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Some conditions for identifiability of spatially-varying parameters in a class of distributed systems

by

ZBIGNIEW SUFLETA

Control Systems Group
Institute of Engineering Cybernetics
Technical University of Wrocław
Wrocław, Poland

This paper discusses the problem of the identifiability of spatially-varying parameters in systems described by initial-boundary value problems for linear, one-dimensional parabolic partial differential equations. For cases involving distributed, noise-free measurements, two questions related to identifiability problem are considered: how to check whether the available input-output data allows unique determination of the unknown system parameters, and how to chose the system input to accomplish this with a priori knowledge? Under suitable assumptions, several identifiability conditions are obtained.

1. Introduction

This paper is in the line of previous papers by Chavent [1], [2], [3], Kitamura and Nakagiri [5] and Pierce [7], and deals with the identifiability problem for systems described by linear, second-order, one-dimensional parabolic partial differential equations with spatially-varying parameters. The identifiability porblem in its deterministic version leads to two basic questions:

1) How to check whether the given noise-free observation of an input-output pair allows unique determination of unknown system parameters?

2) How to choose the input signal in order to a priori guarantee uniqueness of parameters determination?

The results presented in [3], [5] refer only to the first question and are obtained for the case in which the spatially-varying coefficient preceding the time-derivative of the state is absent in the system equation. Moreover, the identifiability conditions presented there contain requirements with respect to the spatial derivatives of the observed output, which can lead to some difficulties in application.

The results presented in [7] refers to very important case of pointwice observations but they apply to the normal form of the diffusion equation and cannot be used directly in practical situation because the substitution of variables leading to the normal form uses unknown parameters.

The general method for studying whether the inverse problem for an abstract linear operator equation is wellposed has been proposed by Romanov [8]; however, to find the effective identifiability conditions for particular distributed systems using Romanov's concepts, a number of detailed problems must be solved in any case.

This paper extends the results obtained in [3], [5], [7] in regard to the first question and formulates some identifiability conditions for the second question.

2. Statement of the problem

Consider a class of systems described by the following initial-boundary value problem:

$$a_1(x)\frac{\partial y(x,t)}{\partial t}-\frac{\partial}{\partial x}\left[a_2(x)\frac{\partial y(x,t)}{\partial x}\right]+a_3(x)\,y(x,t)=u_D(x,t),$$

$$x\in(0,L),\ t>0 \tag{1}$$

$$y(x,0)=u_I(x),\ x\in[O,L] \tag{2}$$

$$\begin{aligned}\alpha_O\, y(0,t)-\beta_O\left.\frac{\partial y(x,t)}{\partial x}\right|_{x=O}&=u_{BO}(t),\ t>0\\ \alpha_L\, y(L,t)+\beta_L\left.\frac{\partial y(x,t)}{\partial x}\right|_{x=L}&=u_{BL}(t),\ t>0.\end{aligned} \tag{3}$$

Let us denote:

$$u\triangleq[u_I(\cdot),u_{BO}(\cdot),u_{BL}(\cdot),u_D(\cdot,\cdot)],\quad a\triangleq[a_1(\cdot),a_2(\cdot),a_3(\cdot)], \tag{4}$$

$$U\triangleq\Big\{u:u_I(\cdot)\in C^2[0,L];\ u_{BO}(\cdot),u_{BL}(\cdot)\in C^2[0,\infty),$$

$$\alpha_O u_I(O)-\beta_O\frac{du_I(0)}{dx}=u_{BO}(0),\ \alpha_L u_I(L)+\beta_L\frac{du_I(L)}{dx}=u_{BL}(0), \tag{5}$$

$$u_D(\cdot,\cdot)\in C^1([0,L]\times[0,\infty))\Big\}$$

$$A\triangleq\{a:a_1(\cdot),a_3(\cdot)\in C^1[0,L],a_2(\cdot)\in C^2[0,L];$$

$$a_1(x),a_2(x)>0,\ a_3(x)\geqslant0 \text{ for each } x\in[0,L]\}. \tag{6}$$

It is assumed that α_O, β_O, α_L, β_L are a priori known nonnegative constants such that $\alpha_O^2+\beta_O^2\neq0$, $\alpha_L^2+\beta_L^2\neq0$. For each fixed $u\in U$ and $a\in A$, the solution of (1), (2), (3) in the class $Y\triangleq\{y(\cdot,t)\in C^2[0,L]$ for each $t\geqslant0$, $y(x,\cdot)\in C^1[0,\infty)$ for each $x\in[0,L]\}$ [6] will be denoted by $y(\cdot,\cdot\,|a,u)$.

