Control
and Cybernetics
VOL. 11 (1982) No. 1–2

A Phase Transition Problem with Delay

by

AUGUSTO VISINTIN*)
Istituto di Analisi Numerica del C.N.R.
Corso Carlo Alberto, 5
27100 Pavia, Italy

A delay is introduced into the jump function in the weak formulation of the Stefan problem similarly to what happens for super-cooling and super-heating effects. An existence result is proved.

Introduction

Let $u_1, u_2 \in \boldsymbol{R}$ $(u_1 < u_2)$ be given. Let the variables $u, w\colon [0, T] \to \boldsymbol{R}$ be related by a "jump condition with delay" according to the following conditions: for a genetic $t \in [0, T]$

$$\left\{\begin{array}{l} \text{If } u(t) < u_1 \text{ (respect. } u(t) > u_2\text{), then } w(t) = -1 \\ \text{(respect. } w(t) = 1\text{)} \\ \text{if } u_1 \leqslant u(t) \leqslant u_2, \text{ then } -1 \leqslant w(t) \leqslant 1 \\ \text{if } u_1 < u(t) < u_2, \text{ then } w(t) \text{ is constant in a neighbourhood of } u(t) \\ \text{if } u(t) = u_1 \text{ (respect. } u(t) = u_2\text{), then } w(t) \text{ is non-increasing} \\ \text{(respect. is non-decreasing). (see fig. 1)} \end{array}\right. \tag{1}$$

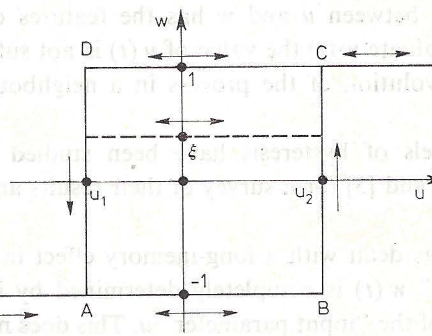

Fig. 1. $u_1, u_2 \in \mathrm{R}$, $u_1 < u_2$ Arrows indicate direction of movement of $(u(t), w(t))$ as t increases. $\xi \in [-1, 1]$ is generic

(*) — This work has been realized during a stay supported by the Polish Academy of Sciences at the Systems Research Institute of P.A.N. in Warsaw in September-October 1981.

Therefore w is controlled by the function u, but the specification of the latter is not sufficient to determine the former.

Let D be an open bounded subset of R^N $(N\geqslant 1)$, $T>0$. Set $Q=D\times[0,T]$.

Relation (1) is assumed to hold in Q and is coupled with the equation

$$\frac{\partial}{\partial t}(u+w)-\Delta u=f \text{ in } Q\,. \tag{2}$$

$\left(\text{where } f \text{ is a datum, } \Delta=\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}\right)$, with suitable initial and boundary conditions.

If $u_1=u_2$ then (1) degenerates into the usual jump condition and we get the weak formulation of the Stefan problem (cf. [4], pag. 196–204, e.g.).

This last is a model for several phase transition phenomena; an example is given by change of state (transition between water and ice, say), with u temperature and $u+w$ enthalpy. In this physical setting generalization (1) corresponds to a water freezing temperature u_1 strictly less than ice melting temperature u_2, as it arises in super-cooling and super-heating.

The transition set is characterized by $u_1\leqslant u\leqslant u_2$ and it is to be expected to have the same dimension of the space.

Also here free boundaries are present, corresponding to $u=u_i$ $(i=1,2)$; formally the following jump conditions hold

$$[w]\cdot\nu_t=[\nabla u]\cdot\bar{\nu}_x \quad \text{on } S_i=\{(x,t)\in Q\,|\,u(x,t)=u_i\} \quad (i=1,2) \tag{3}$$

(where $\bar{\nu}=(\bar{\nu}_x,\nu_\tau)$ is normal to S_i, which is assumed regular enough, and $[\,\cdot\,]$ denotes the jump across S_i), with

$$-2\leqslant[w]\leqslant 2 \quad \text{on } S_i\ (i=1,2) \tag{4}$$

and w decreasing (increasing) w.r.t. time across S_1 (S_2 respect.).

Still formally the diffusion equation $\frac{\partial u}{\partial t}-\Delta u=f$ holds in $Q\setminus(S_1\cup S_2)$, in particular in the transition set $\mathcal{T}=\{(x,t)\in Q\,|\,u_1<u(x,t)<u_2\}$. However notice that this setting does not correspond to so-called "three-phase problem".

The above relation between u and w has the features of hysteresis: for every instant t in order to evaluate $w(t)$ the value of $u(t)$ is not sufficient, but information about the preceding evolution of the process in a neighbourhood of t is required (short-memory effect).

Mathematical models of hysteresis have been studied by Krasnosel'skii and co-workers (e.g., cf. [2] and [3] for a survey of their results and for a large collection of references).

In [6] the author has dealt with a long-memory effect in which at any instant t the "output parameter" $w(t)$ is completely determined by its initial value and by the evolution in $[0,T]$ of the "input parameter" u. This does not hold for relation (1).

In this work we give a weak formulation of (1), (2) (§1); for this formulation we prove an existence result by approximation, at first by time-discretization (§2), then by smoothing the jumps of (1) and using a result of [6] (§ 3). Finally we show that letting $u_1-u_2\to 0$, we get the usual Stefan problem (§ 4).

