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A singularly perturbed optimal control problem with fixed final state and constrained control

by

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An optimal control problem for a linear system with a small parameter in the derivatives and with an integral performance index is considered. The initial and the final states are fixed, the feasible controls belong to a convex and compact set. The convergence properties of the optimal performance are studied.

1. Introduction

Consider the following optimal control problem:
minimize the functional

$$J(x(\cdot), y(\cdot), u(\cdot))=\int_0^T f(x(t), y(t), u(t), t)\, dt \tag{1}$$

subject to the constraints

$$\begin{aligned} \dot{x}&=A_1(t)\,x+A_2(t)\,y+B_1(t)\,u(t),\\ \lambda\dot{y}&=A_3(t)\,x+A_4(t)\,y+B_2(t)\,u(t), \end{aligned} \tag{2}$$

$$x(0)=x^0, \quad x(T)=x^1, \tag{3a}$$

$$y(0)=y^0, \quad y(T)=y^1, \tag{3b}$$

$$u(\cdot)\in U=\{u(\cdot)\in L_\infty(R^r, (0, T)),\ u(t)\in V\subset R^r \text{ a.e. } t\in(0, T)\}, \tag{4}$$

where $x(t)\in R^m$, $y(t)\in R^n$, the final time T is fixed, λ is a "small" positive scalar, which represents the singular perturbation. Assuming that $A_4^{-1}(t)$ exists, for $\lambda=0$ the system (2) becomes

$$\dot{x}=A_0(t)\,x+B_0(t)\,u(t), \tag{5a}$$

$$y(t)=-A_4^{-1}(t)\,(A_3(t)\,x(t)+B_2(t)\,u(t)), \tag{5b}$$

where $A_0=A_1-A_2A_4^{-1}A_3$, $B_0=B_1-A_2A_4^{-1}B_2$. The so-called reduced problem will have the form: minimize (1) subject to (3a), (4) and (5).

Recently, the order reduction of control systems is extensively studied, because it is connected with the simplification of system models, cf. e.g. [1], [2]. The present paper is related to the results of [3], where a similar problem but with unconstrained controls and a stiictly convex functional is analysed.

2. Convergence

The main result of this Section, given in Theorem 1 is obtained under the following conditions:

(i) The components of the matrices A_1, A_2, B_1 are in $L_1(0, T)$, and the components of A_3, A_4, B_2 are in $C[0, T]$. The eigenvalues of the matrix $A_4(t)$ have negative real parts for all $t\in[0, T]$.

(ii) The set V is convex and compact.

(iii) The function f is continuous, the functional J is lower semicontinuous in the uniform topology for x and in the weak L_2 topology for (y, u) (for sufficient conditions see [4]).

(iv) Let P be the attainable set for the system (5a) and

$$R=\int_0^{+\infty}\exp\big(A_4(T)s\big)B_2(T)\,V ds,$$

where the integral of the multifunction is in the sense of Aumann,

Then $x^1\in \operatorname{Int}P$, $y^1+A_4^{-1}(T)A_3(T)x^1\in\operatorname{Int}R$. (5)

Observe that from (i) the set R is well-defined convex and compact set. The sets P and R have nonempty interiors if, for example Int $V\neq\emptyset$, the system (5a) is controllable and the pair $\big(A_4(T), B_2(T)\big)$ satisfies the Kalman rank condition.

By a standard argument, cf. e.g. [4], there exists an optimal control $\hat{u}_0(\cdot)$ for the reduced problem. Moreover, from Lemma 1 in [6] we conclude that for small λ the target point (x^1, y^1) belongs to the attainable set of the system (2), hence there exists an optimal control $\hat{u}_\lambda(\cdot)$ for the perturbed problem. In the sequel it is assumed that λ is sufficiently small. By $\omega(g, \delta)_1$ we denote the modulus of the continuity of the function $g(\cdot)$ in $L_1(0, T)$.