The identification problem consists in determining an unknown parameter a^* belonging to the known set $A_E \subseteq A$, on the basis of the input vector u and the output observation $y(x, t|a^*, u)$ for $(x, t) \in \Omega \subseteq [0, L] \times [0, \infty)$.

The ability to get a unique solution of the above identification problem depends on the amount of a priori information given by the set A_E and the measurement abilities defined by the set Ω. In this paper an emphasis is put on the case in which the set A_E is infinite dimensional, particularly when $A_E = A$. In such a case the solution of the identification problem can be nonunique even if $\Omega = [0, L] \times [0, \infty)$ [3], [5], and it seems to be impossible to get the uniqueness when Ω contains a finite number of points. For that reason, the paper like the previous works [3], [5] deals only with the case in which $O = [0, L] \times [0, T]$, $0 \leqslant T \leqslant \infty$, and estabilishes a kind of preliminary identifiability study in a view of practical needs.

To consider precisely the two identifiability questions mentioned in the introduction, let us introduce the following definitions.

DEFINITION 1. *Let $a^* \in A_E$. The system described by* (1), (2), (3) *is called $\langle u, A_E|a^*\rangle$-identifiable if for each $a \in A_E$ the identity $y(\cdot, t|a, u) = y(\cdot, t|a^*, u)$, $t \in [0, T]$ implies $a = a^*$.*

DEFINITION 2. *The system described by* (1), (2), (3) *is called $\langle u, A_E\rangle$-identifiable if it is $\langle u, A_E|a^*\rangle$-identifiable for each $a^* \in A_E$.*

3. The sufficient conditions for $\langle u, A_E|a\rangle$-identifiability

The results presented in this section refers to the first question formulated in the introduction, i.e. they can be applied to check whether the given input-output pair determine a unique system parameters. Let us denote:

$$A_1 \triangleq \{a \in A : a_1(x_1) = b_1\}, \quad A_2 \triangleq \{a \in A : a_2(x_2) = b_2\} \tag{7}$$

where $x_1, x_2 \in [0, L]$ and $b_1, b_2 > 0$ are fixed known numbers. The necessity to consider the cases of $A_E = A_1$ or $A_E = A_2$ becomes clear when the system to be identified has no distributed input, i.e. $u(\cdot, \cdot) = 0$ (when, for instance, the equation (1) describes the heat diffusion in an one-dimensional medium, this means the lack of internal heat sources). In the above case the system is not $\langle u, A|a\rangle$-identifiable for each $u \in U$ and each $a \in A$.

THEOREM 1. *Let $u \in U$ and $a^* \in A$. If there exist $t_1, t_2, t_3 \in [0, T]$ such that $u_D(\cdot, t_1) = u_D(\cdot, t_2) = u_D(\cdot, t_3) = 0$ and the functions $y(\cdot, t_1|a^*, u)$, $y(\cdot, t_2|a^*, u)$, $y(\cdot, t_3|a^*, u)$ are linearly independent then the system described by* (1), (2), (3) *is:*

(T1) *$\langle u, A_1|a^*\rangle$-identifiable and $\langle u, A_2|a^*\rangle$-identifiable.*

If, moreover, $u_D(\cdot, \cdot)$ is nonzero function then the system is:

