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On Heuristic Strategies for Some Interception Problem

by

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We examine a class of the interception problem for two players under the assumption that each knows only his own kinetic model and the other player's state. Models of the players are linear. Heuristic strategies are proposed. If aims of the players are opposite we consider a pursuit-evasion game in another case we call it a cooperative game.

1. Introduction

There are two players I and II whose dynamic systems are given by equations (1) and (2).

$$\dot{x}_1(t)=A_1(t)\,x_1(t)+B_1(t)\,u(t), \qquad x_1\in R^n, \tag{1}$$

$$\dot{x}_2(t)=A_2(t)\,x_2(t)+B_2(t)\,v(t), \qquad x_2\in R^n. \tag{2}$$

The player I uses a sequence of controls

$$u(t)=\{u^0(t), u^1(t), \dots, u^M(t)\}$$

where $u^i(t)\in U\subset R^r$ for $i\delta'\leqslant t\leqslant(i+1)\,\delta'$, $i=0, 1, 2, \dots, M-1$, δ' is a sampling interval; the player II uses another sequence of controls

$$v(t)=\{v^0(t), v^1(t), \dots, v^{N-1}1(t)\}$$

where $v^j(t)\in V\subset R^s$ for $j\delta''\leqslant t\leqslant(j+1)\,\delta''$, $j=0, 1, 2, \dots, N-1$, δ'' is a sampling interval. Each $u^i(t)$ and $v^j(t)$ are admissible controls associated with general linear

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systems. The game starts at time $t=0$, $x_1(0)$ and $x_2(0)$ are given, and is defined over the period $[O, T]$. Integer numbers M and N indicate numbers of observations of player II by player I and player I by player II, respectively. The game runs under the following assumption:

Assumption: The player I does not know the model of the player II and vice versa. Each player can observe the other player's state vector continually.

After having divided the period $[O, T]$ into M equal subperiods the player I builds his controls $u^i(t)$ between observations using both his own and the second player's actual state vector and properties of his reachable regions, similarly the player II builds his controls $v^j(t)$.

Let the condition of interception be

$$Fx_1(t_f)=Gx_2(t_f) \tag{3}$$

where $t_f \leqslant T$ and F, G are row vectors. We shall consider two kinds of the interception problem. The first kind—when aims of both players are opposite, it means the player I (P—pursuer) wants to execute the condition (3) in minimum time, while the player II (E—evader) attempts to maximize the time of interception. In this case we have a differential game of pursuit-evasion, called the Game of Kind subject to the Assumption. The interception problem of the second kind occurs when both players want to execute the condition (3) in minimum time—we can talk also about a team problem game subject to the Assumption.

Because of the lack of knowledge of the models it seems that the performance index for the problem can be

$$\|Fx_1(t)-Gx_2(t)\|, \tag{4}$$

which determines a distance between the players at each time instant $t \in [O, T]$.

Such stated interception problem is quite new and the authors could not find any similar one. Small similarity can be found with problems of keeping the trajectory of one system sufficiently close to the prescribed target trajectory.

Since the pioneer work on differential games of R. Isaacs [1] and many valuable works (for instance Friedman [4], Varaiya and Lin [6]) the main attention has been paid to games in which players know both models of the motion and the full history of the game. Our game is different.

In section 2 the interception problem is shown briefly, in the next section the heuristic strategies are introduced, the section 4 contains a numerical example and the section 5 contains the conclusions.

2. Interception problem

The players described by equations (1) and (2) want to optimize, according to their own choice, the following performance index

$$\|Fx_1(t)-Gx_2(t)\| \tag{4}$$

at each time instant $t \in [O, T]$.

Two cases of the interception problem can be distinguished:

(i) Pursuit—evasion game

The player I, called Pursuer, uses his admissible controls $u^0(t), u^1(t), ..., u^{M-1}(t)$ to attempt the capture of the player II, called Evader. Evader uses the controls $v^0(t), v^1(t), ..., v^{N-1}(t)$ to attempt to avoid the capture. This problem can be expressed as

$$J = \min_u \max_v \|Fx_1(t) - Gx_2(t)\| \tag{5}$$

at each time $t \in [O, T]$ subject to the equations (1) and (2).

