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Algebraic $\sum_k$—assignment problems

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The assignment problems are formulated in an abstract ordered semiring. Also a polynomial algorithm solving these problems is presented.

1. Introduction.

In recent years much interest was involved in algebraization of mathematical programming problems. Their formulation in the terms of abstract algebraic systems enables investigations of manifold connections between varions classical optimization problems and a deeper insight into the algorithms.

This work is devoted to the algebraic Σ_k — assignment problems. In [4] the classical Σ_k — assignment problem for n-men and n-jobs is formulated. For $k=n$ this problem nccurs to be linear and for $k=1$ to be bottleneck assignment problem. On the other hand there are many works (e.g. [1], [2], [3], [5], [7], [8]) in which the linear and the bottleneck assignment problems are treated as the algebraic optimization problem in a suitable ordered semigroup.

In [6] the following question was posed: what kind of an algebraic structure should be introduced in order to put the Σ_k — objectives and the algebraic objective into one common framework?

In section 2 the basic definitions, properties and examples of the considered algebraic systems are presented.

In section 3 we prove the theorem on which the proposed algorithm is based and the algorithm itself is presented. Also a simple numerical example is given.

2. Assignment problems

The classical formulation of the linear assignment problem is as follows: let n jobs and n men be given and let $A=[a_{ij}]$ $i, j=1, \ldots, n$ be the cost matrix. a_{ij} stands here for the cost of performing i-th job by j-th man. Let Π be the set of all permutations of the set $I=\{1, \ldots, n\}$. Find a permutation $\pi_0 \in \Pi$ which minimizes $\sum_{i=1}^{n} a_{i\pi(i)}$ over all $\pi \in \Pi$

The bottleneck assignment problem consists in finding such a permutation $\pi_0 \in \Pi$ which minimizes $\max_{i \in I} a_{i\pi(i)}$ over all $\pi \in \Pi$.

The Σ_k — assignment problem, $1 \leqslant k \leqslant n$, is: find $\pi_0 \in \Pi$ which minimizes over all $\pi \in \Pi$ the sum of k maximal elements among $a_{i\pi(i)}$.

It is easy to observe that for $k=1$ we obtain the bottleneck and for $k=n$ — the linear assignment problem respectively. Both can be treated as a general linear assignment problem in a suitable ordered semigroup.

Let $(G, *, \leqslant)$ be a semigroup with some commutative binary composition $*$ and a linear order relation $\leqslant$. Let $A=[a_{ij}]_{i,\, j=1,\ldots,n}$ be $n \times n$ matrix with the entries $a_{ij} \in G$. The general linear assignment problem in the semigroup G, as it is defined in [2], reads: find a permutation π_0 for which

$$\mathop{*}_{i \in I} a_{i\pi(i)} := a_{1\pi(1)} * a_{2\pi(2)} * \ldots * a_{n\pi(n)} \quad \text{is minimal.} \tag{1}$$

For the semigroup G being the set of real numbers with the arithmetic addition taken as the composition $*$ and the natural order relation we get the linear assignment problem.

If we take as G the set of real numbers with the composition

$$a * b = \max(a, b)$$

and the natural order relation $\leqslant$ — we get bottleneck assignment problem.

Some other optimization problems can be also considered as a minimization of the objective function of the form (1) in a suitable semigroup. Such situation is e.g. for the lexicographic multicriteria assignment problem, p-norm assignment problem and others (cf. [1], [2], [3], [5]).

Unfortunately the Σ_k — assignment problem can not be placed within this bounds. We need a bit more complicated algebraic system. The most natural is an ordered semiring.

DEFINITION 1. *A* set S with two binary operations $\oplus$, $\otimes$ such that:

(i) with respect to each of them S is a commutative semigroup,

(ii) the distributivity law holds:

$$\forall_{a,b,c \in S} \quad a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c)$$

is called a semiring.

If moreover in S is a linear order relation such that:

(iii) $\forall_{a,b,c \in S} \quad a \leqslant b \quad a \oplus c \leqslant b \oplus c$

(iv) $\forall_{a,b,c \in S} \quad a \leqslant b \quad a \otimes c \leqslant b \otimes c$

then $(S, \oplus, \otimes, \leqslant)$ is an ordered semiring.

