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Constraint modification methods for linear integer programming problems

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Some constraint modifications for linear integer programming problems are considered. These modifications are aimed at maximal feasible solution set reductions of associated linear programming problems (e.g. of original problems with integrality constraints relaxed) while preserving the set of integer solutions. The minimal description problem is formulated and the method for determining approximate solutions for this problem by separate constraint modification is presented. Existing methods for separate constraint modifications are surveyed and a new method is introduced. Properties of these methods and their usefuluess in constructions of efficient linear integer programming algorithms are investigated.

1. Introduction

In linear integer programming (LIP) problems. feasible solution sets are described by linear constraints (equalities, inequalities) and nonnegativity of solutions. By neglecting integrality conditions LIP problems transform to associated linear programming (ALP) problems. Without loss of generality it can be assumed that linear constraints are of inequality type only. Geometrically this inequalities correspond to hyperplanes which describe a polyhedral set of ALP feasible solutions.

Many of solution methods for LIP problems exploit ALP's relaxations to get different types of easy computable bounds. It is intuitively clear that the closet an ALP problem aproximates the corresponding LIP problem the stronger are bounds. Thus it is of a great interest to construct metods which would modify descriptions of ALP polyhedral solutions sets in such a way that resulting sett would be as small as possible or equivalenty, that resultng descriptions would be m i n i m a l.

The first time this intuition was confirmed by numerical experiments described in [1, 3, 15]. In all three cases only hand computations for problem reformulation were applied. Thus, it was not possible to assess savings (if any) in total optimization time (including time spent on modifications) eventually caused by them.

In this paper the problem of minimal description is formally introduced. It turns out however that solving solving this problem is a task at least as hard as solving LIP problems. Therefore we seek solutions of the relaxed minimal description problem. These solutions can be obtained by independent modifications of separate constraints. In subsequent paragraphs two existing methods for separate constraint modifications are presented and a new method is proposed. The particular attention has been devoted to the constraint rotation method which is the most efficient one.

The constraint modification methods are particularly useful in cutting planes algorithms because a cutting plane can be always treated as an ordinary constraint, thus it can be modified during cutting plane algorithm computations.

2. The minimal description problem

We consider the following LIP problem

$$\text{maxime} \sum_{j\in N} c_j x_j$$

$$x \in D(A, a_0) = \left\{ x \;\middle|\; \begin{array}{ll} \sum_{j\in N} a_{ij} x_j \leqslant a_{i0}, & i \in \{1, 2, ..., m\} = M \\ x_j \geqslant 0, & j \in \{1, 2, ..., n\} = N \\ x_j \in Z, & j \in N \end{array} \right\} \tag{1}$$

where

$$c_j \in Z,\ j \in N$$
$$a_{ij} \in Z,\ i \in M,\ j \in N$$
$$a_{i0} \in Z,\ i \in M$$

Z—set of integer numbers.
It is well known that every LIP problem with rational coefficients can be transformed to the LIP problem (1).
We assume that $D(A, a_0)$ is bounded and nonempty.

The following sets correspond with (1).

$$D_c(A, a_0) = \left\{ x \;\middle|\; \begin{array}{ll} \sum_{j\in N} a_{ij} x_k \leqslant a_{i0}, & i \in M \\ x_j \geqslant 0, & j \in N \end{array} \right\}$$

$$D^i(a^i, a_{i0}) = \left\{ x \;\middle|\; \begin{array}{ll} \sum_{j\in N} a_{ij} x_j \leqslant a_{i0} & \\ x_j \geqslant 0, & j \in N \\ x_j \in Z, & j \in N \end{array} \right\}$$

$$D^i_c(a^i, a_{i0}) = \left\{ x \;\middle|\; \begin{array}{ll} \sum_{j\in N} a_{ij} x_j \leqslant a_{i0} & \\ x_j \geqslant 0, & j \in N \end{array} \right\}$$

For the case $M=m=1$ we write $D(a, a_0)$ and $D\ (a, a_0)$ omitting constraint indices. Let A^*, $\hat{A}$, a_0^*, $\hat{a}_0$, where a_{ij}^*, $\hat{a}_{kj}$ a_{i0}, $\hat{a}_{k0} \in Z$, $i \in \{1, 2, ..., m'\}$, $j \in N$, $k \in \{1, 2, ..., m''\}$, $m' \geqslant m''$ be à priori unknown matrices and vectors (A, a_0 are given).