1. Variational formulation

We introduce some notations:

$$\alpha(\xi) = (\xi - u_2)^+ - (\xi - u_1)^-, \quad \forall \xi \in \boldsymbol{R} \tag{1.1}$$

$$\beta(\xi) = \begin{cases} u_1 & \text{if } \xi \leqslant u_1 \\ \xi & \text{if } u_1 < \xi < u_2, \\ u_2 & \text{if } \xi \geqslant u_2 \end{cases} \quad \forall \xi \in \boldsymbol{R} \tag{1.2}$$

$$S(\xi) = \begin{cases} \{-1\} & \text{if } \xi < 0 \\ [-1, 1] & \text{if } \xi = 0, \\ \{1\} & \text{if } \xi > 0 \end{cases} \quad \forall \xi \in \boldsymbol{R} \tag{1.3}$$

$$R(\xi) = \begin{cases} [-\infty, 0] & \text{if } \xi \leqslant u_1 \\ \{0\} & \text{if } u_1 < \xi < u_2, \\ [0, +\infty[& \text{if } \xi \geqslant u_2 \end{cases} \quad \forall \xi \in \boldsymbol{R} \tag{1.4}$$

all of these being maximal monotone graphs, and

$$\psi(\xi) = \begin{cases} \{0\} & \text{if } \xi \neq u_1, u_2 \\ [-\infty, 0] & \text{if } \xi = u_1, \\ [0, +\infty] & \text{if } \xi = u_2 \end{cases} \quad \forall \xi \in \boldsymbol{R}, \tag{1.5}$$

this graph being non-monotone.

By (1) we get

$$w \in S(\alpha(u)) \quad \text{in } Q \tag{1.6}$$

$$\frac{\partial w}{\partial t} \in \psi(u) \quad \text{in } Q; \tag{1.7}$$

the last yields

$$\frac{\partial w}{\partial t} \in R(\beta(u)) \quad \text{in } Q; \tag{1.8}$$

notice that in this deduction no information has been lost, as the behavior of $\frac{\partial w}{\partial t}$ for $u \notin [u_1, u_2]$ may be obtained by (1.6).

Both $S \circ \alpha$ and $R \circ \beta$ are non-monotone graphs; as subsets of $\boldsymbol{R}^2$ they have a non-empty interior, which corresponds to a lack of information. However it is meaningful to compare (1.6) with (1.8), as the relation between w and $\frac{\partial w}{\partial t}$ is one-to-one (at least for "smooth" functions); moreover the informations given by them are complementary, in the sense that $S \circ \alpha$ is single-valued where $R \circ \beta$ is multi-valued and conversely, with the exceptions of $u = u_1$ and $u = u_2$.

Therefore (1.6) and (1.8) seem to describe suitably the relation between u and w. The fact that they are expressed by means of maximal monotone graphs (as S, α, R, β are) will be useful for the study of a weak formulation.

Set $V=H_0^1(D)$, Hilbert space with the norm $\|v\|_V=\|\nabla v\|_{[L^2(D)]^N}$. Let

$$f\in L^2(0,T;V') \tag{1.9}$$

$$u^0\in L^2(D),\quad w^0\in L^\infty(D)\quad \text{be such that } w^0\in S(\alpha(u^0))\quad \text{a.e. in } D \tag{1.10}$$

and

$$g=\int_0^t f(\tau)\,d\tau+u^0+w^0\quad \text{in } H^1(0,T;V'). \tag{1.11}$$

The problem under consideration takes then the form

(P): — *Find* $u\in L^2(0,T;V)\cap H^1(0,T;V')\ (\subset C^0([0,T];L^2(D)))$ *such that*

$$\iint_Q \{u[v-\alpha(u)]+\nabla\int_0^t u(x,\tau)\,d\tau\cdot\nabla[v-\alpha(u)]\}\,dx\,dt+$$

$$+\iint_Q [|v|-|\alpha(u)|]\,dx\,dt\geqslant\int_0^T {}_{V'}\langle g,v-\alpha(u)\rangle_V\,dt,\qquad \forall v\in L^2(0,T;V) \tag{1.12}$$

$$\int_0^T {}_{V'}\left\langle\frac{\partial u}{\partial t},\ v-\beta(u)\right\rangle_V dt+\iint_Q \nabla u\cdot\nabla[v-\beta(u)]\,dx\,dt\geqslant\int_0^T {}_{V'}\langle f,v-\beta(u)\rangle_V\,dt$$

$$\forall v\in L^2(0,T;V) \text{ such that } u_1\leqslant v\leqslant u_2 \text{ a.e. in } Q \tag{1.13}$$

$$u(0)=u^0\quad \text{a.e. in } D. \tag{1.14}$$

S is the subdifferential of the convex functional $L^2(0,T;V)\to \boldsymbol{R}$ defined by $v\mapsto$ $\mapsto\iint_Q^s |v|\,dxdt$; therefore (1.12) can be written in the form

$$u-\Delta\int_0^t u(x,\tau)\,d\tau+S(\alpha(u))\ni g\quad \text{in } L^2(0,T;V') \tag{1.15}$$

which can be justified by integrating (2) w.r.t. t and using (1.6).

By (2) and (1.8) we get

$$\frac{\partial u}{\partial t}-\Delta u-R(\beta(u))\ni f\quad \text{in } Q \tag{1.16}$$

which formally corresponds to (1.13).

