Control
and Cybernetics
VOL. 10 (1981) No. 3–4

Some properties of multivariable control systems with diagonal controller

by

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The paper considers consequences of theorems given in [2, 3, 4]. Certain degenerate cases and the range of the synthesis method of diagonal controller described in [2, 4] are studied. The possibilities of adding nondiagonal elements to diagonal controller are assessed.

Notations:

$\det(G)_i$ ... minor of the matrix G created by deleting i-th row and j-th column.

$$m(f)=\inf_{\omega\in[0,\,\omega_a]} |f(j\omega)| \qquad \text{where } j^2=-1.$$

$$M(f)=\sup_{\omega\in[0,\,\omega_a]} |f(j\omega)|$$

$$\|q(j\omega)\|=M(q_1)+\ldots+M(q_n) \qquad \text{for} \qquad q(j\omega)=(q_1(j\omega),\ \ldots,\ q_m(j\omega))$$

$$f(q)=o(q^p) \qquad \text{iff} \qquad p=\min\left\{k\text{-integer} \mid \lim_{\|q\|\to 0} \frac{|f(q)|}{\|q\|^k}=0\right\}$$

1. Introduction

The paper constitutes a continuation of the article [4], which regarded the synthesis of multivariable linear control systems in the frequency domain. The disturbance damping over the given frequency range $[0, \omega_a]$ was accepted there as a performance index and the requirements were formulated in a quantitative way. The sequential method of diagonal controller design was proposed, which warrants that the closed-loop control system will be stable and will satisfy specified demands. In connection with this the following questions could be stated:

— Can the strong disturbance attenuation be achieved for all plants?

— If we considere plants, for which strong disturbance damping can be achieved, does the procedure described in [4, 2] apply to all of them?
— Will the increased number of "degrees of freedom", that is using the same method for the design of the controller containing more nonzero elements than the diagonal one, improve characteristics of the system?

In the presented paper we shall try answer these questions. At the begining, in order to prepare foundations, we shall remaind some results from [2] (part 2) and describe the generalized on the case of full controller sequential return difference method (part 3). Than we shall discuss the first two questions (part 4) and the last one (part 5).

2. Basic relationships

Let us consider the standard control system with feedback loops and disturbances acting additively on outputs. Assume, that the plant is described by the transfer-function matrix

$$G(s)=[g_{ij}(s)]_{i,j=1,\dots,m} \tag{1}$$

The antidisturbance performance of the system is represented by the matrix

$$Q(s)=[q_{ij}(s)]_{i,j=1,\dots,m}=(I+G(s)\,R(s))^{-1} \tag{2}$$

where $R(s)=\operatorname{diag}(r_{11},\dots,r_{mm})$ refers to the controller.

From now on we shall omit the argument $s=j\omega$ if it does not lead to misunderstanding.

In [2] the notion of main loops was introduced, which are created by the controller and diagonal elements of matrix G when interactions are cancelled ($g_{ij}=0$ for $i\neq j$). For these loops the vector-valued function

$$q=(q_1, q_2, \dots, q_m), \text{ where } q_i=1/(1+g_{ii}\,r_{ii}) \tag{3}$$

of autonomous dampings was built.

In the original formulation of the synthesis problem [2, 4], the diagonal elements of matrix Q, or dampings achieved in main loops after taking into account interactions, play the decisive role. The vector built of them we shall call diagonal dampings and denote with

$$q_d=(q_{11}, q_{22}, \dots, q_{mm}) \tag{4}$$

For both of these vectors we define norms $\|q\|$ and $\|q_d\|$ (see Notations).

In [23] the fundamental for subsequent considerations theorem was proved.

THEOREM 1. *Let*

$$A=\frac{\det G}{\prod\limits_{k=1}^{m} g_{kk}};\qquad A_i=\frac{\det (G)_i^i}{\prod\limits_{\substack{k=1\\k\neq i}}^{m} g_{kk}} \tag{5}$$

$$B=\frac{\det(I+GR)}{\prod_{k=1}^{m}(1+g_{kk}\,r_{kk})} \tag{6}$$

Then for the given control system the following relationships hold:

$$q_{ii}=\frac{A_i}{B}\,q_i+o(q), \qquad i=1, \ldots, m \tag{7}$$

$$q_{ij}=(-1)^{i+j}\frac{\det(G)^i_j}{\det(G)^i_i}\,q_{ii}+o(q), \qquad i\neq j \tag{8}$$

$$B=A\left(1-\sum_{i=1}^{m}q_i\right)+\sum_{i=1}^{m}A_i\,q_i+o(q) \tag{9}$$

On the basis of this theorem we are able to formulate the sufficient conditions of satisfying inequalities

$$M(q_{ii})\leqslant\delta_i, \qquad i=1, \ldots, m. \tag{10}$$

which, with the requirement that the system should be stable constitute our control problem.

