L=(P-RF)^{-1}\,RG$$

This is equivalent to

$$DU=P-RF \tag{6a}$$

$$DL=RG, \tag{6b}$$

where D is some $p\times p$ nonsingular polynomial matrix. System (6) can be written in more concise form

$$DS_T^0=S_H^0\begin{bmatrix}1_p & 0\\ -F & G\end{bmatrix}$$

Since (1) and closed loop system are assumed to be controllable and observable, the state spaces of both systems should have the same dimensions. Thus det $D\in R$—$\{0\}$ because

$$\deg(\det U)=\deg(\det P)+\deg(\det D)$$

and deg (det U) as well as deg (det P) are dimensions of minimal realisations of T and H respectively. Let denote

$$\begin{bmatrix}1_p & 0\\ -F & G\end{bmatrix}=M \tag{7}$$

We get

$$DS_T^0=S_H^0M, \tag{8}$$

where M is constant matrix of the form (7) and D is unimodular polynomial matrix. We see that equation (8), is linear with respect to elements of matrices F and G and the left side of it is parametrised by coefficients of elements of D. (8) is much more attractive for computional purpose than (4) but so far one might doubt in advantages of it since elements of the unimodular matrix remain unspecified and can be of arbitrary high degree.

It was shown however ([15]) that if there exists a solution to (8) satisfying the assumptions of the problem statement then the matrix D should have the following form

$$D=\begin{bmatrix}D_1 & 0 & \dots & 0\\ U_{21} & D_2 & \dots & 0\\ \dots & \dots & \dots & .\\ U_{t1} & U_{t2} & \dots & D_t\end{bmatrix} \tag{9}$$

Step 5. Check if condition (10) is satisfied. If not then there is no solution to the problem. Otherwise choose maximal number of independent rows of matrix

$$\begin{bmatrix} R_1 \\ E \end{bmatrix}$$

take corresponding rows of matrix

$$\begin{bmatrix} U_1 - P_1 X & P_1 Y - L_1 \\ 1_p - X & Y \end{bmatrix}$$

and solve resulting linear noncontradictory equation with respect to matrices F and G.

4. Example

Let

$$H(s) = g_1^{-1} \begin{bmatrix} s^4+2s^2-3s & 0 & s^4+s^3+2s^2-s-3 \\ s^4-s^3+2s^2-5s+3 & s^3+2s-3 & 0 \\ s^2-2s+1 & 0 & s^4-s^3-s^2+s \end{bmatrix}$$

$$T(s) = g_2^{-1} \begin{bmatrix} (s+3)(-2.5s^3-3.5s^2+1.5s-0.5) & (s+3)(5.5s^3+6.5s^2+7.5s+0.5) \\ 1.5s^4+22.5s^3+30.5s^2+14s-3.5 & 7.5s^4+25.5s^3+37.5s^2+19.5s \\ (s+3)(-4s^3-3s^2+5s+2) & (s+3)(-2s^3-3s^2+4s+1) \end{bmatrix}$$

where $g_1 = (s-1)^2(s^2+s+3)$ and $g_2 = (s+3)(9s^3+10s^2+5.5s+0.5)$. Compute minimal observability matrices

$$S_H^0 = \begin{bmatrix} s-1 & 0 & 0 & s & 0 & s+1 \\ 0 & s-1 & 0 & s-1 & 1 & 0 \\ 0 & 0 & s^2+s+3 & 1 & 0 & s^2+s \end{bmatrix}$$

$$S_T^0 = \begin{bmatrix} 4s+1 & 0 & -2.5s & -1 & 3s+1 \\ s-3 & s+3 & -2.5s+1.5 & s+2 & 2s \\ 2s^2+2s+1 & 0 & s^2+s+0.5 & -s^2-s+1 & s^2+s+2 \end{bmatrix}$$

We see that row indexes of both polynomial matrices constitute the same set $\{1, 1, 2\}$. Polynomial unimodular matrix D has the form

$$D = \begin{bmatrix} d_{11} & d_{12} & 0 \\ d_{21} & d_{22} & 0 \\ a_1 s+b_1 & a_2 s+b_2 & d \end{bmatrix}$$

where d_{11}, d_{12}, d_{21}, d_{22}, a_1, b_1, a_2, b_2, d are real numbers such that $(d_{11} d_{22} + - d_{21} d_{12}) d \neq 0$.