2. An existence result

THEOREM 1: *Assume that* (1.10), (1.11′) *hold and*

$$u^0\in V \tag{2.1}$$

$$f=f_1+f_2,\ \text{with } f_1\in L^2(Q),\ f_2\in W^{1,1}(0,T;V'). \tag{2.2}$$

Then (P) has at least one solution such that moreover

$$u\in H^1(0,T;L^2(D))\cap L^\infty(0,T;V). \tag{2.3}$$

Proof: i) *Approximation*:

Let $m \in N$, $k = \frac{T}{m}$. Set

$$\begin{cases} f^n_m = f^n_{1m} + f^n_{2m}, \quad f^n_{1m}(x) = \frac{1}{k} \int\limits_{(n-1)k}^{nk} f_1(x,t)\,dt \quad \text{a.e. in } D, \\ f^n_{2m} = f_2(nk) \text{ in } V' \quad \text{for } n = 1, \dots, m \end{cases} \tag{2.4}$$

$$K(\xi,\eta) = \begin{cases} \{-1\} & \text{if } \xi < u_1 \\ [-1,\eta] & \text{if } \xi = u_1 \\ \{\eta\} & \text{if } u_1 < \xi < u_2 \\ [\eta, 1] & \text{if } \xi = u_2 \\ \{1\} & \text{if } \xi > u_2 \end{cases} \quad \forall \xi \in R, \ \forall \eta \in [-1,1]. \tag{2.5}$$

We introduce a **time-discretized problem** (P_m): — *Find* $u^n_m \in V$, $w^n_m \in L^\infty(D)$ *for* $n = 1, \dots, m$ *such that* — *setting* $u^0_m = u^0$, $w^0_m = w^0$ *a.e. in* D —

$$\frac{u^n_m - u^{n-1}_m}{k} + \frac{w^n_m - w^{n-1}_m}{k} - \Delta u^n_m = f^n_m \quad \text{in } V', \quad \text{for } n = 1, \dots, m, \tag{2.6}$$

$$w^n_m \in K(u^n_m, w^{n-1}_m) \quad \text{a.e. in } D, \text{ for } n = 1, \dots, m. \tag{2.7}$$

For every $m \in N$ we solve (P_m) step by step. Fix $n \in \{1, \dots, m\}$ and assume that u^{n-1}_m and w^{n-1}_m are known. $K(., w^{n-1}_m(x))$ is a maximal monotone graph a.e. in D, therefore there exists a convex, lower semi-continuous functional $L^n_m : L^1(D) \to R$ such that $K(., w^{n-1}_m) = \partial L^n_m$ a.e. in D.

Introduce the coercive, strictly convex, lower semi-continuous functional $J^n_m : V \to R$, defined by

$$v \mapsto \frac{1}{2} \|v\|^2_{L^2(D)} + L^n_m(v) + \frac{k}{2} \|v\|^2_V - \int\limits_D (u^{n-1}_m + w^{n-1}_m) v\,dx - k_{V'} \langle f^n_m, v \rangle_V, \tag{2.8}$$

which has a unique minimizing argument, denoted by u^n_m. We have

$$\partial J^n_m(u^n_m) = u^n_m + K(u^n_m, w^{n-1}_m) - (u^{n-1}_m + w^{n-1}_m) - k(\Delta u^n_m + f^n_m) \ni 0 \quad \text{in } V'; \tag{2.9}$$

therefore defining w^n_m by means of (2.6) we get (2.7).

Solution of (P_m) is unique. Numerical resolution of (P_m) can be performed by standard space-discretization methods.

ii) *Estimates*:

Fix a generic $l \in \{1, \dots, m\}$; multiply (2.6) by $u^n_m - u^{n-1}_m$ and sum for $n = 1, \dots, l$. Notice that

$$\sum_{n=1}^{l} \int\limits_D \frac{(u^n_m - u^{n-1}_m)}{k} (u^n_m - u^{n-1}_m)\,dx = k \sum_{n=1}^{l} \left\| \frac{u^n_m - u^{n-1}_m}{k} \right\|_{L^2(D)}, \tag{2.10}$$

$$\sum_{n=1}^{l} \int\limits_D \frac{(w^n_m - w^{n-1}_m)}{k} (u^n_m - u^{n-1}_m)\,dx \geqslant 0 \quad (\text{by (2.7)}), \tag{2.11}$$

$$\sum_{n=1}^{l} {}_{V'}\langle -\Delta u_m^n, u_m^n - u_m^{n-1}\rangle_V = \sum_{n=1}^{l} \int_D \nabla u_m^n \cdot \nabla (u_m^n - u_m^{n-1})\, dx \geqslant$$

$$\geqslant \frac{1}{2} \sum_{n=1}^{l} (\| \nabla u_m^n \|^2_{[L^2(D)]^N} - \| \nabla u_m^{n-1} \|^2_{[L^2(D)]^N}) = \frac{1}{2} \| u_m^l \|_V^2 - \frac{1}{2} \| u^0 \|_V^2 \quad (2.12)$$

$$\sum_{n=1}^{l} \int_D f_{1m}^n (u_m^n - u_m^{n-1})\, dx \leqslant \| f_1 \|_{L^2(Q)} \cdot \left(k \sum_{n=1}^{l} \left\| \frac{u_m^n - u_m^{n-1}}{k} \right\|^2_{L^2(D)} \right)^{\frac{1}{2}}, \quad (2.13)$$

$$\sum_{n=1}^{l} {}_{V'}\langle f_{2m}^n, u_m^n - u_m^{n-1}\rangle_V = {}_{V'}\langle f_{2m}^l, u_m^l\rangle_V - {}_{V'}\langle f_{2m}^1, u^0\rangle_V -$$

$$- \sum_{n=1}^{l} {}_{V'}\langle f_{2m}^n - f_{2m}^{n-1}, u_m^{n-1}\rangle_V \leqslant \text{Const.}\, \| f_{2m} \|_{W^{1,1}(O,T,V')} \cdot \max_{n=0,\dots,l} \| u_m^n \|_V . \quad (2.14)$$