The minimal description problem (Problem 1)

For given m' find A^*, a_0^* satisfying the condition

$$D(A^*, a_0^*)=D(A, a_0)$$

such that*)

i) $D_c(A^*, a_0^*) \subset D_c(A, a_0)$

ii) for each $\hat{A}$, $\hat{a}_0$ others than A^*, a_0^* satisfying $m' \geqslant m''$ and $D(\hat{A}, \hat{a}_0)=D(A^*, a_0^*)$ the relation $D_c(\hat{A}, \hat{a}_0) \subsetneq D_c(A^*, a_0^*)$ does not hold.

The feasible solution set description for the LIP problem (1) given by A^*, a_0^* is called the minimal description and depends on m'. When m' is sufficiently large the solution of the minimal description problem is the hull of feasible points.

The number m_c of convex hull inequalities is usually large when compared with the number of inequalities in an original LIP problem. Moreover, rules for convex hull constructing exists for a few particular cases of LIP problems only. Thus there is an interest in seeking solutions for the minimal description problem when $m' < m_c$. It is well known that any increase of constraint number is, from computational point of view, very unpleasant. To avoid this we consider in what ollows only the $m=m'=m''$ case.

This assumption excludes the "most tight" feasible solution set descriptions for LIP problems but it keeps number of inequalities unchanged. Taking this into account we can formulate the following lemma.

LEMMA 1. *If*

$$\forall_{i} \ \exists_{a^{*i}, a_{i0}^*} \ D^i(a^{*i}, a_{i0}^*)=D^i(a^i, a_{i0}) \wedge D_c^i(a^{*i}, a_{i0}^*) \subset D_c^i(a^i, a_{i0})$$

then

$$D(A^*, a_0^*)=D(A, a_0) \wedge D_c(A^*, a_0^*) \subset D_c(A, a_0)$$

Proof. The straightforward proof is based on the following relation $(A \subset B) \cap \cap (C \subset D) \Rightarrow (A \cap C) \subset (B \cap D)$ where A, B, C, D are any sets and the fact that $D(A, a_0)= = \bigcap_{i=1}^{m} D^i(a^i, a_{i0})$ and $D_c(A, a_0)=\bigcap_{i=1}^{m} D_c^i(a^i, a_{i0})$ Q.E.D. ■

Basing on the Lemma 1 we can find solutions for a relaxed minimal description problem (condition ii) relaxed) by modifying separate constraints independently.

*) In this paper we use the following notation: $\subset$ — set inelvsion, $\not\subset$ — negation of set inclusion, $\subsetneq$ — proper inclusion.

This contrasts favorably with the fact the exact solutions require (implicit at least) determination of feasible solution sets for LIP problems. Thus exact solutions of the minimal description problem can not improve efficiency of any LIP problems solution method.

For separate constraint modifications we formulate now one-dimensional equivalent of Problem 1.

The boundedness assumption implies that any inequality $\sum_{j\in N} \tilde{a}_j x_j \leqslant \tilde{a}_0$ can be reduced to $\sum_{j\in N} a_j x_j \leqslant a_0$, where $a_j, a_0 \in Z_+$ (Z_+—nonnegative integers) and $a_j \leqslant a_0$, $j \in N$. The nonnegativity of coefficients we achive by the following transformation

$$\tilde{x}_j = U_j - x_j \quad \text{when } \tilde{a}_j < 0,$$
$$\tilde{x}_j = x_j \quad \text{otherwise,}$$

where U_j is an upper bound for x_j variable. The variables for which $a_j > a_0$ can be eliminated (fixed to value 0). To simplify the notation we will assume in the sequel that all inequalities under consideration have been reduced as above.