(T2) *$\langle u, A|a^*\rangle$-identifiable.*

Proof. Note that since for each $a \in A$ we have $a_1(x)>0$, $x \in [0, L]$, the equation (1) can be rewritten in the form:

$$\frac{\partial y(x,t)}{\partial t}-\frac{a_2(x)}{a_1(x)}\frac{\partial^2 y(x,t)}{\partial x^2}-\frac{a_2'(x)}{a_1(x)}\frac{\partial y(x,t)}{\partial x}+$$

$$+\frac{a_3(x)}{a_1(x)}y(x,t)=\frac{u_D(x,t)}{a_1(x)};\ x \in (0, L),\ t>0 \tag{8}$$

where the prime stands for a derivative with respect to x. Consider an arbitrary $a \in A$ such that $y(\cdot, t|a, u)=y(\cdot, t|a^*, u)$ for $t \in [0, T]$. Taking into account (8) we can write:

$$\left[\frac{a_2(x)}{a_1(x)}-\frac{a_2^*(x)}{a_1^*(x)}\right]\frac{\partial^2 y(x,t|a^*,u)}{\partial x^2}+\left[\frac{a_2'(x)}{a_1(x)}-\frac{a_2^{*\prime}(x)}{a_1^*(x)}\right]\frac{y(x,t|a^*,u)}{\partial x}-$$

$$-\left[\frac{a_3(x)}{a_1(x)}-\frac{a_3^*(x)}{a_1^*(x)}\right]y(x,t|a^*,u)=\left[\frac{1}{a_1(x)}-\frac{1}{a_1^*(x)}\right]u_D(x,t),\ (x,t) \in \Omega. \tag{9}$$

By assumption there exist $t_1, t_2, t_3 \in (O, T)$ such that $u_D(\cdot, t_i)=0$, $i=1, 2, 3$, i.e. the functions $y(\cdot, t_i|a^*, u)$, $i=1, 2, 3$ are the solutions of the equation:

$$\left[\frac{a_2(x)}{a_1(x)}-\frac{a_2^*(x)}{a_1^*(x)}\right]v''(x)+\left[\frac{a_2'(x)}{a_1(x)}-\frac{a_2^{*\prime}(x)}{a_1^*(x)}\right]v'(x)-$$

$$-\left[\frac{a_3(x)}{a_1(x)}-\frac{a_3^*(x)}{a_1^*(x)}\right]v(x)=0,\ x \in (0, L) \tag{10}$$

which is a linear, second order, ordinary differential equation (homogeneous one) having no more than two linearly independent solutions. But the functions $y(\cdot, t_i|a^*, u)$, $i=1, 2, 3$ are assumed to be linearly independent and they satisfy (10). It is possible only if:

$$\frac{a_2(x)}{a_1(x)}=\frac{a_2^*(x)}{a_1^*(x)},\ \frac{a_2'(x)}{a_1(x)}=\frac{a_2^{*\prime}(x)}{a_1^*(x)},\ \frac{a_3(x)}{a_1(x)}=\frac{a_3^*(x)}{a_1^*(x)},\ x \in (0, L) \tag{11}$$

which implies $a=ca^*$ with a certain $c \in R$.

If $a, a^* \in A_1$ then $a_1(x_1)=a_1^*(x_1)=b_1$ and $c=\dfrac{a_1^*(x_1)}{a_1(x_1)}=1$, i.e. $a=a^*$. Thus, the system is $\langle u, A_1|a^*\rangle$-identifiable.

Similarly the $\langle u, A_2|a^*\rangle$-identifiability holds. Moreover, if $a=ca^*$ then the left hand side of the equation (9) vanishes and we have $\left[\dfrac{1}{a_1(x)}-\dfrac{1}{a_1^*(x)}\right]u_D(x,t)=0$, $(x, t) \in \Omega$ For nonzero $u_D(\cdot, \cdot)$ this implies $a_1(x)=a_1^*(x)$ for a certain $x \in (O, L)$ and as a consequence $c=1$, i.e. $a=a^*$. Thus the system is $\langle u, A|a^*\rangle$-identifiable. Q.E.D. ∎

Note that the identifiability conditions obtained in Theorem 1 are expressed in terms of the pair u, $y(\cdot, t|a^*, u)$, $t \in [0, T]$ which, in spite of the fact that a^*

is unknown, is known from the experiment. To check these conditions the linear independence of the functions $w_i(\cdot)=y(\cdot, t_i|a^*, u)$, $i=1, 2, 3$ must be verified which can lead to some numerical troubles. However, the conditions presented seem to be checkable more easily than those proposed in [3], [5], because no properties of the spatial derivatives of the output need be verified.