(ii) Team problem.

Both players have the same aim to minimize the distance

$$J = \min_u \min_v \|Fx_1(t) - Gx_2(t)\| \tag{6}$$

at each time $t \in [O, T]$ subject to the equations (1) and (2).

3. Heuristic strategies

Let the time period of the game $[O, T]$ be divided into M equal subintervals for the player I and into N, also equal, subintervals for the player II. Without loss of generality we can assume that $M=N$.

The player I observes the second player's state vector $x_2(t)$ at each discrete time $t=k\delta$, where $k=0, 1, 2, ..., N$, and δ is the length of the subinterval, viz. $T=N\delta$. Simultaneously the second player observes the first player's state vector $x_1(t)$ at $t=k\delta$. The game begins at an initial time $t=0$ with an initial distance between I and II player

$$D_0 = \|Fx_1(0) - Gx_2(0)\|. \tag{7}$$

Let $x(k\delta) \equiv x(k)$.

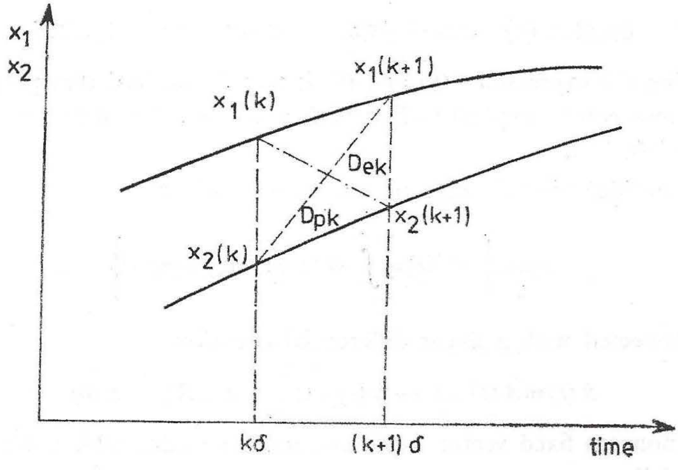

Fig. 1 Optimized distances.

At each time-interval $[k\delta, (k+1)\,\delta]$ the player I minimizes the following distance

$$D_{pk}=\min_{u^k\in U}\ \|Fx_1\,(k+1)-Gx_2\,(k)\| \tag{8}$$

and the player II
for the pursuit—evasion game, maximizes the distance

$$D'_{ek}=\max_{v^k\in V}\ \|Fx_1\,(k)-Gx_2\,(k+1)\| \tag{9}$$

or for the cooperative game, minimizes the same distance

$$D''_{ek}=\min_{v^k\in V}\ \|Fx_1\,(k)-Gx_2\,(k+1)\|. \tag{9a}$$

The sets of admissible controls U and V are defined as follows

$$U=\Big\{u^k\,(t):\ \|u^k\,(t)\|_{p1}=\Big(\int\limits_{k\delta}^{(k+1)\delta}\ \sum_{i=1}^{r}\ |u_i^k\,(t)|^{p1}\,dt)^{1/p1}\leqslant C_p,\quad \text{for}\ 1\leqslant p1<\infty$$

$$=\ \operatorname*{ess\,sup}_{k\delta\leqslant t\leqslant(k+1)\delta}\ |u_i^k(t)|,\quad \text{for}\ p1=\infty\Big\} \tag{10}$$

where $1/\mathrm{pl}+1/q1=1$

and

$$V=\Big\{v^k\,(t):\|v^k\,(t)\|_{p2}=\int\limits_{k\delta}^{(k+1)\delta}\ \sum_{i=1}^{r}\ |v_i^k\,(t)|^{\,p2}\,dt)^{1/p2}\leqslant C_e,\quad \text{for}\ 1\leqslant p2<\infty$$

$$=\ \operatorname*{ess\,sup}_{\delta k\leqslant i\leqslant(k+1)\delta}\ |v_i^k(t)|,\quad \text{for}\ p2=\infty\Big\} \tag{11}$$

where $1/p2+1/q2=1$.