The one-dimensional minimal description problem (Problem 2)

Find a^*, a_0^* satisfying the condition

$$D(a^*, a_0^*) = D(a, a_0)$$

such that

i) $D_c(a^*, a_0^*) \subset D_c(a, a_0)$

ii) for each $\hat{a}$, $\hat{a}_0$ others then a^*, a_0^* and satisfying

$$D(\hat{a}, \hat{a}_0) = D(a^*, a_0^*)$$

the relation

$$D_c(\hat{a}, \hat{a}_0) \subset D_c(a^*, a_0^*)$$

does not hold.

The feasible solution set description for the LIP problem (1) ($|M|=1$ case) given by a^*, a_0^* is called the minimal description.

The following three paragraphs present methods of separate constraint modifications.

3. The parallel shifting method

This method introduced by Salkin and Breining [13] is based on the well known theorem.

THEOREM 1 ([13]). *A hyperplane with an integer coefficient equation*

$$\sum_{j\in N} a_j x_j = a_0$$

containes integer points iff g.c.d. $(a_1, a_2, ..., a_n)$ divides a_0. If this hyperplane containes one integer point it contains infinite number of them.
Given an inequality

$$\sum_{j\in N} a_j x_j \leqslant a_0$$

then using theorem 1 we can check whether the corresponding hyperplane

$$\sum_{j\in N} a_j x_j = a_0$$

contains integer points. If not, the right hand side of the inequality can be decreased by one without eliminating any integer point from the half-space described by it.

This procedure can be repeated as many times as possible shifting each time the hyperplane towards integer points until it touches them. Such an approach has been alrelady suggested for cutting planes in the method of integer forms [13] but it can be applied to problem constraints in the same way. For this method

$$D(a, a_0^*) = D(a, a_0)$$

and

$$D_c(a, a_0^*) \subset D_c(a, a_0).$$

The above relations are valid for $a_0^* \leqslant a_0$. For $a_0^* < a_0$ the last relation takes form:

$$D_c(a, a_0^*) \subsetneq D_c(a, a_0).$$

This in turn guarantees that if for the case $m > 1$ $\underset{i}{\exists}$ $a_{i0}^* < a_{i0}$ then

$$D_c(A, a_0^*) \subsetneq D_c(A, a_0).$$

The question is how often changes of right hand side coefficients occur in practice. The answer is given by the following theorem.

THEOREM 2 ([9]). *If u and v are randomly generated numbers then the probability of g.c.d. $(u, v) = 1$ equals $(6/\pi^2)$ 0,61.*

The fact that g.c.d. $(a_i, a_j) = 1$ implies g.c.d. $(a_1, a_2, ..., a_n) = 1$, so g.c.d. $(a_1, a_2, ...$ $..., a_n)$ divides a_0. Therefore parallel shifting method might not change right hand side coefficients very often. On the other hand there exists efficient algorithms for finding g.c.d. of n integers. They are based on the following relations

$$\text{g.c.d. } (a_1, a_2, a_3) = \text{g.c.d. } (\textit{g.c.d. } (a_1, a_2), a_3)$$
$$\vdots$$
$$\text{g.c.d. } (a_1, a_2, ..., a_n) = \text{g.c.d. } (\text{g.c.d. } (a_1, a_2, ..., a_{n-1}), a_n)$$

The g.c.d. of two numbers can be found by *the Euclidean algorithm* [10].

THEOREM 3. ([10]). *The number of iterations of the Euclidean algorithm for two integers is never greater than five times the number of digits in the smaller number.*

Let a_1 be the smallest number among n positive integer $a_1, \ldots, a_n$. Then, when the above scheme of computing the g.c.d. of n numbers is used, the following theorem holds.