Let us introduce for autonomous dampings denotions

$$x_k=M(q_k), \qquad k=1, \ldots, m. \tag{11}$$

Then the sufficient conditions of satisfying (10) form the set of inequalities

$$\frac{M(A_i)\,x_i}{m(A)\,(1-\sum_{k=1}^{m}x_k)-\sum_{k=1}^{m}M(A_k)\,x_k}\leqslant\delta_i \tag{12}$$

$$x_i>0 \qquad i=1, \ldots, m$$

$$m(A)\left(1-\sum_{k=1}^{m}x_k\right)-\sum_{k=1}^{m}M(A_k)\,x_k>0. \tag{13}$$

3. Return differences

The return differences method consists in consecutive joining of single controllers into feedback loops in a way which ensures stability of the final closed-loop system. The procedure described here, based on [3], is slightly more general than in [5].

i) Let R_0 be a zero matrix and $R(\alpha, \beta)$ the matrix which has the only nonzero entry $r_{\alpha\beta}$ on the place (α, β) $\alpha, \beta\leqslant m$. $r_{\alpha\beta}$ is the transfer function of the appropriate controller.

Let us arrange all the controllers we want to add into sequence

$$\{r_{i_1 j_1}, r_{i_2 j_2}, \dots, r_{i_s j_s}\} \qquad s \leqslant m^2 \tag{14}$$

The final matrix R can be expressed as

$$R = \sum_{k=1}^{s} R(i_k, j_k) \tag{15}$$

We shall define also the sequence

$$\{R_0, R_1, \dots, R_s\} \tag{16}$$

where

$$R_k = R_{k-1} + R(i_k, j_k), \qquad k = 1, \dots, s \tag{17}$$

To each R_k corresponds matrix T_k, which will play the crucial role in the algorithm,

$$T_k = [t_k(i, j)]_{i, j=1, \dots m} = (I + GR_k)^{-1} G \tag{18}$$

ii) Now we shall calculate the transfer function being seen by the controller $R_{i_k j_k}$ under assumption, that the remaining $k-1$ controllers $R_{i_1 j_1}, \dots, R_{i_{k-1} j_{k-1}}$ have been already joined to the system.

It turns out [3], that it is simply the proper element of the matrix T_{k-1}

$$t_{k-1}(j_k, i_k) \tag{19}$$

Let us form the cone-dimensional return difference

$$f_k = 1 + r_{i_k j_k} t_{k-1}(j_k, i_k), \qquad k = 1, \dots, s. \tag{20}$$

As it can be easily proved [5, 3], if no one of the return differences has zeros in the right half-plane, then the whole system is stable.

The selection of the controller $R_{i_k j_k}$ for transfer function $t_{k-1}(j_k, i_k)$, by means of standard methods, we shall call synthesis of the k-th equivalent loop.

If we are designing the system with diagonal controller then the sequence (14) takes simple form:

$$\{r_{i_k j_k}\}_{k=1}^{s} = \{r_{11}, \dots, r_{mm}\}. \tag{21}$$

To fascilitate calculations of succesive matrices T_k the reccursive formulas can be worked out [5, 3]:

$$t_k = t_{k-1} - \frac{r_{i_k j_k}}{f_k} t_{k-1}(\cdot, i_k)\, t_{k-1}(j_k, \cdot) \tag{22}$$

where

$t_{k-1}(\cdot, i_k)$ — i_k-th column of the matrix T_{k-1}

$t_{k-1}(j_k, \cdot)$ — j_k-th row of the matrix T_{k-1}.

4. The range of the design procedure

In the above considerations the autonomous dampings occupy central place. They are intermediate link between requirements imposed on the system (from which we calculate $x_1, \ldots, x_m$) and scalar dampings in equivalent loops (which are calculated from $x_1, \ldots, x_m$. Details see [4]). By damping in k-th equivalent loop we mean $M(1/f_k)$. Hence the relations between vector of autonomous dampings and vector of diagonal dampings occuring in the problem formulation (10) are of great importance.

At the beginning let us notice that sufficient conditions (12), (13) can be satisfied only when $m(A)>0$ and $M(A_i)<\infty$ $i=1, \ldots, m$. Generally speaking functions A and A_i have decisive influence on the properties of the control system. It can be expressed in the following theorem:

THEOREM 2.

i) *If* $m(A)>0$ and $M(A_i)<\infty$, $i=1, \ldots, m$ then

$$(\|q\| \to 0) \Rightarrow (\|q_d\| \to 0) \tag{23}$$

ii) If $m(A_i)>0$ for $i=1, \ldots, m$ and $M(A)<\infty$, $M(A_i)<\infty$ for $i=1, \ldots, m$ then

$$(\|q_d\| \to 0) \Rightarrow (\|q\| \to 0) \tag{24}$$

iii) If $m(A)>0$, $M(A)<\infty$ and $m(A_i)>0$, $M(A_i)<\infty$, $i=1, \ldots, m$ then

$$(\|q_d\| \to 0) \Leftrightarrow (\|q\| \to 0) \tag{25}$$

Proof. Based on theorem 1, see [3].