Axioms (iii) and (iv) secure the compatibility of the order relation with the algebraic operations in S.

REMARK. The set of all positive real numbers with the natural addition and multiplication and natural order relation fulfills all axioms (i)–(iv), however in the set of all real numbers (iv) fails.

One can obtain a broad class of ordered semirings in the following way: let S be an ordered semigroup with the multiplication $\otimes$ and order relation $\leqslant$ fulfilling the compatibility condition:

$a \leqslant b$ then for every $c \in S$ $a \otimes c \leqslant b \otimes c$. Introducing in S the addition $\oplus = =\max$ defined as

$$a \oplus b = \max(a, b)$$

we get on S the structure of an ordered commutative semiring.

The distributivity law holds in $(S, \oplus, \otimes, \leqslant)$:

$$c \otimes (a \oplus b) = (c \otimes a) \oplus (c \otimes b)$$

For the proof of this equality let us assume that $a \leqslant b$, that is $a \oplus b = b$. Then

$$c \otimes (a \oplus b) = c \otimes b \geqslant c \otimes a$$

which implies:

$$(c \otimes a) \oplus (c \otimes b) = c \otimes b = c \otimes (a \oplus b),$$

in the case $b \leqslant a$ the proof is similar

Moreover the addition $\oplus = \max$ is compatible with the order relation $\leqslant$ in S, that is if $a \leqslant b$ then for every $c \in S$ $a \oplus c \leqslant b \oplus c$

REMARK. Quite similarly one can prove that taking min as addition, in S, that is

$$a \oplus b = \min(a, b)$$

one can obtain a structure of an ordered semiring on S.

Now we will give some examples of the ordered semigroups in which the multiplication $\otimes$ is compatible with the order relation.

EXAMPLES. (cf. [8])

1. Let R_+ (Z_+) be the set of nonnegative real (integer) numbers with the usual order relation $\leqslant$ and multiplication defined as

$$a \otimes b = a + b$$

Compatibility condition is naturally valid.

2. The unit interval $\langle 0,1 \rangle$ with the multiplication:

$$a \otimes b = a + b - ab$$

and the usual order relation.

Let us order this sequence in such a way, that:

$$a_{i_1\pi(i_1)} \geqslant a_{i_2\pi(i_2)} \geqslant \ldots \geqslant a_{i_n\pi(i_n)}$$

In order to simplify the indices let us denote the j-th element,

$$j \in I, \quad a_{i_j\pi(i_j)} \quad \text{by} \quad \bar{a}_{j,\pi}.$$

Proposition 2.

If $b=\bar{a}_{k,\pi}$ and $B=[b_{ij}]_{i,\,j=1,\ldots,n}$ where $b_{ij}=a_{ij} \oplus b$ then

$$G_B(\pi)=F_A(\pi) \otimes b^{n-k}$$

Proof.

$$G_B(\pi)=\bigotimes_{i=1}^{n}(a_{i\pi(i)} \oplus b)=\bigotimes_{j=1}^{n}(\bar{a}_{j,\pi} \oplus b)$$

The second equality is valid because of the commutativity of the multiplication $\otimes$.

$$\bar{a}_{j,\pi} \oplus b=\begin{cases}\bar{a}_{j,\pi} & \text{for} \quad j \leqslant k \\ b & \text{for} \quad j > k\end{cases}$$

$$G_B(\pi)=\bar{a}_{1,\pi} \otimes \bar{a}_{2,\pi} \otimes \ldots \otimes \bar{a}_{k,\pi} \otimes b^{n-k}.$$

On the other hand

$$F_A(\pi)=\bigoplus_{\substack{I_k \subset I \\ |I_k|=k}} \bigotimes_{i \in I_k} a_{i\pi(i)}=\bigoplus_{\substack{I_k \subset I \\ |I_k|=k}} \bigotimes_{j \in I_k} \bar{a}_{j,\pi}=\bar{a}_{1,\pi} \otimes \bar{a}_{2\pi} \otimes \ldots \otimes \bar{a}_{k,\pi}$$

The rearrangement of the elements in the second equality is possible because of the commutativity and the associativity of the multiplication $\otimes$.