Thus we get

$$k \sum_{n=1}^{m} \left\| \frac{u_m^n - u_m^{n-1}}{k} \right\|^2_{L^2(D)} \leqslant \text{Const. (indep. of } m), \quad (2.15)$$

$$\max_{n=0,\dots,m} \| u_m^n \|_V \leqslant \text{Const. (indep. of } m). \quad (2.16)$$

Denote by $u_m(x, t)$ the function obtained interpolating linearly the values $u_m(x, nk) = u_m^n(x)$ for $n=0, \dots, m$ a.e. in D; define w_m similarly. Set $\hat{u}_m(x, t) = u_m^n(x)$, $\hat{w}_m(x, t) = w_m^n(x)$ a.e. in D and $\hat{f}_m(t) = f_m^n$ in V' if $(n-1)k < t \leqslant nk$, for $n=1, \dots, m$.

Then (2.6) becomes

$$\frac{\partial}{\partial t}(u_m + w_m) - \Delta \hat{u}_m = \hat{f}_m \text{ in } V', \text{ a.e. in } [0. T]; \quad (2.17)$$

(2.15) and (2.16) yield

$$\| u_m \|_{H^1(0,T;L^2(D)) \cap L^\infty(0,T;V)} \leqslant \text{Const. (indep. of } m), \quad (2.18)$$

$$\| \hat{u}_m \|_{H^\tau(0,T:L^2(D)) \cap L^\infty(0,T;V)} \leqslant \text{Const. (indep. of } m)\ \forall \tau < \frac{1}{2}, \quad (2.19)$$

moreover

$$\| w_m \|_{L^\infty(Q)} \leqslant 1 . \quad (2.20)$$

iii) *Limit:*

By (2.18) and (2.20) there exist u, w such that — possibly taking subsequences —

$$u_m \to u \quad \text{in } H^1(0, T; L^2(D)) \cap L^\infty(0, T; V) \text{ weak star}, \quad (2.21)$$

$$w_m \to w \quad \text{in } L^\infty(Q) \text{ weak star} \quad (2.22)$$

whence

$$\hat{u}_m \to u \quad \text{in } H^\tau(0, T; L^2(D)) \cap L^\infty(0, T; V) \text{ weak star}, \ \forall \tau < \frac{1}{2}, \quad (2.23)$$

and, as α and β are Lipschitz-continuous,

$$\alpha(\hat{u}_m)\to\alpha(u) \quad \text{in } H^\tau(0,T;L^2(D))\cap L^\infty(0,T;V) \text{ weak star}, \ \forall\tau<\frac{1}{2}, \tag{2.24}$$

$$\beta(\hat{u}_m)\to\beta(u) \quad \text{in } H^\tau(0,T;L^2(D))\cap L^\infty(0,T;V) \text{ weak star}, \ \forall\tau<\frac{1}{2}. \tag{2.25}$$

Notice that

$$\hat{f}_m\to f \quad \text{in } L^2(0,T;V') \text{ strong}; \tag{2.26}$$

integrating (2.17) w.r.t. t and taking $m\to\infty$ we get

$$u+w-\Delta\int_0^t u(x,\tau)\,d\tau=g \quad \text{in } V', \quad t\in[0,T]. \tag{2.27}$$

(2.7) yields

$$\hat{w}_m\in S(\alpha(\hat{u}_m)) \quad \text{a.e. in } Q, \tag{2.28}$$

that is for every $v\in L^2(Q)$

$$\int\int_Q (|\alpha(\hat{u}_m)|-|v|)\,dx\,dt\leqslant\int\int_Q \hat{w}_m[\alpha(\hat{u}_m)-v]\,dx\,dt, \tag{2.29}$$

whence taking $m\to\infty$ and using (2.22), (2.24)

$$\int\int_Q (|\alpha(u)|-|v|)\,dx\,dt\leqslant\int\int_Q w[\alpha(u)-v]\,dx\,dt; \tag{2.30}$$

(2.27) and (2.30) yield (1.12).

By (2.7) we have

$$\begin{cases} w_m^n\leqslant w_m^{n-1} & \text{if } u_m^n\leqslant u_1, \quad \text{i.e.} \quad \beta(u_m^n)=u_1, \\ w_m^n=w_m^{n-1} & \text{if } u_1<u_m^n<u_2, \quad \text{i.e.} \quad u_1<\beta(u_m^n)<u_2, \\ w_m^n\geqslant w_m^{n-1}. & \text{if } u_m^n\geqslant u_2, \quad \text{i.e.} \quad \beta(u_m^n)=u_2, \end{cases} \tag{2.31}$$

and by (2.6)

$$k\sum_{n=1}^m\int_D\left\{\frac{(u_m^n-u_m^{n-1})}{k}[v-\beta(u_m^n)]+\nabla u_m^n\cdot\nabla[v-\beta(u_m^n)]\right\}dx-$$

$$-k\sum_{n=1}^m {}_{V'}\langle f_m^n, v-\beta(u_m^n)\rangle_V=\sum_{n=1}^m\int_D(w_m^n-w_m^{n-1})[\beta(u_m^n)-v]\,dx\,dt\geqslant 0, \tag{2.32}$$

$$\forall v\in L^2(0,T;V) \text{ such that } u_1\leqslant v\leqslant u_2 \quad \text{a.e. in } Q;$$

notice that, setting $B(\xi)=\int_0^\xi \beta(\eta)\,d\eta \quad \forall \xi \in \mathbf{R}$,

$$\overline{\lim_{m\to\infty}}\left\{-\sum_{n=1}^{m}\int_D (u_m^n-u_m^{n-1})\,\beta(u_m^n)\,dx\right\}\leqslant$$