Point iii) of the theorem 2 specifies conditions under which the regulation of main loops is equivalent to the regulation of the whole system. Let us explore, what will happen if some of these constraints are violated. Using once again theorem 1 in [3] it was proved:

THEOREM 3.

i) If $m(A)=0$ *then there exists frequency* $\omega_0 \in [0, \omega_a]$ *such that*

$$q_{11}(j\omega_0)+\ldots+q_{mm}(j\omega_0)=1 \tag{26}$$

exactly to $o(q)$.

ii) *If some of the quantities* $M(A_i)$ *do not exist, but* $M(A)$ *exists, then for certain frequency* $\omega_1 \in [0, \omega_a]$

$$\lim_{\omega \to \omega_1} (q_{11}(j\omega)+\ldots+q_{mm}(j\omega))=1 \tag{27}$$

exactly to $o(q)$.

This theorem describes the class of plants, for which no one diagonal controller can make it possible to obtain satisfying disturbance dampings. At the same time

Recalling the formula for matrix inversion, we obtain

$$t_m(k,\alpha)=\begin{cases} o(q) & k\neq\alpha \\ \dfrac{\det G}{\det (G)_k^k}\, q_{kk}+o(q), & k=\alpha \end{cases} \tag{40}$$

From the definition of $o(q)$ it results that

$$o(q^k)\, o(q^p)=o(q^{k+p+1}) \tag{41}$$

because we have to do only with polynomials of q_i, $i=1, \ldots, m$. This allows us to obtain from (33) final approximations:

$$\tilde{q}_{ij}=o(q^0), \qquad i\neq\alpha, \beta; \qquad j=1, \ldots, m \tag{42}$$

$$\tilde{q}_{\alpha j}=o(q^{-1}), \qquad j=1, \ldots, m \tag{43}$$

As it can be seen, we have lost control over the impact of disturbances on α-th output (because $o(q^{-1})$ is the function practically independent on $\|q\|$). Let us check now what is the cost of deeper damping in the β-th row (34). From (40) $t_m(\beta, \alpha)=o(q)$ and this implies that in order to satisfy inequality (37) the controller $r_{\alpha\beta}$ must have immense gain (for $\|q\|\approx 0.1$, $|r_{\alpha\beta}|\approx 1000$). The regularity can be demonstrate on simple example. Let

$$G=\begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 1 & 3 & 2 \end{bmatrix}$$

Applying $R_3=\mathrm{diag}(10, 10, 10)$ we obtain

$$Q=10^{-1}\begin{bmatrix} 0.5 & 0.06 & -0\,2 \\ -0.4 & 0.25 & 0.3 \\ 0.3 & -0\,4 & 0.2 \end{bmatrix}$$

After adding $r_{12}=k$

$$\tilde{Q}=\frac{1}{52k+14261}\begin{bmatrix} 52k+741 & -32k+90 & -40k+310 \\ -520 & 341 & 410 \\ 390 & k-530 & 2k+231 \end{bmatrix}$$

To have any influence, k must be great, $k\gg 300$.
If it is so, than

$$\lim_{k\to\infty}\tilde{Q}=10^{-1}\begin{bmatrix} 1 & -6 & -8 \\ 0 & 0 & 0 \\ 0 & 0.2 & 0.4 \end{bmatrix}$$

The result is in accordance with our conclusions. From the above it follows that it does not pay to introduce into the system more than m controllers. It is even possible to formulate statement, that the only reasonable control system configuration constitutes the final controller of the form

$$R = KR_d \tag{44}$$

where K — transfer function matrix of the compensator selected in order to obtain compensated plant matrix $G_k = GK$ with suitable from the point of view of regulation properties. R_d is a transfer function matrix of the diagonal controller. It elements depend immediately on requirements imposed on the system and may be designed with [2, 4] method.

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Received, November 1980.

Pewne własności wielowymiarowych układów regulacji z regulatorem diagonalnym.

W pracy rozważane są konsekwencje twierdzeń podanych w [2, 3, 4]. Badane są pewne przypadki zdegenerowane i zakres stosowalności metody syntezy regulatora diagonalnego podane w [2, 4]. Rozważane są możliwości dodania elementów pozadiagonalnych do regulatora diagonalnego,

Некоторые свойства многомерных систем регулирования с диагональным регулятором

В работе обсуждаются следствия теорем, представленных в [2], [3], [4]. Изучаются некоторые вырожденные случаи и область применения представленного в [2], [4] метода синтеза диагонального регулятора. Рассматриваются возможности дополнения диагонального регулятора недиагональными элементами.