The game terminates if

$$0\leqslant\|Fx_1\,(k)-Gx_2\,(k)\|\leqslant\varepsilon,\quad \text{for some}\ k=0,1,\ldots,N. \tag{12}$$

Examining the expressions (8) and (9) it may be noticed that properties of reachable regions can be applied to find both u^k and v^k. (For definition of reachable regions see [10, 11]).

Let us consider the following maximization problem

$$\max_{\hat{u}}\ \Big\{\lambda x^d\ (t)=\int\limits_0^t\lambda\Phi\,(t,\tau)\,B\,(\tau)\,u\,(\tau)\,dt\Big\} \tag{13}$$

which is connected with a linear differential equation

$$\dot{x}\,(t)=A\,(t)\,x\,(t)+B\,(t)\,u\,(t),\qquad x\in R^n,\qquad x\,(0), \tag{14}$$

and λ is a nonzero fixed vector in R^n and $u\in U$ an admissible control associated with general linear systems.

The solution of (14) is following

$$x(t)=\Phi(t)\,x(0)+\int_0^t \Phi(t,\tau)\,B(\tau)\,u(\tau)\,d\tau. \tag{15}$$

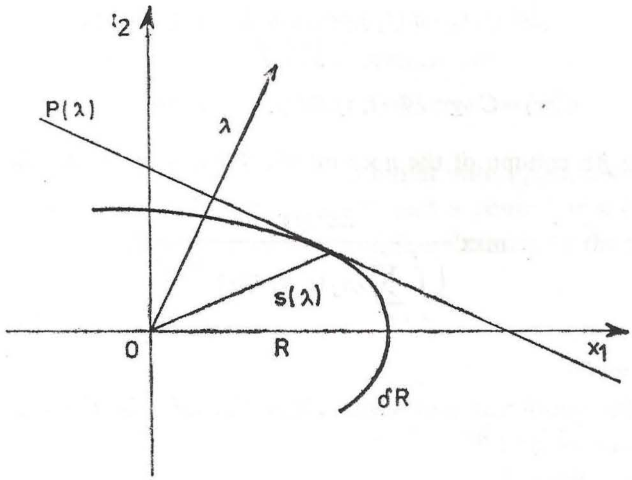

Fig. 2. Illustration: R—reachable region, δR—boundary of the reachable region, $P(\lambda)$—supporting hyperplane, $s(\lambda)$—contact function.

Defining the difference

$$x^d(t)=x(t)-\Phi(t)\,x(0) \tag{16}$$

and applying properties of a supporting hyperplane $P(\lambda)$ to the reachable region

$$R(0,t)=\left\{x^d(t):\ x^d(t)=\int_0^t \Phi(t,\tau)\,B(\tau)\,u(\tau)\,d\tau\right\}, \tag{17}$$

and next using Hölder inequality for integrals with the condition

$$\operatorname{sgn} u(\tau)=\operatorname{sgn} \lambda\Phi(t,\tau)\,B(\tau), \qquad 0\leqslant\tau\leqslant t \tag{18}$$

the (13) may be written as follows

$$\lambda x^d(t)=\left(\int_0^t |\lambda\Phi(t,\tau)\,B(\tau)|^q d\tau\right)^{1/q} C, \tag{19}$$

where C is expressed as $\|u\|_p$ and $1/p+1/q=1$. Finally it is easy to obtain the well--known expressions

$$u_j^*(\tau)=\frac{C\lambda h_j(t,\tau)^{q-1}\operatorname{sgn} h_j(t,\tau)}{\left(\int_0^t \sum_{i=1}^r (|\lambda h_i(t,\theta)|^q d\theta\right)^{(q-1)/q}}, \tag{20}$$