$$\leqslant -\underline{\lim}_{m\to\infty}\sum_{n=1}^{m}\int_D [B(u_m^n)-B(u_m^{n-1})]\,dx=-\underline{\lim}_{m\to\infty}\int_D [B(u_m^m)-B(u^0)]\,dx\leqslant$$

$$\leqslant \text{(as } B \text{ is convex and lower semicontinuous)}\leqslant$$

$$\leqslant -\int_D [B(u(T))-B(u^0)]\,dx=-\int_0^T {}_{V'}\!\left\langle \frac{\partial u}{\partial t},\,\beta(u)\right\rangle_V dt\,; \tag{2.33}$$

$$\overline{\lim_{m\to\infty}}\left\{-k\sum_{n=1}^{m}\int_D \nabla u_m^n\cdot\nabla\beta(u_m^n)\,dx\right\}=-\lim_{m\to\infty} k\sum_{n=1}^{m}\int_D |\nabla\beta(u_m^n)|^2\,dx\leqslant$$

$$\leqslant -\iint_Q |\nabla\beta(u)|^2\,dx\,dt=-\iint_Q \nabla u\cdot\nabla\beta(u)\,dx\,dt\,; \tag{2.34}$$

taking the superior limit as $m\to\infty$ in (2.32) and using (2.26), (2.33) and (2.34), we get (1.13). ■

3. Another approximation procedure

It appears natural to approach the relation between u and w by means of the one sketched in fig. 2 and then to take $\lambda\to+\infty$.

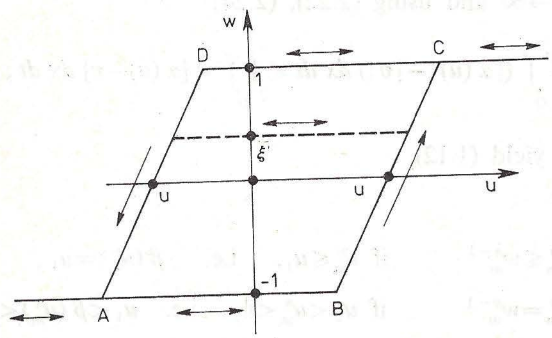

Fig. 2. The slope of BC and AD is $\lambda>0$. Arrows indicate direction of movement of $(u(t),\, w(t))$ as t increases. $\xi\in[-1,1]$ is generic.

The situation of fig. 2 has already been considered in [6] (see § 5, example 2).

Denote by $\mathscr{R}_\lambda$ the union of the closed parallelogram ABCD with the half-lines δ_1, δ_2 and by $\mathscr{S}_\lambda$ the corresponding multi-application $\mathbf{R}\to\mathscr{S}(\mathbf{R})$. Set

$$g_l=\begin{cases}\lambda & \text{on }]A,D]\\ 0 & \text{in } \mathscr{R}_\lambda\setminus]A,D]\,,\end{cases} \tag{3.1}$$

$$g_r=\begin{cases}\lambda & \text{on } [B,C[\\ 0 & \text{in } \mathscr{R}_\lambda\setminus[B,C[\,.\end{cases} \tag{3.2}$$

For all $u \in C^1([0, T_j])$ and $w^0 \in [-1, 1]$ with $w^0 \in \mathscr{S}_\lambda(u(0))$, the relation sketched in fig. 2 can be expressed as follows

$$\begin{cases} \dfrac{\partial w}{\partial t} = g_r(u, w)\left(\dfrac{\partial u}{\partial t}\right)^+ - g_l(u, w)\left(\dfrac{\partial u}{\partial t}\right)^- \quad \text{a.e. in }]0, T[\\ w(0) = w^0 . \end{cases} \tag{3.3}$$

As it has been shown in [6], this Cauchy problem can be integrated, yielding

$$w(t) = \mathscr{F}_\lambda(u, t, w^0) \tag{3.4}$$

where $\mathscr{S}_\lambda$ and $\mathscr{F}_\lambda$ fulfill the following conditions:

$$\begin{cases} \forall (v, t, \xi) \text{ such that } v \in C^0([0, T]),\ t \in [0, T],\ \xi \in \mathscr{S}_\lambda(v(0)) \\ \mathscr{F}_\lambda(v, t, \xi) \in \mathscr{S}_\lambda(v(t)) ; \end{cases} \tag{3.5}$$

$$\begin{cases} \forall v \in C^0([0, T]),\ \forall \xi \in \mathscr{S}_\lambda(v(0)), \text{ the function } t \mapsto \mathscr{F}_\lambda(v, t, \xi) \\ \text{is continuous in } [0, T] ; \end{cases} \tag{3.6}$$

$$\forall v \in C^0([0, T]),\ \forall \xi \in \mathscr{S}_\lambda(v(0)),\ \mathscr{F}_\lambda(v, 0, \xi) = \xi ; \tag{3.7}$$

$$\begin{cases} \forall \bar{t} \in]0, T[,\ \forall v_1, v_2 \in C^0([0, T]) \text{ such that } v_1 = v_2 \text{ in } [0, \bar{t}], \\ \forall \xi \in \mathscr{S}_\lambda(v_1(0)),\ \mathscr{F}_\lambda(v_1, t, \xi) = \mathscr{F}_\lambda(v_2, t, \xi) . \end{cases} \tag{3.8}$$

Assume that (1.9), ..., (1.11) hold. For every $\lambda < 0$, set

$$w_\lambda^0 = w^0,\quad u_\lambda^0 = u^0 - \frac{w^0}{\lambda} \quad \text{a.e. in } D. \tag{3.9}$$