4. The choice of imputs warranting system identifiability

The second important problem refers to the question of how to warrant $\langle u, A_E|a^*\rangle$-identifiability before the experiment by means of a suitable input design. Since the appropriate requirements with regard to the input vector u may generally depend on the unknown parameter a^*, the satisfactory choice of u must make the system $\langle u, A_E|a\rangle$-identifiable for all parameters expected to appear in the system, i.e. $\langle u, A_E\rangle$-identifiable. Several sufficient identifiability conditions for solving the above problem can be found using Theorem 1. To this end let us write the solution of (1), (2), (3) for given a and u in the form [9]:

$$y(x, t|a, u)=\sum_{n=1}^{\infty} h_n(t|a, u)\,\varphi_n(x|a),\quad x\in(0, L),\ t>0 \tag{12}$$

where:

$$h_n(t|a, u)=\int_0^L a_1(x)\,u_I(x)\,\varphi_n(x|a)\,dx\exp[-\lambda_n(a)\,t]+$$

$$+\int_0^t\Big[\int_0^L u_D(x, t)\,\varphi_n(x|a)\,dx+g_O^{(n)}u_{BO}(s)+g_L^{(n)}u_{BL}(s)\Big]\exp[-\lambda_n(a)(t-s)]\,ds \tag{13}$$

and:

$$g_O^{(n)}=-a_2(0)\frac{\varphi_n(0|a)+\varphi_n'(0|a)}{\alpha_O+\beta_O},\quad g_L^{(n)}=a_2(L)\frac{\varphi_n(L|a)-\varphi_n'(L|a)}{\alpha_L+\beta_L} \tag{14}$$

Here $\lambda_n(a)$, $\varphi_n(\cdot|a)$, $n=1, 2, \ldots$ are eigenvalues and eigenfunctions, respectively, obtained as a solution of the Sturm-Liouville problem:

$$\frac{d}{dx}\left[a_2(x)\frac{d\varphi_n(x)}{dx}\right]+[\lambda_n a_1(x)-a_3(x)]\,\varphi_n(x)=0,\quad x\in(0, L) \tag{15}$$

$$\alpha_O\,\varphi_n(0)-\beta_O\,\varphi_n'(0)=0,\qquad \alpha_L\,\varphi_n(L)+\beta_L\,\varphi_n'(L)=0 \tag{16}$$

with the condition $\int_O^L \varphi_n^2(x)\,dx=1$, $n=1, 2, \ldots$.

LEMMA 1. *Let $y(\cdot, \cdot|a, u)$ be expressed by (12) with an arbitrary $a\in A$ and $u\in U$. If there exist n_1, n_2, n_3 such that the functions $f_i(\cdot)=h_{n_i}(\cdot|a, u)$, $i=1, 2, 3$ are linearly independent on $(0, T)$ then there exist $t_1, t_2, t_3\in(O, T)$ such that the functions $z_i(\cdot)=y(\cdot, t_i|a, u)$, $i=1, 2, 3$ are linearly independent.*

Proof. If $f_i(\cdot)$, $i=1, 2, 3$ are l.i. (linearly independent) then there exist t_1, t_2, t_3 such that the vectors $[f_1(t_i), f_2(t_i), f_3(t_i)]$, $i=1, 2, 3$ are l.i. and, hence, the sequences $\{h_n(t_i|a, u)\}$ are l.i.. Since for each $t>0$ the sequence $\{h_n(\cdot|a, u)\}$ defines a unique decomposition of $y(\cdot, t|a, u)$ with respect to the model basis $\{\varphi_n(\cdot|a)\}$ [4], the functions $z_i(\cdot)=y(\cdot, t_i|a, u)$, $i=1, 2, 3$ are also l.i., Q.E.D.. ∎