for $1<p<\infty$,

$$u_j^*(\tau)=\begin{cases}0, & j\neq j^\circ\\ C\cdot\text{sgn}\,[\lambda h_{j^\circ}(t, t_j)]\,\delta(t-t_{j^\circ}), & j=j^\circ,\end{cases} \tag{21}$$

for $p=1$,

where

$$|\lambda\Phi(t, t_{j^\circ})\,B(t_{j^\circ})|=\max_{\substack{1\leqslant j\leqslant m\\ 0\leqslant\tau\leqslant t}}\lambda\Phi(t,\tau)(B(\tau),$$

$$u_j^*(\tau)=C\,\text{sgn}\,\lambda\Phi(t,\tau)\,B(\tau),\qquad \text{for } p=\infty, \tag{22}$$

where h_j is the jth column of the $n\times r$ matrix ΦB and the vector λ is such that

$$\max\frac{\lambda x^d(t)}{\left(\int\limits_0^t\sum\limits_{i=1}^r\lambda h_j(t,\theta)^q d\theta\right)^{1/q}}=C, \tag{23}$$

for $j=1, 2, 3, \ldots, m$.

It should be mentioned that some difficulties arise in the case when corners and flat portions occur [11].

Strategy for the player I

For the given system (1) with the initial state $x_1(k)$, $k=0, 1, 2, \ldots, N-1$ and the terminal target $x_1(k+1)\in R_1(k, k+1)$ and the control set U the player I chooses an admissible control $u^{k*}(t)$ which transfers the state x_1 such that
(i) if $x_2(k)\in R_1(k, k+1)$, then $u^{k*}(t)$ transfers the state of the system to a point $x_1(k+1)$ such that

$$Fx_1(k+1)=Gx_2(k) \tag{24}$$

subject to

$$\frac{\lambda^k[x_2(k)-\Phi_1(k+1, k)\,x_1(k)]}{\left(\int\limits_k^{k+1}\sum\limits_{i=1}^r|\lambda^k h_{1i}(k+1,\tau)|^{q1}d\tau\right)}=C_p', \tag{25}$$

Notice, that the equation (25) is similar to (23) and if $C_p'=C_p$ (from (10)) then $x_1(k+1)\in\delta R_1(k, k+1)$—the boundary of the reachable region for the player I.
(ii) if $x_2(k)\notin R_1(k, k+1)$, then $u^{k*}(t)$ transfers the state of the system to a point $x_1(k+1)$ such that

$$\min_{x_1\in\delta R_1(k,k+1)}\|Fx_1(k+1)-Gx_2(k)\|. \tag{26}$$

The solution of the problem (26) requires to solve the following nonlinear programming problem

$$\max_{|\lambda^k|=1}\lambda^k[x_2(k)-\Phi_1(k+1, k)\,x_1(k)+s_1^k(\lambda^k)] \tag{27}$$

where

$$s_1^k(\lambda^k) = \int_k^{k+1} \Phi_1(k+1, \tau) B(\tau) u^k(\tau, \lambda^k) d\tau \in \delta R_1(k, k+1) \tag{28}$$

denotes the contact function. In general (27) is a strongly nonlinear function of λ^k and iterative methods have to be applied to find the solution [7, 10]. The optimal λ^{k*} found from (27) allows us to find the optimal control using the expressions (20–22).