By theorem 1 of [6], there exists at least one

$$u_\lambda \in H^1(0, T; L^2(D)) \cap L^\infty(0, T; V) \tag{3.10}$$

such that

$$u_\lambda(x, 0) = u_\lambda^0(x) \quad \text{a.e. in } D \tag{3.11}$$

and, if

$$w_\lambda(x, t) = \mathscr{F}_\lambda(u_\lambda(x, .), t, w^0(x)) \quad \forall t \in [0, T], \quad \text{a.e. in } D, \tag{3.12}$$

then $w_\lambda \in H^1(0, T; L^2(D)) \cap L^\infty(0, T; V)$ and

$$\frac{\partial}{\partial t}(u_\lambda + w_\lambda) - \Delta u_\lambda = f \quad \text{in } V', \quad \text{a.e.} \quad \text{in }]0, T[. \tag{3.13}$$

THEOREM 2: *For all $\lambda \in R^+$, let u_λ, w_λ be such as in* (3.10), ..., (3.13); *then there exists at least one u such that, possibly after taking a subsequence.*

$$u_\lambda \to u \text{ in } H^1(0, T; L^2(D)) \cap L^\infty(0, T; V) \textit{ weak star} \tag{3.14}$$

Moreover such u is a solution of problem (P).

Proof: Multiply (3.13) against $\frac{\partial u_\lambda}{\partial t}$; by a standard procedure this yields

$$\|u_\lambda\|_{H^1(0,T;L^2(D))\cap L^\infty(0,T;V)} \leq \text{Const. (indep. of } \lambda), \tag{3.15}$$

therefore there exist u, w such that, possibly taking subsequences,

$$u_\lambda \to u \text{ in } H^1(0,T;L^2(D))\cap L^\infty(0,T;V) \text{ weak star} \tag{3.16}$$

$$w_\lambda \to w \quad \text{in } L^\infty(Q) \text{ weak star.} \tag{3.17}$$

Taking $\lambda \to +\infty$ in (3.14) we get

$$\frac{\partial}{\partial t}(u+w) - \Delta u = f \quad \text{in } V', \text{ a.e. in }]0,T[, \tag{3.18}$$

whence by time integration

$$u + w - \Delta \int_0^t u(\tau)\, d\tau = g \quad \text{in } V', \quad t \in [0,T]. \tag{3.19}$$

Notice that

$$\frac{\partial w_\lambda}{\partial t} = 0 \quad \text{if } u_\lambda - \frac{w_\lambda}{\lambda} \neq u_i \quad (i=1,2), \tag{3.20}$$

whence

$$\frac{\partial}{\partial t}\left(u_\lambda - \frac{w_\lambda}{\lambda}\right) = \begin{cases} \dfrac{\partial u_\lambda}{\partial t} & \text{if } \quad u_\lambda - \dfrac{w_\lambda}{\lambda} \neq u_i \quad (i=1,2), \\ 0 & \text{if } \quad u_\lambda - \dfrac{w_\lambda}{\lambda} = u_i \quad (i=1,2), \end{cases} \tag{3.21}$$

therefore

$$u_\lambda - \frac{w_\lambda}{\lambda} \to u \quad \text{in } H^1(0,T;L^2(D)) \text{ weak ;} \tag{3.22}$$

similarly we get

$$u_\lambda - \frac{w_\lambda}{\lambda} \to u \quad \text{in } L^\infty(0,T;V) \text{ weak star,} \tag{3.23}$$

and thus, as α and β are Lipschitz-continuous,

$$\alpha\left(u_\lambda - \frac{w_\lambda}{\lambda}\right) \to \alpha(u) \quad \text{in } H^1(0,T;L^2(D))\cap L^\infty(0,T;V) \text{ weak star,} \tag{3.24}$$

$$\beta\left(u_\lambda - \frac{w_\lambda}{\lambda}\right) \to \beta(u) \quad \text{in } H^1(0,T;L^2(D))\cap L^\infty(0,T;V) \text{ weak star.} \tag{3.25}$$

We have

$$w_\lambda \in S\left(\alpha\left(u_\lambda - \frac{w_\lambda}{\lambda}\right)\right) \quad \text{a.e. in } Q\,; \tag{3.26}$$

by a standard procedure, (3.17) and (3.24) yield

$$w \in S(\alpha(u)) \quad \text{a.e. in } Q, \tag{3.27}$$

which together with (3.19) gives (1.12).

By (3.13) and (3.20) we have

$$\iint_Q \frac{\partial u_\lambda}{\partial t}\left[v-\beta\left(u_\lambda-\frac{w_\lambda}{\lambda}\right)\right]dx\,dt+\iint_Q \nabla u_\lambda\cdot\nabla\left[v-\beta\left(u_\lambda-\frac{w_\lambda}{\lambda}\right)\right]dx\,dt-$$

$$-\int_0^T {}_{V'}\left\langle f, v-\beta\left(u_\lambda-\frac{w_\lambda}{\lambda}\right)\right\rangle_V dt=\iint_Q \frac{\partial w_\lambda}{\partial t}\left[v-\beta\left(u_\lambda-\frac{w_\lambda}{\lambda}\right)\right]dx\,dt\geqslant 0, \tag{3.28}$$

$$\forall v \in V \text{ such that } u_1\leqslant v\leqslant u_2 \quad \text{a.e. in } D;$$