This gives us the equality we need.

Now let π_0 be an optimal permutation for F_A which means, that:

$$\forall_{\pi \in \Pi} \; F_A(\pi) \geqslant F_A(\pi_0).$$

Let as before $\bar{a}_{k,\pi_0}$ be the k-th element in the ordered sequence determined by π_0.

$$B_0=[a_{ij} \oplus b_0] \text{ where } b_0=\bar{a}_{k,\pi 0}.$$

If π_1 is an optimal permutation for G_{B_0} then:

$$G_{B_0}(\pi_1) \leqslant G_{B_0}(\pi_0)=F_A(\pi_0) \otimes b_0^{n-k} \leqslant F_A(\pi_1) \otimes b_0^{n-k}.$$

On the other hand:

$$G_{B_0}(\pi_1) \geqslant F_A(\pi_1) \otimes b_0^{n-k}.$$

This gives:

$$G_{B_0}(\pi_1)=F_A(\pi_1) \otimes b_0^{n-k}.$$

Observe now, that π_0 is optimal for G_{B_0}:

$$\forall_{\pi \in \Pi} \; G_{B_0}(\pi) \geqslant F_A(\pi) \otimes b_0^{n-k} \geqslant F_A(\pi_0) \otimes b_0^{n-k}=G_{B_0}(\pi_0)$$

so finally we get:

$$F_A(\pi_0) \otimes b_0^{n-k} = F_A(\pi_1) \otimes b_0^{n-k}.$$

Now let us take an ordinal decomposition of the ordered semigroup $(S, \otimes, \leqslant)$ into the indexed family of subsemigroups $(S_\lambda; \lambda \in \Lambda)$ (cf. [3], [8]). Here Λ is the linearly ordered set and S_λ, $\lambda \in \Lambda$, are mutually disjoint, linearly ordered commutative subsemigroups of S!

For $s \in S$ let $\lambda(s)$ denote the index of this unique subsemigroup which contains element s.

The ordinal decomposition has the property: if $s \leqslant s'$ then $\lambda(s) \leqslant \lambda(s')$. So we get $\lambda(F_A(\pi_0)) \geqslant \lambda(b_0)$.

Using the Proposition 4.19 (6) and (1) from [8] we get

$$\lambda(b_0) = \lambda(b_0^{n-k})$$

$$\text{and } F_A(\pi_0) = F_A(\pi_1).$$

This last equality menas that π_1 is optimal for F_A.

So we get the following theorem

THEOREM. *Let π_0 be a permutation which minimizes F_A and b_0 be the k-th element in the sequence $\bar{a}_{1,\pi_0} \geqslant \bar{a}_{2,\pi_0} \geqslant \ldots \geqslant \bar{a}_{n,\pi_0}$. Then a permutation minimizing G_{B_0} is also optimal for F_A.*

REMARK. In the case of the ordinary Σ_k — assignment problem the above theorem gives Dinic result (cf. [4]).

In order to solve the algebraic Σ_k — assignment problem (SAP) it is sufficient to find optimal permutations π_l' for the problems:

(LAP) $$\min_{\pi \in \Pi} G_{B_l}(\pi), \qquad l = 1, \ldots, n^2$$

where $B_l = [a_{ij} \oplus a_l]$ $i, j = 1, \ldots, n$

and a_l, $l = 1, \ldots, n^2$, is an entry of the matrix A. Then one may find the values $F_A(\pi_l')$.

A permutation, for which this value is minimal, is optimal for (SAP).

For the simplicity let us assume that there is an element $e \in S$ such that for all $a \in S$ $a \otimes e = a$.

The problems (LAP) can be solved by the algorithm developed in [5]. This algorithm is of order $0(n^3)$. In the worst case we have to solve n^2 (LAP) problems. So we get $0(n^5)$ estimation for the whole algorithm.

Let $B = [b_{ij}]$ $i, j = 1, \ldots, n$ be a matrix with $b_{ij} \in S$. The algorithm finds a permutation π which minimizes G_B.