Strategy for the player II

Similarily, or the given system (2) with an initial state $x_2(k)$, $k=0, 1, 2, \ldots, N-1$ and a terminal target $x_2(k+1) \in R_2(k, k+1)$ and a control $v^k \in V$ the player II chooses an admissible control $v^{k*}(t)$ which transfers the state to the point $x_2(k+1)$ such that

(i) for the pursuit—evasion game

$$\max_{x_2 \in R_2(k,k+1)} \|Fx_1(k) - Gx_2(k+1)\| \tag{29}$$

which correspondens to the following programming problem

$$\max_{|\xi^k|=1} \xi^k [-x_1(k) + \Phi_2(k+1, k) x_2(k) + s_2^k(\xi^k)] \tag{30}$$

(ii) for the cooperative game

$$\min_{x_2 \in R_2(k,k+1)} \|Fx_1(k) - Gx_2(k+1)\| \tag{31}$$

or equivalently

$$\min_{|\xi^k|=1} \xi^k [x_1(k) - \Phi_2(k+1, k) x_2(k) + s_2^k(\xi^k)], \tag{32}$$

where ξ^k is a nonzero vector in R and $s_2^k(\xi^k)$ is a contact function.

Remarks are similar—the solution of (32) requires the usage of iterative methods. Again applying (20–22) it is easy to find the control for the player II.

4. Illustrative examples

In this section some numerical results are presented to illustrate the method explained in Section 3.

As an example let be considered a pursuit-evasion game governed by the following equations

$$\dot{x}_1 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x_1 + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} u, \quad x_1(0) = \begin{bmatrix} 0.0 \\ 0.0 \end{bmatrix} \tag{33}$$

$$\dot{x}_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x_2 + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} v, \quad x_2(0) = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix} \tag{34}$$

and controls of both players are assumed to be constrained in amplitude $\|u^k\|_\infty \leqslant 3.0$ and $\|v^k\|_\infty \leqslant 1.0$ (equations (10–11)). The initial time $t=0$ and the final time $T=1.0$ are assumed. The results:

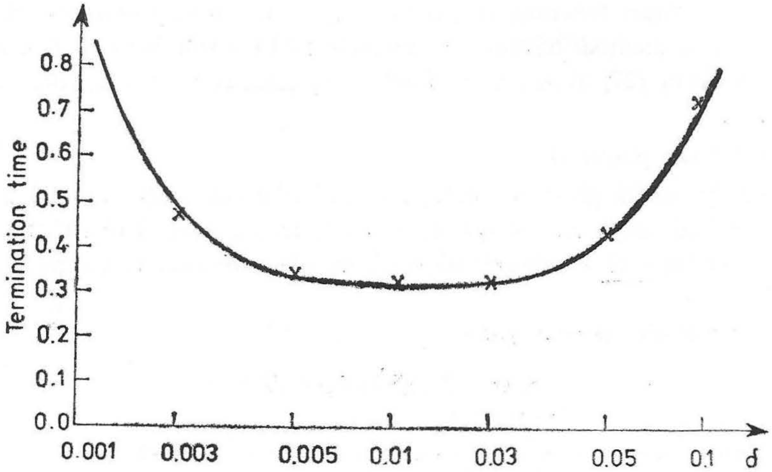

Fig. 3. Termination time as a function of length of sampling.

(i) Termination time versus the sampling length ($\varepsilon=0.1$ in (12)). The first decreasing part of the curve is effected by computation errors and the rising part shows errors resulting from the applied heuristic strategies.

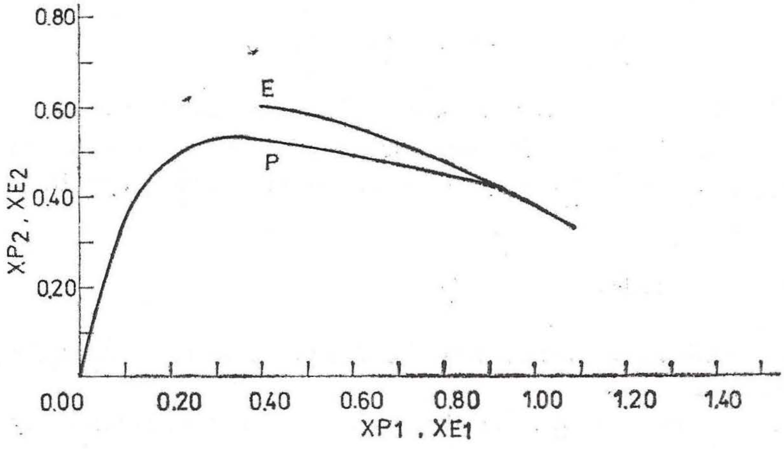

Fig. 4. State trajectories (Pursuer wins).