(3.16) and (3.25) yield

$$\iint_Q \frac{\partial u_\lambda}{\partial t}\beta\left(u_\lambda-\frac{w_\lambda}{\lambda}\right)dx\,dt \to \iint_Q \frac{\partial u}{\partial t}\beta(u)\,dx\,dt, \tag{3.29}$$

notice that

$$\nabla\left(u_\lambda-\frac{w_\lambda}{\lambda}\right)=\begin{cases}\nabla u_\lambda & \text{if } u_\lambda-\dfrac{w_\lambda}{\lambda}\neq u_i \quad (i=1,2),\\[2ex] 0 & \text{if } u_\lambda-\dfrac{w_\lambda}{\lambda}=u_1 \quad (i=1,2),\end{cases} \tag{3.30}$$

and then

$$\varliminf_{\lambda\to+\infty}\iint_Q \nabla u_\lambda\cdot\nabla\beta\left(u_\lambda-\frac{w_\lambda}{\lambda}\right)dx\,dt=$$

$$=\varliminf_{\lambda\to+\infty}\iint_Q \nabla\left(u_\lambda-\frac{w_\lambda}{\lambda}\right)\cdot\nabla\beta\left(u_\lambda-\frac{w_\lambda}{\lambda}\right)dx\,dt=$$

$$=\varliminf_{\lambda\to+\infty}\left\|\beta\left(u_\lambda-\frac{w_\lambda}{\lambda}\right)\right\|^2_{L^2(0,T;V)}\geqslant(\text{by } 3.25)\ \|\beta(u)\|^2_{L^2(0,T;V)}=$$

$$=\iint_Q \nabla u\cdot\nabla\beta(u)\,dx\,dt; \tag{3.31}$$

thus taking the upper limit as $\lambda\to+\infty$ in (3.28) we get (1.13). ■

4. Other results

Let $u_{1j}\leqslant 0\leqslant u_{2j}$ for every $j\in N$; let $u_{ij}\to 0$ (in R) as $j\to\infty$, for $i=1,2$; accordingly for every $j\in N$ define α_j and β_j similarly to (1.1) and (1.2), define also (P_j) as (P), with α and β replaced by α_j and β_j.

THEOREM 3: *For every $j \in N$, let u_j be a solution of (P_j). Assume that $g \in L^2(Q)$. Then*

$$\int_0^t u_j(x, \tau)\, d\tau \to U \quad in \quad H^1(0, T; L^2(D)) \cap L^2(0, T; H^2(D)) \; weak, \tag{4.1}$$

where U is the unique solution of the following variational inequality
(VI): — *Find $U \in H^1(0, T; L^2(D)) \cap L^2(0, T; H^2(D) \cap V)$ such that*

$$\iint_Q (U_t - \Delta U - g)(v - U_t)\, dx\, dt + \iint_Q (|v| - |U_t|)\, dx\, dt \geqslant 0, \; \forall v \in L^2(0, T; V). \tag{4.2}$$

$$U(0) = 0 \quad a.e. \quad in \; D. \tag{4.3}$$

REMARK: (VI) is a weak formulation of the classical Stefan problem (cf. [1]).

Proof: For every $j \in N$, set $U_j(x, t) = \int_0^t u_j(x, t)\, d\tau$ a.e. in Q; the corresponding (1.12) yields

$$\| U_{jt} - \Delta U_j \|_{L^2(Q)} \leqslant \text{Const}. \tag{4.4}$$

whence (cf. [5], chap. 4)

$$\| U_j \|_{H^1(0, T; L^2(D)) \cap L^2(0, T; H^2(D) \cap V)} \leqslant \text{Const}. \tag{4.5}$$

Therefore there exists U such that, possibly taking a subsequence,

$$U_j \to U \quad \text{in} \quad H^1(0, T; L^2(D)) \cap L^2(0, T; H^2(D) \cap V) \; \text{weak} \tag{4.6}$$

whence, as $\alpha_j \to$ Identity uniformly in $\boldsymbol{R}$,

$$\alpha_j(U_{jt}) \to U \quad \text{in} \quad L^2(Q) \; \text{strong}. \tag{4.7}$$

For every $j \in N$, multiply the corresponding (1.12) against $v - u_j$ and take the upper limit as $j \to \infty$; notice that for any choice of $\varphi_j : Q \to \boldsymbol{R}$ measurable such that $\varphi_j \in S(\alpha_j(u_j))$ a.e. in Q, by (4.7)

$$\overline{\lim_{j\to\infty}} \iint_Q \varphi_j (v - u_j)\, dx\, dt = \overline{\lim_{j\to\infty}} \iint_Q \varphi_j [v - \alpha_j(u_j)]\, dx\, dt +$$
$$+ \overline{\lim_{j\to\infty}} \iint_Q \varphi_j [\alpha_j(u_j) - u_j]\, dx\, dt \leqslant$$
$$\leqslant \overline{\lim_{j\to\infty}} \iint_Q [|v| - |\alpha_j(u_j)|]\, dx\, dt + 0 \leqslant (\text{by } (4.7)) \leqslant \iint_Q [|v| - |U_t|]\, dx\, dt; \tag{4.8}$$

thus we get (4.2). As the solution of (VI) is unique, the whole sequence $\{U_j\}$ converges to U. ■

PROPOSITION 1: *Assume that* (1.9), ..., (1.11) *hold. If*

$$f \leqslant 0 \quad in \quad \mathscr{D}'(Q), \quad u_2 \geqslant 0, \quad u^0 \leqslant u_2 \quad a.e. \quad in \; D \tag{4.9}$$

then for any solution u of problem (P)

$$u \leqslant u_2 \quad a.e. \quad in \; Q. \tag{4.10}$$

Similarly if $f \geqslant 0$ in $\mathscr{D}'(Q)$, $u_1 \leqslant 0$ and $u_1 \leqslant u^0$ a.e. in D, then $u_1 \leqslant u$ a.e. in Q.