Step 1. For $i = 1, \ldots, n$ $\quad u_i := e$
For $j = 1, \ldots, n$ $\quad w_j := e$
$V_j = \min(b_{ij};\ i \in I)$

V_j is a minimal element in the j-th column, say $v_j=b_{f(j)j}$. This defines a function $f: I\to I$. Construct $\psi: I\to I\cup\{0\}$ such that

1. $\psi|_{f(I)}\neq 0$
2. if $\psi(i)=\psi(i')\neq 0$ then $i=i'=f(\psi(i))$.

This implies that if $i\notin f(I)$ then $\psi(i)=0$.

Step 2. Find i_0 such that $\psi(i_0)=0$.

For every $j=1, \ldots, n$ $m_j:=b_{i\ \ j}\otimes w_j$

$q(j):=i_0$

$J:=\emptyset \quad K:=\{i_0\}$.

If there is no i_0 with $\psi(i_0)=0$ then $\psi: I\to I$ is an optimal permutation.

Step 3. For every $j\notin J$ take $d_j=m_j/v_j$ $d_{j_0}:=\min(d_j, j\notin J)$.

Step 4. For $i\in K$ $u_i:=u_i\otimes d_{j_0}$.

For $j\in J$ $w_j:=w_j\otimes d_{j_0}$

If $j_0\notin\psi(I)$ go to Step 6.

Step 5. For every $j\in J\cup\{j_0\}$

$$m_j'=v_j\otimes d_j/d_{j_0}$$

$$m_j''=(b_{f(j_0)j}\otimes w_j/u_{f(j_0)}\otimes v_j)\otimes V_j$$

$m_j=\min(m_j', m_j'')$

Define $q(j):=f(_{j0})$ for these j for which $m_j=m_j''$.

$K:=K\cup\{f(j_0)\}$, $J:=J\cup\{j_0\}$ go to Step 3.

Step 6. In the bipartite graph $G=(K, J\cup\{j_0\}; E)$ where the set of edges is

$$E=\{(q(j), j):\quad j\in J\cup\{j_0\}\cup\{(j, f(j)); j\in J\}$$

construct the path $(i_0, j', i_1, j_1, \ldots, i_s, j_0)$ from i_0 to j_0. This path defines the new values of f and ψ:

$$\psi(i_0)=j' \quad \psi(i_l)=j_l \quad \psi(i_s)=j_0$$

$$f(j)=i_0 \quad f(j_l)=i_l \quad f(j_0)=i_s$$

$$\text{for } l=1, \ldots, \quad s=1$$

go to step 2.

Example

To illustrate how this algorithm works let us solve the linear assignment problem with the matrix G_{B_1}.

$$G_{B_1}=\begin{bmatrix} 1 & 5 & 3 & 0 & 1 \\ 2 & 0 & 1 & 3 & 1 \\ 4 & 3 & 2 & 1 & 2 \\ 3 & 0 & 4 & 2 & 1 \\ 1 & 2 & 1 & 5 & 0 \end{bmatrix}$$

Step 1. $u_1=u_2=u_3=u_4=u_5=0$

$w_1=w_2=w_3=w_4=w_5=0$

$v_1=b_{11}=1,\ v_2=b_{22}=0,\ v_3=b_{23}=0,\ v_4=b_{14}=0,\ v_5=b_{55}=0$

$f(1)=1 \quad f(2)=2 \quad f(3)=2 \quad f(4)=1 \quad f(5)=5$

$\psi(1)=4 \quad \psi(2)=2 \quad \psi(3)=0 \quad \psi(4)=0 \quad \psi(5)=5$

Step 2. $i_0=3 \quad \psi(3)=0$

$m_1=4,\ m_2=3,\ m_3=2,\ m_4=1,\ m_5=2$

$q(j)=3 \quad j=1, \ldots, 5$

$K=\{3\} \quad J=\emptyset$

Step 3.

$d_1=3,\ d_2=3,\ d_3=1,\ d_4=1,\ d_5=2$

$d_4=\min_j (d_j)=1 \quad j_0=4$

Step 4.

$u_3=1 \quad j_0=4 \in \psi(I)$

Step 5.