(ii) Figures (4–6) show trajectories of both players, Fig. (7–8) show strategies used by the players. Notice that the player I (Pursuer) from time $t \simeq 0.4$ does not use his full control because the state of the player II (Evader) lies within his reachable region. Fig. 9 shows the decreasing distance between the players versus time.

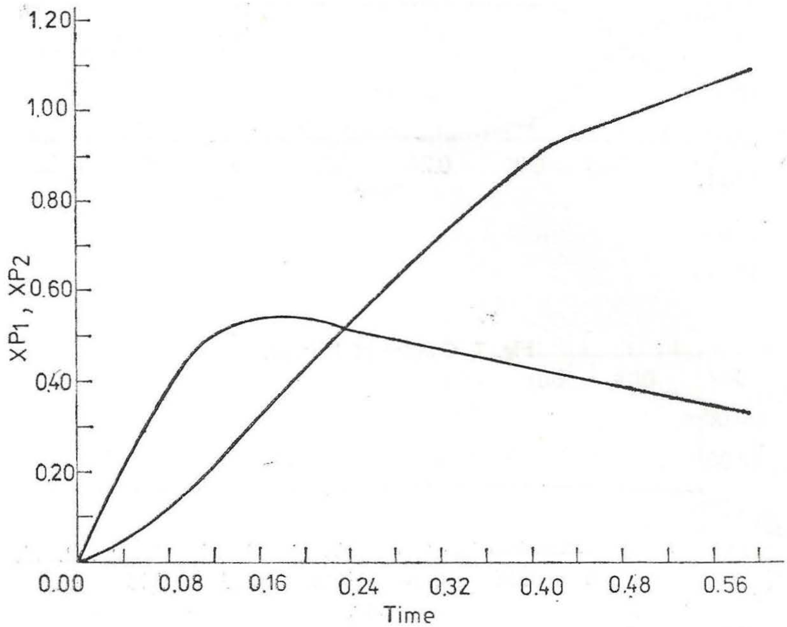

Fig. 5. Time trajectories of Pursuer.

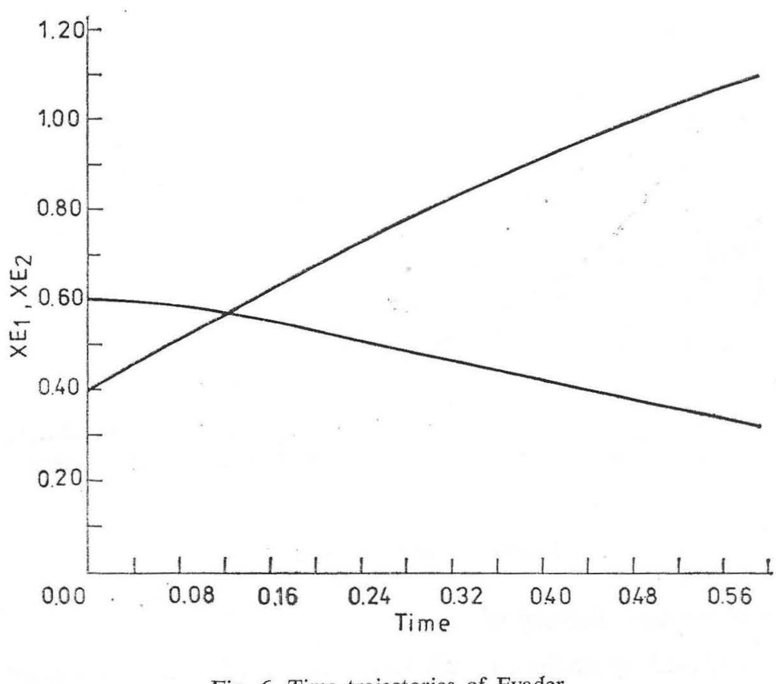

Fig. 6. Time trajectories of Evader.