THEOREM 4. ([2]). *The iteration number of the Euclidean algorithm for n integers is never greater than n-2 plus five times the number of digits in the smallest number.*

4. A new method for separate constraint modifications

In the parallel shifting method right hand side coefficients can be decreased only when the corresponding hyperplanes do not contain integer points. Let us note that because of the LIP problem formulation (1) we are interested in those integer points which are in R^n_+ (the nonnegative orthant of R^n). This gives rise to the following method for constraint modifications.

Find the smallest value $k=k^*$, $k=0, 1, \ldots, a_0-1$, such that a hyperplane $\sum_{j\in N} a_j x_j = a_0 - k$ contains at least one integer point in R^n_+.

An algorithm for finding k^* is based on results from [11]. By so called generalized Euclidean procedure we find a general solution of an (diophantine) equation $\sum_{j\in N} a_j x_j = a_0 - k$ in the following form $x = x^*(k) + Fy$, where $x^*(k)$ is a particular solution, F—a fundamental (integer) $nx(n-1)$ matrix, y—any integer vector. The condition $x \geqslant 0$ implies $x^*(k) + Fy \geqslant 0$. Then the Fourier-Motzkin elimination is applied to check whether there exists a vector y such that the set of inequalities is satisfied. The Fourier-Mortzkin method must be slightly modified for only integer values of y_i's are acceptable. Then in the substitution phase of Fourier-Motzkin method some values of y_i's may lead to inconsistency and to avoid this the simple branch and bound procedure must be applied. For determination of k^* it sufficies to apply the Fourier-Motzkin elimination only once. This results from the folloving theorems.

THEOREM 5 ([11]) *Let k_0, k_1 $(k_1 > k_0)$ be the two smallest nonnegative integer values of k such that an equation $\sum_{j\in N} a_j x_j = a_0 - k$ has integer solutions. Then this equation has integer solutions only for $k \in K = \{k | k = k_0 + tm_0$, t a nonnegative integer$\}$, where $m_0 = k_1 - k_0$.*

THEOREM 6 ([11]) *Let k_0, k_1 $(k_1 > k_0)$ be the two smallest nonnegative integer values of k for which an equation $\sum_{j\in N} a_j x_j = a_0 - k$ has integer solutions, denoted by $x(k_0)$ and $x(k_1)$, respectively.* Then, there exists an integer solution vector $x(k)$ of the form

$$x(k) = x(k_0) - (k - k_0) f$$

only for those $k = k_0 + tm_0$ where f is a constant integer vector, called the difference vector, $m_0 = k_1 - k_0$ and t is any positive integer.

By the two above theorems there is no need for solving $\sum_{j\in N} a_j x_j = a_0 - k$ for all values of k. After finding k_0 and k_1 the difference vector

$$f = -(x(k_1) - x(k_0))/(k_1 - k_0)$$

and k-dependent particular solutions

$$x(k) = x(k_0) - (k - k_0) f$$

can be constructed. With $x(k_0)$ and f known, the presented method can be reduced to the problem of finding the smallest k for which the set of inequalities

$$x(k_0) - (k - k_0) f + Fy \geqslant 0$$

has integer solutions y.

The analysis of relaticns between $D(a, a_0)$ and $D_c(a, a_0)$ sets before and after constiaint modifications given for the parallel shifting method is also valid for the method just described. The gain, in the later case, are greater possible changes of right hand side coefficients.

The characteristic feature of the Fourier-Motzkin method is a fast grow of number of inequalities in the elimination process. It can be proved however that by triangularizing fundamental matrices of single diophantine equalions ($n \times (n-1)$ size) number of inequalities at each step of the elimination process decreases. It should be noted however that because of the backtrack procedure which at the worst case requires exponential in n number of additions and comparisions, the new method presented here might be much more time consuming that the previous one.