$m_1'=3,\ m_2'=2,\ m_3'=1,\ m_5'=1$

$m_1''=1,\ m_2''=5,\ m_3''=3,\ m_5''=1$

$m_1=1,\ m_2=2,\ m_3=1,\ m_5=1$

for $j=1 \ m_j'>m_j''$

$q(1)=f(4)=1$

$I=\{3\}\cup\{1\} \quad J=\emptyset\cup\{4\}$

Step 3.

$d_1=0, \quad d_2=2, \quad d_3=0, \quad d_5=1$

$d_1=\min (d_j)=0 \quad j_0=1$

Step 4.

$u_3=2, \quad u_1=1, \quad w_4=1$

$j_0=1 \notin \psi(I)$

Step 6.

Bipartite directed graph $G=\{\{3, 1\}, \{4\}\cup\{1\}, E\}$ $E=\{(3, 4), (1, 1)\}\cup\{(4, 1)\}$

$$\begin{tikzcd} 3 \arrow[r] & 4 \arrow[dl] \\ 1 & 1 \arrow[l] \end{tikzcd}$$

The unique path from $i_0=3$ to $j_0=1$ is $(3, 4, 4, 1, 1, 1)$.

The new function f is defined as

$f(1)=1,\ f(2)=2,\ f(3)=2,\ f(4)=3,\ f(5)=5$

The new ψ is:

$\psi(1)=1,\ \psi(2)=2,\ \psi(3)=4,\ \psi(4)=0,\ \psi(5)=5$

Step 2.

$\psi(4)=0 \quad i_0=4$

$m_1=3, \quad m_2=0, \quad m_3=4, \quad m_4=3, \quad m_5=1$

$q(j)=4 \quad j=1, \ldots, 5$

$K=\{4\}, \quad J=\emptyset$

Step 3.

$d_1=2,\ d_2=0,\ d_3=3,\ d_4=3,\ d_5=1$

$d_2=\min\limits_j (d_j)=0 \quad j_0=2$

Step 4.

$u_4=0 \quad 2 \in \psi(I)$

Step 5.

$m_1'=3,\ m_3'=4,\ m_4'=3,\ m_5'=1$

$m_1''=2,\ m_3''=1,\ m_4''=4,\ m_5''=1$

$m_1=2,\ m_3=1,\ m_4=3,\ m_5=1$

For $j=1, 3 \quad m_j'>m_j''$

$q(1)=f(2)=2, \quad q(3)=f(2)=2$

$K=\{4\}\cup\{2\} \quad J=\{2\}$

Step 3.

$d_1=1,\ d_3=0,\ d_4=3,\ d_5=1$

$d_3=\min\limits_j (d_j)=0 \quad j_0=3$

Step 4.

$u_4=0,\ w_2=0 \quad 3 \notin \psi(I)$

Step 6.

Bipartite graph $G=(\{4, 2\}, \{2\}\cup\{3\}, E)$

$E=\{(4, 2), (2, 3)\}\cup\{(2, 2)\}$

$$\begin{tikzcd} 4 \arrow[r] & 2 \arrow[dl] \\ 2 & 3 \arrow[l] \end{tikzcd}$$

The unique path from $i_0=4$ to $k=3$ is (4, 2, 2, 2, 2, 3).
The new ψ is

$\psi(1)=1,\ \psi(2)=3,\ \psi(3)=4,\ \psi(4)=2,\ \psi(5)=5$

Step 2.

There is no i_0 with $\psi(i_0)=0$.
So ψ is the optimal permutation.
The optimal value is

$$G_{B_1}(\psi)=b_{11}+b_{23}+b_{34}+b_{42}+b_{55}=3$$

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Algebraiczne $\sum_k$ – zagadnienie przydziału.

Podane jest sformułowanie zadań przydziału jako algebraicznych zadań przydziału w abstrakcyjnym półpierścieniu.
Zaproponowany jest również algorytm rozwiązujący te zadania.

Алгебраические $\sum_k$ – задачи о назначении

Задачи о назначении формулируются в виде алгебраических задач в абстрактном полукольце. Предложен алгоритм решения этих задач.