Proof: Assume that (4.9) holds. Let $t \in]0, T]$ and $k \in]0, T-t]$. By (1.12) (cf. also (1.15)) we get

$$\int_D \left\{ \frac{[u(t+k)-u(t)]}{k} [u(t+k)-u_2]^+ + \right.$$

$$\left. + \frac{l}{k} \left(\nabla \int_t^{t+k} u(x,\tau)\, d\tau \right) \cdot \nabla [u(t+k)-u_2]^+ \right\} dx -$$

$$- {}_{V'}\left\langle \frac{l}{k} \int_t^{t+k} f(\tau)\, d\tau, [u(t+k)-u_2]^+ \right\rangle_V =$$

$$= \int_D (\varphi' - \varphi'') [u(t+k)-u_2]^+ \, dx, \qquad (4.11)$$

with $\varphi' \in S(\alpha(u(t)))$, $\varphi'' \in S(\alpha(u(t+k)))$. The last term is non-positive, as it is easy to check; taking $k \to 0$ we get

$$\int_D \left\{ \frac{1}{2} \frac{\partial}{\partial t} [(u(t)-u_2)^+] + |\nabla (u(t)-u_2)^+|^2 \right\} dx =$$

$$= {}_{V'}\langle f(t), (u(t)-u_2)^+ \rangle_V \leqslant 0, \qquad (4.12)$$

whence we get the thesis after integration w.r.t. t. ■

Uniqueness of solution of problem (P) is an open question. We are only able to prove the following result.

PROPOSITION 2: *Assume that* (1.9), ..., (1.11) *hold and that*

$$f=0 \quad a.e. \ \ in\ Q; \quad u_1 \leqslant 0 \leqslant u_2, \quad u_1 \leqslant u^0 \leqslant u_2 \quad a.e.\ in\ D. \qquad (4.13)$$

Then problem (P) *has at most one solution.*

Proof: By proposition 1, for any solution of (P) we have $u_1 \leqslant u \leqslant u_2$ a.e. in Q; therefore $\beta(u) = u$ and (1.13) reduces to a standard variational inequality, having at most one solution. ■

GENERALIZATIONS: The above developments can be generalized in many ways.

V can be replaced by a Dirichlet space, other boundary conditions may be taken into account. If the constants u_1, u_2 are replaced by two functions $u_i \in L^\infty(D) \cap V$ $(i=1, 2)$ with $u_1 \leqslant u_2$, then the above results still hold.

It is also possible to modify the constitutive relation (1); in some cases this is equivalent to replacing $\frac{\partial}{\partial t}(u+w)$ by $\frac{\partial}{\partial t}[a(u)+w]$ in (2), with $a: R \to R$ non-decreasing. More generally the linear operator $\frac{\partial}{\partial t} - \Delta$ can be replaced by a non-linear parabolic one of the form $u \mapsto \frac{\partial}{\partial t} a(u) - \Delta b(u)$; if $0 < a_1 < a < a_2 < +\infty$, $0 < b_1 < b < b_2 < +\infty$ (with a_1, a_2, b_1, b_2 constant) in a neighbourhood of $[u_1, u_2]$, under regularity assumptions the above results can be extended.

One may also couple (1) with the following hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2}+\frac{\partial w}{\partial t}-\Delta u=f \quad \text{in} \quad Q\,; \tag{4.14}$$

an existence result for the corresponding weak formulation does not seem immediate. Notice that for approximating problem obtained by smoothing the jumps of (1) existence is a consequence of the results of [6].

References

[1] DUVAUT G. The solution of a two-phase Stefan problem by a variational inequality. [in]: Moving Boundary Problems in Heat Flow and Diffusion J. R. Ockendon and W. R. Hodgkins, Eds. Oxford Clarendon Press 1975, 173–181.

[2] KRASNOSEL'SKII M. A. Equations with nonlinearities of hysteresis type — VII Int. Konf. Nichtlineare Schwing, Berlin 1975, Bd I, 1; Abh. Akad. Wiss. DDR, Jahrg. 1977, 3N. 437–458 (Russian) — English abstract in Zentralblatt für Mathematik 406–93032, 1977.

[3] KRASNOSEL'SKII M. A., POKROVSKII A. V. Operators representing nonlinearities of hysteresis type [in]: Theory of operators in functional spaces. G. P. Akilov, Ed. Novosibirsk, Nauka 1977. (Russian).

[4] LIONS J. L. Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires Paris Dunod, Gauthier-Villars 1969.

[5] LIONS J. L., MAGENES E. Non-Homogeneous Boundary Value Problems — vol. I, Grund. Math. Wiss. **181** Berlin Springer 1972.

[6] VISINTIN A. A model for hysteresis of distributed systems — (to appear).

[7] VISINTIN A. Hystérésis dans les systèmes distribués. Paris, C. R. Acad. Sc. **293** (1981), Série I, 625–628.

Received, December 1981

Zagadnienie przemiany fazowej z opóźnieniem

Do funkcji skokowej występującej w słabym sformułowaniu zagadnienia Stefana wprowadzone zostaje opóźnienie, co odpowiada uwzględnieniu zjawisk przechłodzenia i przegrzania. Udowodniony zostaje wynik dotyczący istnienia rozwiązania.

Проблема фазового перехода с запаздыванием

Вводится запаздывание в скачкообразную функцию выступающую в слабой формулировке проблемы Стефана, что похоже на ситуацию характеристическую для явлений переохлаждения и перенагрева.

Доказывается существование решения проблемы.