5. The constraint rotation method

Any LIP problem with additional constraints $0 \leqslant x_j \leqslant 1$, $j \in N$, is called a linear binary programming (LBP) problem. Every LIP problem can be reduced to an LBP problem. We confine ourselves for a while to LBP problems and we extent the results oi this paragraph to LIP problems, at the end of it.

As it was mentioned before each constraint of a LBP problem treated separately can be reduced to the form

$$\sum_{j\in N} a_j x_j \leqslant a_0, \qquad x_j = 0 \text{ or } 1,\ j \in N \tag{2}$$

where $0 \leqslant a_j \leqslant a_0$.

If the following condition

$$\underset{r\in N}{\forall} \ \underset{x\in D(a,a_0)}{\exists} \ \sum_{j\in(N/\{r\})} a_j x_j + a_r = a_0 \tag{3}$$

holds for (2) then without eliminating any point from $D(a, a_0)$ no stronger constraint can be obtained (we call a constraint $ax' \leqslant b'$ stronger than $a''x \leqslant b''$ if

$\forall_i a_i' \geqslant a_i''$, $b' \leqslant b''$ and at least one inequality is strict). Constriants satisfying (3) are called the strongest constraints [14].

The constraint rotation method constructs the strongest constraints (Kianfar [7]).

For any $r \in N$ we, can write (2) in the form

$$a_r x_r \leqslant a_0 - \sum_{j \in (N/\{r\})} a_j x_j = a_0 - b_r .$$

The following relation holds

$$\forall_{x \in D(a, a_0), x_r = 1} \quad b_r \leqslant a_0 - a_r$$

Let

$$b_x^r = \max_{\substack{x \in D(a, a_0) \\ x_r = 1}} \sum_{j \in (N/\{r\})} a_j x_j \tag{3a}$$

As a new value of a_r we can take $a_r^* = a_0 - b_r^*$ for (3a) implies that the constraint

$$\sum_{j \in (N/\{r\})} a_j x_j + a_r^* x_r \leqslant a_0$$

does not eliminate any element of $D(a, a_0)$. In the case when $b_r^* < a_0 - a_r$, what implies $a_r^* > a_r$, the hyperplane

$$\sum_{j \in (N/\{r\})} a_j x_j + a_r^* x_r = a_0$$

containes at least one feasible integer point more than the hyperplane $\sum_{j \in N} a_j x_j = a_0$.

The value of b_r^* is computed as the maximal element of the set

$$B_r = \{b_r | b_r = \sum_{j \in J} a_j, \ J \subseteq (N \setminus \{r\}), \ b_r \leqslant a_0 - a_r\}$$

The sets B_r, $r = 1, 2, \ldots, n$ are built by a dynamic programming procedure [7, 14]. If at the step r of this procedure $b_r^* < a_0 - a_r$ then $a_r^* = a_0 - b_r^*$, otherwise $b_r^* = a_0 - a_r$ and $a_r^* = a_r$. In both cases we proceed to step $r+1$. As a result we get a constraint

$$\sum_{j \in N} a_j x_j^* \leqslant a_0, \qquad x_j = 0 \quad \text{or } 1, \ j \in N \tag{4}$$

which has the following properties

1. $D(a, a_0) = D(a^*, a_0)$.
2. $a_j \leqslant a_j^* \leqslant a_0$, $j \in N$.
3. If $\sum_{j \in N} a_j \bar{x}_j = a_0$ and $\bar{x} \in D(a, a_0)$ then $\sum_{j \in N} a_j^* \bar{x}_j = a_0$.
4. Let $K = \{x \in D(a^*, a_0) | \sum_{j \in N} a_j x_j = a_0\}$,

   $K^* = \{x \in D(a^*, a_0) | \sum_{j \in N} a_j^* x_j = a_0\}$.