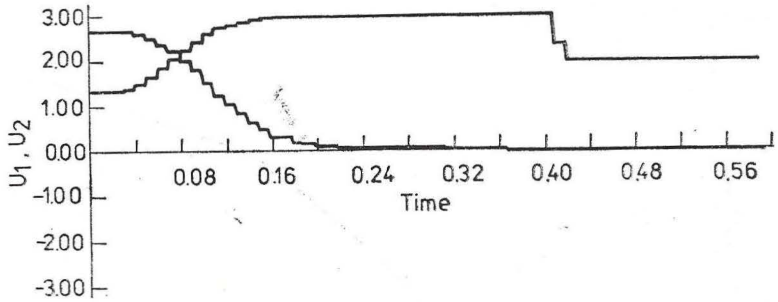

Fig. 7. Control of Pursuer.

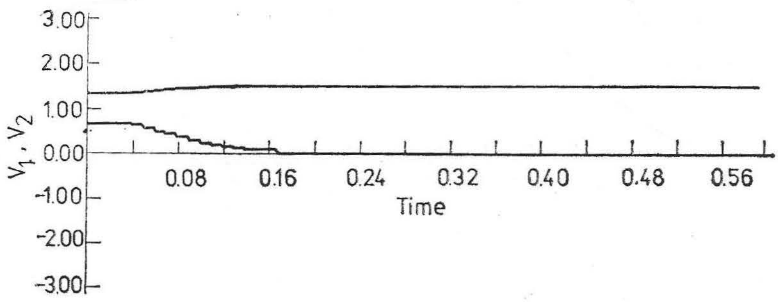

Fig. 8. Control of Evader.

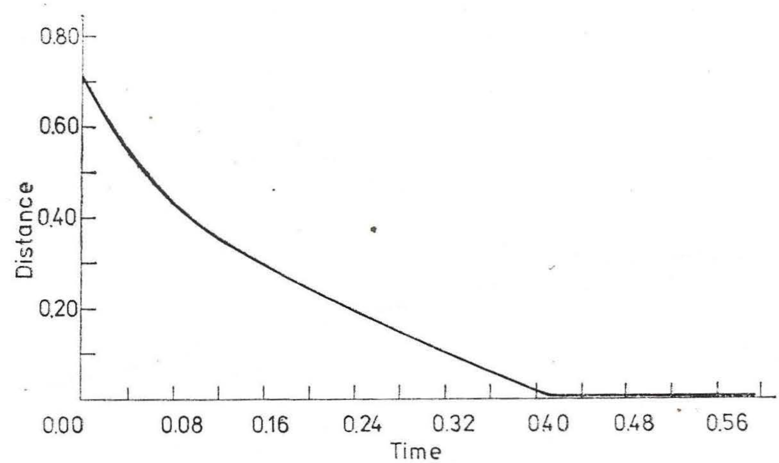

Fig. 9. Distance between players.

In the above case Pursuer wins.

(iii) Figures (10–12) show the case when Evader wins. In this case the constraints are different than in previouse example, namely: $\|u^k\|_\infty \leqslant 3.0$ and $\|v^k\|_\infty \leqslant 1.5$.