LEMMA 2. *Let $y(\cdot, \cdot|a, u)$ be expressed by (12) with an arbitrary $a \in A$ and $u \in U$. If there exist n_1, n_2, n_3 such that for each $t_1, t_2, t_3 \in (O, T)$ the vectors $[h_{n_1}(t_i|a, u), h_{n_2}(t_i|a, u), h_{n_3}(t_i|a, u)]$ $i=1, 2, 3$ are linearly independent then the functions $y(\cdot, t_i|a, u)$, $i=1, 2, 3$ are linearly independent for each $t_1, t_2, t_3 \in (O, T)$.*
The proof of Lemma 2 is similar to that of Lemma 1.

LEMMA 3. *If $\lambda_n(a)$ is a sequence of eigenvalues corresponding to an arbitrary $a \in A$ and $v(\cdot) \in L^1[t_0, T]$ is nonzero function then for each n_1, n_2, n_3 the functions $w_{n_i}(\cdot)$, $i=1, 2, 3$ given by: $w_n(t)=\int_{t_0}^{t} v(s)\exp[-\lambda_n(a)(t-s)]\,ds$, $t \in [t_0, T]$, $n=1, 2, \ldots$ are linearly independent.*

LEMMA 4. *If $g_O^{(n)}$ and $g_L^{(n)}$ are defined by (14) for an arbitrary $a \in A$ then $g_O^{(n)} \neq 0$ and $g_L^{(n)} \neq 0$ for each $n=1, 2, \ldots$.*

LEMMA 5. *Let $V_0 \subseteq C^1[0, L]$ be the set of nonzero functions each of which vanishes on a certain subinterval of $[0, L]$. If $\{\varphi_n(\cdot|a)\}$ is a sequence of eigenfunctions corresponding to an arbitrary $a \in A$ then the sequence $\{\mu_n\}$ given by*

$$\mu_n=\int_{O}^{L} a_1(x)\,v(x)\,\varphi_n(x|a)\,dx, \qquad n=1, 2, 3, \ldots \tag{17}$$

with $v(\cdot) \in V_0$ contains an infinite number of nonzero elements.

The proofs of Lemmas 3, 4, 5 follow from the known properties of the eigenvalue problem [4] and are omitted.

THEOREM 2. *If the input vector $u \in U$ takes one of the forms:*

(F1) $u=[0, v(\cdot), 0, 0]$ *or* $u=[0, 0, v(\cdot), 0]$ *where $v(\cdot) \in C^2[0, T]$ is an arbitrary nonzero function*

(F2) $u=[0, v_O(\cdot), v_L(\cdot), 0]$ *where $v_O(\cdot), v_L(\cdot) \in C^2[O, T]$ are such that there exist $t_1, t_2 \in (0, T)$, $t_1 < t_2$ for which*

$$v_O(t)=\begin{cases} c_1 v_L(t) & \text{for } t \in [0, t_1] \\ c_2 v_L(t) & \text{for } t \in [t_2, T] \end{cases} \quad \text{with certain } c_1, c_2 \in R,\ c_1 \neq c_2$$

and $v_L(\cdot)$ is nonzero in each of intervals $[0, t_1]$, $[t_2, T]$

(F3) $u=[v(\cdot), 0, 0, 0]$ *where $v(\cdot) \in V_O$ (V_O being defined in Lemma 5)*

then the system described by (1), (2), (3) is $\langle u, A_1\rangle$-identifiable and $\langle u, A_2\rangle$-identifiable.

Proof. For the proof it is sufficient to express the solution $y(\cdot, \cdot|a, u)$ in the form (12) and show by Lemmas 3, 4, 5 that the corresponding functions $h_{n_i}(\cdot|a, u)$, $i=1, 2, 3$ are linearly independent for a certain triple n_1, n_2, n_3 uniformly with respect to $a \in A$. The final step follows from Lemma 1 and Theorem 1.