Then $|K^*|\geq|K|$ and it is proved [14] that $|K^*|\geq 2$.
The property 2 implies

$$D_c(a^*, a_0)\subset D_c(a, a_0)$$

and in the case when $\underset{j}{\exists}\, a_j^* \neq a_j$

$$D_c(a^*, a_0)\subsetneq D_c(a^*, a_0)$$

Thus the constraint rotation method solves the relaxed minimal description Problems 1 and 2 (condition ii/ relaxed).

THEOREM 7. *The constraint rotation method gives the minimal description for Problem* 2.

Proof. See [5].

For the first two constraint modification methods presented here; modified constraints are uniquely determined. For the constraint rotation method the result depends on coefficients ordering. ■

Let us assume that we always start to modify coefficients from the most left one and we proceed to the right. Forther let a s_1 and a s_2 denote the coefficients after constraint rotation with a coefficients ordering s_1 and s_2 respectively.

LEMMA 2. *If for any two coefficients ordering* s_1 *and* s_2

$$\underset{i}{\exists}\; a_i^*(s_1)\neq a_i^*(s_2)$$

then

$$D_c(a^*(s_2), a_0)\nsupseteq D_c(a^*(s_1), a_0)$$

Proof. See [5].

The problem of choosing the most suitable (from computational point of view) coefficients ordering was investigated in [14].

Each change of a coefficient by the constraint rotation method means that a modified equality constraint (a hyperplane) constains one nonnegative integer solution more than the original constraint. The question arises what are the conditions which assume that a modified equality constraint containes $k\leq n$ such points? Let $\{l\}_s$ be any s element subset of N, $|N|=n\geq 2$.

THEOREM 8. *If*

$$\underset{\{l\}_s\in 2^N}{\forall}\; \sum_{j\in\{l\}_s} a_j^* > a_0$$

then the rotated constraint contains at least $k=\left\lceil \frac{n}{s-1} \right\rceil$ *nonnegative integer points, where* $[x]$*—the least integer not less then* x.

Proof. See [5].

Let us note that if a constraint cuts off a part of solution unit hypercube i.e. $\sum_{j \in N} a_j > a_0$, what is always assumed, then $\sum_{j \in N} a_j^* > a_0$. Thus $s = n$ and

$$k = \begin{cases} \left[\frac{2}{1}\right] = 2 & \text{for } n = 2 \\ \left[\frac{n}{n-1}\right] + 1 = 2 & \text{for } n > 2. \end{cases}$$

The Theorem 8 is therefore a generalization of a theorem given in [14] which says that any rotated constraint contains at least two nonnegative integer points.

The worst-case behaviour of an algorithm for rotation of constraints (constructing the strongest constraint) is $0(n^2 a_0)$ additions and comparisions. Methods for improving the average efficiency of the algorithm given in [7] have been proposed [5, 8].

The constraint rotation method can be made applicable for integer (not binary) variables either by an obvious generalization of the strongest constraint construction or by binary expansions of integer variables. It must be stressed however that although the binary expansion of an integer variable is always possible the oposite transformation after coefficients modifications may be not.

In [5] the following problem has been considered: what could be the maximal number of different constraints generated by rotation of a separate constraint with different coefficient orderings

THEOREM 10. *A single constraint with n variables modified by the constraint rotation method can generate maximum R (n) different constraints:*

$$R(1)=1,\ R(2)=2,\ R(3)=4,\ R(4)=4,\ R(3k+a)=3^k a$$

where $k=1, 2, \dots,$ $a \in \{2, 3, 4\}$.

Proof. See [5].

The bound established by the theorem is sharp what can be easily verified by an example (see [5]).