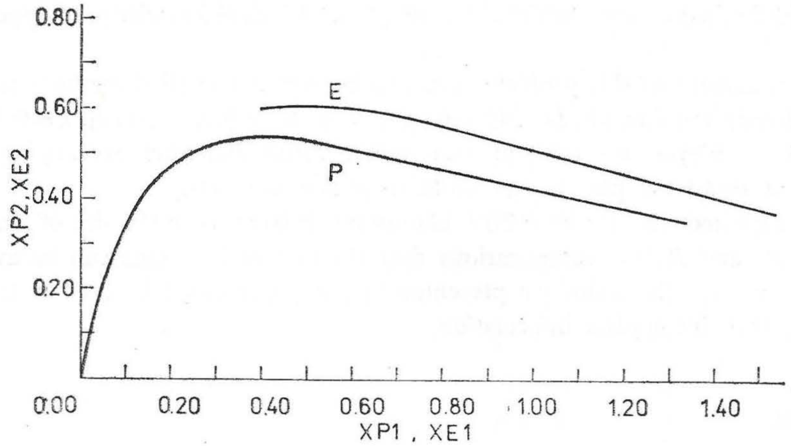

Fig. 10. State trajectories (Evader wins).

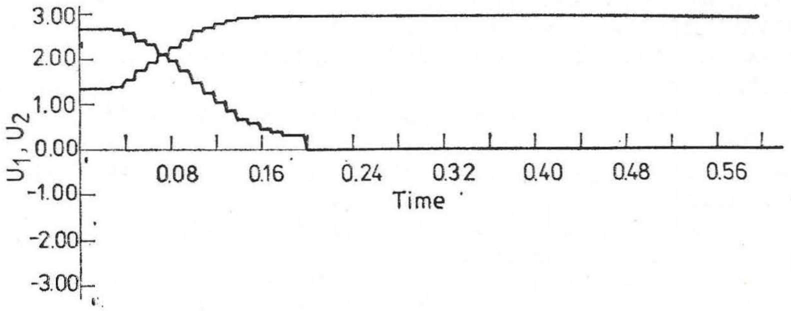

Fig. 11. Control of Pursuer.

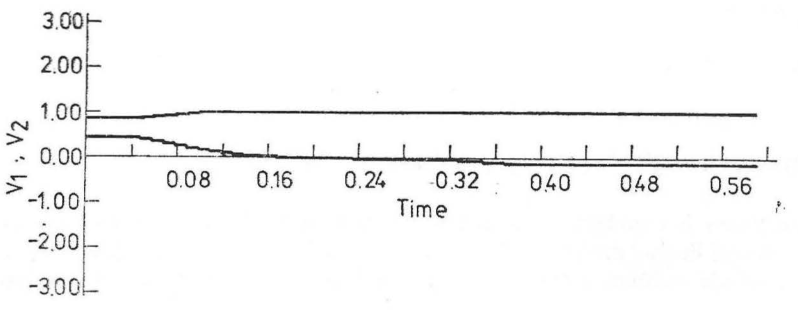

Fig. 12. Control of Evader.

5. Conclusions

The purpose of this paper is to present the heuristic approach to some kind of the interception problem of two players descirbed by linear differential equations with limited controls under the assumption of lack of knowledge of opponent's model.

The statement of this problem seems to be new but applied methods to solve it are already well-known. During each period of time $[k\delta, (k+1)\delta]$, $k=0, 1, 2, \dots$ $\dots, N-1$ the players use optimal open-loop controls and after exchanging information at $t=k\delta$ the players can build their new controls.

It is expected that if ε (in (12)) is chosen much larger than the size of reachable regions R_1 and R_2 the complications near the end of the game can be avoided.

It seems that the technique presented in this paper could be used in tracking problem with incomplete information.

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Heurystyczne strategie dla pewnego problemu przechwytywania

Rozpatrywany jest problem przechwytywania, w którym bierze udział dwu graczy. Każdy z graczy zna swój liniowy model oraz korzysta z informacji o aktualnej pozycji oponenta. Wyróżniono dwa rodzaje problemu przechwytywania — problem ucieczki oraz grę kooperacyjną.

Евристические стратегии некоторой проблемы перехватывания

Рассматривается проблема перехватывания, в которой два игрока. Каждой итгок знает свою линейную модель, а также актуальное положение двугого игрока. Придложены евристические стратегии. Выделяются два рода проблемы — игра убегания и кооперативная игра.