THEOREM 3. *If* $u=[u_I(\cdot), u_{BO}(\cdot), u_{BL}(\cdot), u_D(\cdot, \cdot)]$ *where* $u_I(\cdot)$ *is an arbitrary function from* $C^2[0, L]$ *and*:

(A1) $u_{BO}(t)=\sum_{r=O}^{m_O} \mu_O^{(r)} t^r$, $u_{BL}(t)=\sum_{r=O}^{m_L} \mu_L^{(r)} t^r$, $t \in [0, T]$

(A2) $u_D(x, t)=\sum_{k=1}^{m_D} v_k(x) \sum_{r=O}^{p_k} \nu_k^{(r)} t^r$, $x \in [0, L]$, $t \in [0, T]$,

$v_k(\cdot) \in V_O$, $k=1, 2, \ldots, m_D$ and there exist $t_1, t_2, t_3 \in (O, T)$ such that $u_D(\cdot, t_i)=0$, $i=1, 2, 3$

(A3) one of the following conditions holds:

(C1) $m_L \geq m_O+3$, $m_L \geq p_k+3$, $k=1, 2, \ldots, m_D$, $\mu_L^{(m_L)} \neq 0$

(C2) $m_O \geq m_L+3$, $m_O \geq p_k+3$, $k=1, 2, \ldots, m$, $\mu_O^{(m_O)} \neq 0$,

(C3) there exists r, $1 \leq r \leq m_D$ such that $p_r \geq m_L+3$, $p_r \geq m_O+3$, $p_r \geq p_k+3$, $k=1, 2, \ldots, r-1, r+1, \ldots, m_D$, $\nu_r^{(p_r)} \neq 0$

then the system described by (1), (2), (3) is:

(T1) $\langle u, A_1\rangle$-identifiable and $\langle u, A_2\rangle$-identifiable.

If, moreover $u_D(\cdot, \cdot)$ is nonzero function then the system is

(T2) $\langle u, A\rangle$-identifiable.

Proof. If $u \in U$ satisfies (A1) and (A2) then for each $a \in A$ the functions $h_n(\cdot|a, u)$ appearing in the series (12) are of the form:

$$h_n(t|a, u)=\kappa_O^{(n)} \exp[-\lambda_n(a) t]+\int_O^t \Big[g_O^{(n)} \sum_{r=O}^{m_O} \mu_O^{(r)} s^r+ + g_L^{(n)} \sum_{r=O}^{m_L} \mu_L^{(n)} s^r+\sum_{k=1}^{m_D} \kappa_k^{(n)} \sum_{r=O}^{p_k} \nu_k^{(r)} s^r\Big] \exp[-\lambda_n(a)(t-s)]\, ds, \quad t>0 \tag{18}$$

where:

$$\kappa_O^{(n)}=\int_O^L a_1(x)\, u_I(x)\, \varphi_n(x)\, dx, \quad \kappa_k^{(n)}=\int_O^L v_k(x)\, \varphi_n(x|a)\, dx, \quad k=1, 2, \ldots, m_D. \tag{19}$$

Let us note that for each $\lambda \in R$ we have:

$$\int_O^t s^r \exp[-\lambda(t-s)]\, ds=(-1)^{r+1} \frac{r}{\lambda^{r+1}} [\exp(-\lambda t)-1]+ + \sum_{i=1}^{r} (-1)^{r-1} \frac{r!}{i!\lambda^{r-i+1}} t^i, \quad t>0 \tag{20}$$

Assume that, for example, (C1) holds. Hence, for each triple n_1, n_2, n_3 we can write:

$$h_{n_i}(t|a,u)=\left[\kappa_O^{(n_i)}+g_O^{(n_i)}\pi_{OO}^{(n_i)}+g_L^{(n_i)}\pi_{LO}^{(n_i)}+\sum_{k=1}^{m_D}\kappa_k^{(n_i)}\sigma_{kO}^{(n_i)}\right]\exp[-\lambda_{n_i}(a)t]+$$

$$+g_O^{(n_i)}\sum_{r=O}^{m_O}\pi_{Or}^{(n_i)}t^r+g_L^{(n_i)}\sum_{r=O}^{m}\kappa_{Lr}^{(n_i)}t^r+\sum_{k=1}^{m_D}\kappa_k^{(n_i)}\sum_{r=O}^{p_k}\sigma_{kr}^{(n_i)}t^r+$$