Since $[P]_h = 1_3$ we have $X=[DU]_h$ and $Y=[DL]_h$.

$$R_1=\begin{bmatrix}1 & 0 & 2\\ 0 & 1 & 0\\ 1 & 0 & -3\\ 0 & 0 & 0\end{bmatrix},\qquad E=\begin{bmatrix}1 & 0 & 1\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix},\qquad P_1=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -3\\ 0 & 0 & -1\end{bmatrix}$$

$$X=\begin{bmatrix}4d_{11}+d_{12} & d_{12} & -2.5d_{11}-2.5d_{12}\\ 4d_{21}+d_{22} & d_{22} & -2.5d_{21}-2.5d_{22}\\ 4a_1+a_2+2d & a_2 & -2.5a_1-2.5a_2+d\end{bmatrix},\qquad Y=\begin{bmatrix}d_{12} & 3d_{11}+2d_{12}\\ d_{22} & 3d_{21}+2d_{22}\\ a_2-d & 3a_1+2a_2+d\end{bmatrix}$$

$$U_1=\begin{bmatrix}3d_{12}-d_{11} & -3d_{12} & -1.5d_{12}\\ 3d_{22}-d_{21} & -3d_{22} & -1.5d_{22}\\ 3b_2-b_1-d & -3b_2 & -1.5b_2-0.5d\\ 3a_2-4b_1-a_1-b_2-2d & -b_2-3a_2 & 2.5b_1+2.5b_2-1.5a_2-d\end{bmatrix}$$

$$L_1=\begin{bmatrix}d_{11}-2d_{12} & -d_{11}\\ d_{21}-2d_{22} & -d_{21}\\ b_1-2b_2-d & -2d-b_1\\ a_1+d-b_2-2a_2 & -3b_1-a_1-2b_2-d\end{bmatrix}$$

It is easy to check that

$$\perp\begin{bmatrix}R_1\\ E\end{bmatrix}=\begin{bmatrix}0 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & -1 & -1\\ 0 & 0 & 1 & 0 & -1 & 0 & 4\\ 1 & 0 & 1 & 0 & 0 & -2 & 1\end{bmatrix}$$

We compute now the matrices U_1-P_1X and P_1Y-L_1

$$U_1-P_1X=\begin{bmatrix}-5d_{11}+2d_{12} & -4d_{12} & 2.5d_{11}+d_{12}\\ -5d_{21}+2d_{22} & -4d_{22} & 2.5d_{21}+d_{22}\\ 3b_2+12a_1+3a_2+5d-b_1 & 3a_2-3b_2 & -7.5a_1-7.5a_2-1.5b_2+2.5d\\ 3a_1+4a_2-4b_1-b_2 & -2a_2-b_2 & -2.5a_1-4a_2+2.5b_1+2.5b_2\end{bmatrix}$$

$$P_1Y-L_1=\begin{bmatrix}3d_{12}-d_{11} & 4d_{11}+2d_{12}\\ 3d_{22}-d_{21} & 4d_{21}+2d_{22}\\ 4d-3a_2-b_1-2b_2 & -9a_1-6a_2+b_1-d\\ a_2+b_2-a_1 & 3b_1+2b_2-2a_1-2a_2\end{bmatrix}$$

The unique solution to (10) is:

$$a_1=a_2=b_1=b_2=d_{12}=d_{21}=0,\qquad d_{11}=d_{22}=d=1.$$

In the last step we compute a solution to equation

$$\begin{bmatrix}1 & 0 & 2\\ 0 & 1 & 0\\ 1 & 0 & -3\end{bmatrix}[F\ G]=\begin{bmatrix}-5 & 0 & 2.5 & -1 & 4\\ 2 & -4 & 1 & 3 & 2\\ 5 & 0 & 2.5 & 4 & -1\end{bmatrix}$$

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Ogólny algorytm syntezy układu o zadanej macierzy transmitancji za pomocą proporcjonalnego sprzężenia zwrotnego od wyjścia

W pracy przedstawiona została metoda sprowadzenia nieliniowego równania macierzowego $T=H(1_q-FH)^{-1}G$ (wiążącego macierze transmitancji operatorowych H i T oraz stałe macierze: sprzężenia zwrotnego od wyjścia F i transformacji wektora wejścia G, występującego w problemie syntezy układu o zadanej macierzy transmitancji) do liniowego układu równań o stałych współczynnikach. Przedstawiony algorytm pozwala znaleźć rozwiązanie problemu w przypadku właściwych macierzy transmitancji operatorowych o dowolnej liczbie kolumn. W przypadku, gdy to możliwe naturalną konsekwencją przekształceń jest parametryzacja otrzymanej rodziny rozwiązań.

Общий алгоритм синтеза системы с заданной матрицей передаточной функции с помощью пропорциональной обратной связи с выхода

В работе представлен метод сведения нелинейного матричного уравнения $T=$ $=H(1_q-F\,H)^{-1}\,G$ (связывающего матрицы операторных передаточных функций H и T а также постоянные матрицы: обратной связи с выхода F и преобразования вектора входа G, выступающего в задаче синтеза системы с заданной матрицей передаточной функции) к линейной системе уравнений с постоянными коэффициентами. Представленный алгоритм позволяет найти решение задачи в случае правильных матриц операторных передаточных функций с производьным числом столбцов. В случае, когда это возможно, естественным следствием преобразований является параметризация полученного семейства решений.