6. Formal relations between constraint modification methods

There exist close connections between the parallel shifting methods and the constraint rotation method. To show this relation let us note that if for a given constraint $\sum_{j \in N} a_j x_j \leqslant a_0$ it is possible to decrease a_0 we can try to increase one of coeffcients instead. The condition $a_j \leqslant a_0$, $j \in N$ (paragraph 5) generates bounds for integer variables where $|x|=|x|-1$

$$x_j \leqslant q_j = \left[\frac{a_0}{a_j}\right].$$

For each $k=1, 2, \ldots, n$ we compute

$$b_k^* = \max_{\substack{x\text{-integer},\ x_k \neq 0 \\ x_j \leqslant q_j,\ j \in N}} \left\{ \sum_{j \in N} a_j x_j \;\middle|\; \sum_{j \in N} a_j x_j \leqslant a_0 \right\}.$$

We can select an index, say index k and if only $q_k \neq 0$ and $\varepsilon_k = \left[\frac{a_0 - b_k^*}{q_k}\right] \neq 0$ increase a_k to $\hat{a}_k = a_k + \varepsilon_k$ without loss of any integer point satisfying the original constraint. It may happen that $\varepsilon_k = 0$ for all $k \in N$ so only the parallel shifting methods are applicable. For PBL problems we get always $\varepsilon_k > 0$ for each $k \in N$ (recall that in the case considered the corresponding equality constraint does not contain any integer point from R_+^n and $q_k = 1$ for all $k \in N$). This procedure repeated at most n times constitutes the constraint rotation method.

The construction just described gives rise to the following conclusions. Suppose that an inequality constraint is given.

Case A. The corresponding equality constraint does not pass through any integer point from R_n^+.
i) $x \in Z_n^+$. When the parallel shifting is possible the rotation may not be possible.
ii) $x \in \{0, 1\}^n$. When the parallel shifting is possible the rotation is also possible.

Case B. The corresponding equality constraint passes through one or more integer points from R_+^n. This does not exclude the possibility that $\varepsilon_k > 0$ for some k.
i) $x \in Z_+^n$. When the parallel shifting is impossible the rotation may be possible.
ii) $x \in \{0, 1\}^n$. When the parallel shifting is impossible the rotation may be possible.
The last conclusion follows also from the fact that the parallel shifting method (the new one) guarantees that after modifications the corresponding equality constraint passes through at least one integer point from R_+^n but the constraints rotation method guarantees at least two such points. Thus the constraint shifted might be rotated. As it was mentioned before for LBP problems each coefficient modification calculated by the constraint rotation method increases number of integer (binary) points from R_+^n the corresponding equality constraint satisfies. For LIP problems number of such points increases only when we increase a selected coefficient, e.g. k-th, by $\varepsilon_k = \frac{a_0 - b_k^*}{q_k}$.

Finally, we come to the conclusion that the constraint rotation method is more efficient for solving the relaxed minimal description problem. For this method the worst-case computational complexity is encouraging too. This is why it was selected to numerical experiments described in [5, 6].

7. Modifications of cutting planes

All this what was said about constraint modifications can be applied to cutting planes e.g. the cutting planes used in the method of integer forms of Gomory. It is well known that if only a LIP problem has integer coefficients each Gomory cutting plane can be transformed to original variables and then it is in the form of inequality with integer coefficients [4]. Any modification method applied to constraints can be applied to cutting planes as well.
This will result in problems descriptions much closer to the minimal descriptions than in the case of modifications of constraints only.

8. Conclusions

The minimal description problem introduced in this paper is a formalization of a very important practical problem: how to formulate optimization problems to get their formal descriptions most suitable for given optimization method? Intuitions and computational experimences point to minimal descriptions of problems or to descriptions close to them. All three methods presented in this paper may be used in this context.

The question as practical efficiency of such an approach has been studied experimentally, in [5]. The results of a large numerical experiment with the constraint and cut rotation method combined with the method of integer forms fully confirmed its practical effectiveness. It was shown that rotations of constraints and cuts reduce simultaneously total computation time, number of cuts and number of simplex iterations and all this three values for tested problems were reduced significantly. Thus at least the constraint/cut rotation method may be reccomended to be built into existing optimization codes.