$$+g_L^{(n_i)}\sum_{k=m+1}^{m_L}\left[\sum_{r=k}^{m_L}\mu_L^{(n_i)}(-1)^{r-k}\frac{r!}{k!\,\lambda_{n_i}(a)^{r-k+1}}\right]t^k,\ t\in[0,T],\ i=1,2,3 \tag{21}$$

where:

$$\pi_{Or}^{(n)}=\sum_{k=r}^{m_O}\mu_O^{(k)}(-1)^{k-r}\frac{r!}{k!\,\lambda_n(a)^{k-r+1}},\ r=0,1,\dots,m_O \tag{22}$$

$$\pi_{Lr}^{(n)}=\sum_{k=r}^{m}\mu_L^{(k)}(-1)^{k-r}\frac{r!}{k!\,\lambda_n(a)^{k-r+1}},\ r=0,1,\dots,m \tag{23}$$

$$\sigma_{qr}^{(n)}=\sum_{k=r}^{p_q}\nu_q^{(k)}(-1)^{k-r}\frac{r!}{k!\,\lambda_n(a)^{k-r+1}},\ r=0,1,\dots,p_q,\quad q=1,2,\dots,m_D \tag{24}$$

and $m=\max\{m_O, p_1, \dots, p_{m_D}\}$.

Formula (21) expresses the decomposition of the functions $h_{n_i}(\cdot|a,u)$, $i=1,2,3$ on the basis composed of $\exp[-\lambda_{n_1}(a)(\cdot)]$, $\exp[-\lambda_{n_2}(a)(\cdot)]$, $\exp[-\lambda_{n_3}(a)(\cdot)]$, $1, (\cdot), (\cdot)^2, \dots, (\cdot)^{m^L}$. Let us consider three vectors, each of which is composed of the three last coefficients in the decomposition of the function $h_{n_i}(\cdot|a,u)$ on the mentioned basis.

Since $m \geqslant m+3$, these vectors are of the form:

$$b_i=g_L^{(n_i)}\left[\mu_L^{(m_L-2)}\frac{1}{\lambda_{n_i}(a)}-\mu_L^{(m_L-1)}\frac{m_L-1}{\lambda_{n_i}(a)^2}+\mu_L^{(m_L)}\frac{m_L(m_L-1)}{\lambda_{n_i}(a)^3},\right.$$

$$\left.\mu_L^{(m_L-1)}\frac{1}{\lambda_{n_i}(a)}-\mu_L^{(m_L)}\frac{m_L}{\lambda_{n_i}(a)^2},\ \mu_L^{(m_L)}\frac{1}{\lambda_{n_i}(a)}\right],\ i=1,2,3 \tag{25}$$

It is easy to calculate that the determinant of the matrix composed of these vectors can be expressed by the formula:

$$W=-[\mu_L^{(m_L)}]^3\,m_L(m_L-1)\frac{g_L^{(n_1)}g_L^{(n_2)}g_L^{(n_3)}}{\lambda_{n_1}(a)^3\lambda_{n_2}(a)^3\lambda_{n_3}(a)^3}W_1 \tag{26}$$

where:

$$W_1=\begin{vmatrix}\lambda_{n_1}(a)^2 & \lambda_{n_1}(a) & 1\\ \lambda_{n_2}(a)^2 & \lambda_{n_2}(a) & 1\\ \lambda_{n_3}(a)^2 & \lambda_{n_3}(a) & 1\end{vmatrix}$$

Since W_1 is the Vandermonde determinant and $\lambda_{n_i}(a)$, $i=1, 2, 3$ are different values [4], we have $W_1 \neq 0$. Moreover, $g_L^{(n_i)} \neq 0$, $i=1, 2, 3$ by Lemma 5, and $\mu_L^{(m_L)} \neq 0$ by assumption (C1). Thus $W \neq O$ and consequently the vectors b_1, b_2, b_3 are linearly independent. Since the basis considered for the decomposition of $h_{n_i}(\cdot\,|a, u)$, $i=1, 2, 3$ is composed of exponential and power functions, this implies that for each triple $t_1, t_2, t_3 \in (O, T)$ the vectors $[h_{n_1}(t_i|a, u),\ h_{n_2}(t_i|a, u),\ h_{n_3}(t_i|a, u)]$, $i=1, 2, 3$ are l.*i*.. Hence, the assumptions of Lemma 2 are satisfied (for each $a \in A$) and (*T*1) follows from this lemma and Theorem 1. Assuming $u_D(\cdot, \cdot) \neq 0$ we obtain also (*T*2). Q.E.D. ■

The proof of the theorem under assumptions (C2) or (C3) can be performed analogically.

Theorems 2 and 3 formulate only sufficient conditions for u to warrant the system identifiability. The fundamental problem is a practical realization of inputs belonging to the obtained classes. Some restrictions can appear due to the zero initial conditions required in Theorem 2, which can be nonrealistic in some cases or quite natural in other ones. Choosing, however, the boundary and distributed inputs as polynomials of time-variable (see Theorem 3) it is possible to warrant the identifiability for an arbitrary initial condition.

5. Conclusions

A number of results for identifiability of a certain class of systems described by partial differential equations with spatially-varying coefficients have been presented. The results which refer to the problem of checking whether the given input-output data allow unique determination of system parameters enrich the list of identifiability conditions presented in [1], [2], [3], [5]. These results refer to a more general case in which the coefficient $a_1(\cdot)$ is considered, and seem to be more convenient in application because no requirements with respect to the spatial derivative of the output signal are stated. An attempt is made to solve the problem of the choice of identifying inputs. The results show that it is possible to propose a number of simple inputs sufficient for the identifiability.

Only the one-dimensional spatial domain and a distributed type of observation are considered. Also the coefficients in boundary conditions are assumed to be known. These are the main restrictions of the results.

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Warunki identyfikowalności dla pewnej klasy obiektów o parametrach rozłożonych ze współczynnikami zależnymi od położenia

Praca dotyczy zagadnienia identyfikowalności zależnych od położenia parametrów w obiektach opisywanych liniowymi, jednowymiarowymi równaniami różniczkowymi cząstkowymi typu parabolicznego. Dla przypadku bezszumowego pomiaru wielkości wejściowych i wyjściowych obiektu rozważono dwa podstawowe pytania wiążące się z zagadnieniem identyfikowalności: jak sprawdzić czy pozostająca w dyspozycji informacja pomiarowa pozwala na jednoznaczne określenie parametrów obiektu i jak wybrać sygnał wejściowy obiektu dla zapewnienia takiej możliwości a priori?

Przy odpowiednich założeniach określono szereg rezultatów dających odpowiedź na powyższe pytania. Warunki wystarczające identyfikowalności wiążące się z pierwszym pytaniem wyrażono w postaci wymagań w stosunku do mierzonych sygnałów wejściowego i wyjściowego, podczas gdy odpowiednie warunki wiążące się z drugim pytaniem definiują pewne klasy sygnałów wejściowych zapewniających identyfikowalność.

Условия идентифицируемости для некоторого класса объектов с распределенными параметрами, с коэффициентами зависимыми от положения

Работа касается проблемы идентифицируемости зависимых от положения параметров в объектах описываемых линейными, одномерными дифференциальными уравнениями в частных производных параболического типа. Для случая беспомеховых измерений входных и выходных величин объекта рассмотрены два основные вопроса, связанные с проблемой идентифицируемости: как проверить, позволяет ли имеющаяся измерительная информация однозначно определить параметры объекта и, как выбрать входной сигнал объекта для априорного обеспечения такой возможности?

При соответствующих предпосылках определен ряд результатов, дающих ответ на выше указанные вопросы. Достаточные условия идентифицируемости, связанные с первым вопросом, выражены в виде требований по отношению к измеряемым входному и выходному сигналам, а соответствующие условия, связанные со вторым вопросом, определяют некоторые классы входных сигналов, обеспечивающих идентифицируемость.