References

[1] Beale E. M., Tomlin J. A. An integer programming approach to class of combinatorial problems. *Math. Progr.* **3** (1972).

[2] Bradley G. An algorithm and bound for gcd for n integers. *Communications of ACM* **13** (1970).

[3] Geoffrion A. M., Graves G. W. Multicomodity distribution system design by Benders decomposition. *Manag. Sci.* **20** (1974).

[4] Gomory R. E. An algorithm for integer solutions to linear programs. [In:] W. Graves G. W., Wolfe P. Recent advances in mathematical programming. New York, 1963.

[5] Kaliszewski I. The cutting plane method efficiency in integer programimng. Doctoral Dissertation (in Polish), IBS PAN, 1980.

[6] Kaliszewski I., Walukiewicz S. Tighter equivalent formulations of integer programming problems. Survey of Mathematical Programming. Budapest Akademia Kiado, 1979.

[7] Kianfar F. Stronger inequalities for 0,1 integer programming using knapsack functions. *Oper. Res.* **19** (1971).

[8] KIANFAR F. Stronger inequalities for 0–1 integer programming: computational refinements. *Oper. Res.* **24** (1976).

[9] KNUTH D. E. The art of computer programming. Vol. 2. New York, Addidon–Wesley, 1969.

[10] LAME G. Note sur la limite du nombre des divisions dans la recherche du plus grand common diviseur entre deux nombers entries. *C. R. Acad, Sci.* Paris **19** (1884).

[11] RICHMOND R. T., RAVINDRAN A. A generalized Euclidean procedure for integer linear programs. *Nav, Res. Log. Quart.* **21** (1974).

[12] SAATY T. Optimization in integers and related extremal problems. New York, Mc Graw Hill, 1970.

[13] SALKIN H. M., BREINING P. Integer points on the Gomory fractional cut (hyperplane). *Naval Res. Logist. Quart.* **18** (1971) 4.

[14] WALUKIEWICZ S. Almost linear problems of discrete programming (in polish). Warszawa, 1975. Prace Instytutu Organizacji i Kierownictwa, Ser. B, z. 23.

[15] WILLIAMS H. P. Experiments in the formulation of integer programming problems. *Math. Progr. Study* **5** (1974).

Metody modyfikacji ograniczeń programowania liniowego całkowitoliczbowego

Rozpatrywane są pewne modyfikacje ograniczeń zadań programowania liniowego całkowitoliczbowego. Modyfikacje te mają na celu maksymalną redukcję zbioru rozwiązań dopuszczalnych zadania związanego programowania liniowego (t.j. zadania będącego osłabieniem zadania wyjściowego o warunek dyskretności zmiennych) przy zachowaniu wszystkich dopuszczalnych rozwiązań całkowitoliczbowych. Sformułowany jest problem minimalnego opisu oraz wskazany sposób znajdowania rozwiązań przybliżonych dla tego problemu poprzez modyfikacje pojedynczych ograniczeń. Przytacza się znane metody modyfikacji pojedynczych ograniczeń oraz wprowadza metodę nową. Badane są własności tych metod oraz ich przydatność do konstruowania efektywnych algorytmów rozwiązywania zadań programowania liniowego całkowitoliczbowego.

Методы модыфикации ограничений задач линейного целочисленого программирования

Рассмотрены некоторые модификации ограничений задач линейного целочисленного программирования. Целью этих модификаций является максимальная редукция множества допустимых решений сопряженной задачи линейного программирования (т.е. задачи, являющейся ослаблением начальной задачи, за счет условия дискретности переменных) при сохранении всех допустимых целочисленных решений. Формулируется проблема минимального описания и представлен способ нахождения приближенных решений для этой проблемы, путем модификации отдельных ограничений. Приведены известные методы модификации отдельных ограничений и вводится новый метод. Исследуются свойства этих методов и их пригодность для разработки эффективных алгоритмов решения задач линейного целочисленного программирования.