We present first an auxiliar result, which, however, turns out to be of independent interest.

LEMMA 1: *Consider the reduced system* (5a). *There exist numbers* $\varepsilon_0>0$ *and* $c_0>0$ *such that for every* $\varepsilon\in(0, \varepsilon_0)$ *and* $x'_\varepsilon, x''_\varepsilon\in R^m$, $|x'_\varepsilon-x^1|<\varepsilon$, $|x''_\varepsilon-x^1|<\varepsilon$, *if the feasible control* $u'_\varepsilon(\cdot)$ *drives the state to* x'_ε, *then there exists a feasible control* $u''_\varepsilon(\cdot)$, *which drives the state to* x''_ε, *such that the set*

$$M_\varepsilon=\{t\in[0, T],\ u'_\varepsilon(t)\neq u''_\varepsilon(t)\}$$

consists of no more than $m+1$ *intervals* $\Delta_\varepsilon^1, \ldots, \Delta_\varepsilon^{m+1}$, meas $\bigcup_{i=1}^{m+1} \Delta_\varepsilon^i \leqslant c_0\, \varepsilon$. Moreover

$$\omega(u_\varepsilon'', \delta)_1 \leqslant \omega(u_\varepsilon', \delta)_1 + c_0\, \varepsilon \tag{6}$$

for $\delta \in (0, \varepsilon]$.

Proof: Since $x^1 \in \operatorname{Int} P$ there exists a simplex $S \subset P$ such that $x^1 \in \operatorname{Int} S$. Each of the vertices x_s of S can be reached by a feasible control $u_s(\cdot)$. Let $u_s^h(\cdot)$ be a piecewice constant approximation of $u_s(\cdot)$ with a step length h. Obviously, the corresponding final state x_s^h converges to x_s. Then for some sufficiently small but fixed h the point x^1 belongs to the interior of the simplex S^h with vertices x_s^h. By the representation theorem, there exists $\varepsilon_1 > 0$ such that if $|x - x^1| < \varepsilon_1$, then x is a convex combination of x_s^h. Hence, each x, belonging to the ε_1-neighbourhood of x^1, can be reached by means of a convex combination of $m+1$ piecewise constant controls. We proved that there exists a ε_1-neighbourhood of x^1, each point of which can be reached by a control of variation, bounded uniformly in this ε_1-neighbourhood.

Let S^m be the unit sphere in R^m, and let $\varepsilon_0 \in (0, \varepsilon_1)$ be fixed. There exists $\alpha > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$, $x \in R^m$, $|x - x^1| < \varepsilon$, and $l \in S^m$ the point $x + \alpha l$ satisfies $|x + \alpha l - x^1| < \varepsilon_1$. Let $\varepsilon \in (0, \varepsilon_0)$, $x_\varepsilon' \in R^m$, $x_\varepsilon'' \in R^m$, $|x_\varepsilon' - x^1| < \varepsilon$, $|x_\varepsilon'' - x^1| < \varepsilon$ be arbitrarily chosen, and let the feasible control $u_\varepsilon'(\cdot)$ correspond to x_ε'. For given $l \in S^m$ one can choose a control $u_l(\cdot) \in V$ of bounded variation (uniformly in the ε_0-neighbourhood of x^1) which drives the state of (5a) to $x_\varepsilon' + \alpha l$. Denoting $\Delta u_l(\cdot) = u_l(\cdot) - u_\varepsilon'(\cdot)$ we have

$$\int_0^T \varphi(T, t)\, B_0(t)\, \Delta u_l(t)\, dt = \alpha l,$$

where $\varphi(t, \tau)$ is the fundamental matrix solution of (5a) normalized at $t = \tau$. Hence

$$\int_0^T l^* \varphi(T, t)\, B_0(t)\, \Delta u_l(t)\, dt = \alpha, \tag{7}$$

where * denotes the transposition. Let $\delta = \min\{3(T+1)\, \varepsilon/\alpha, 1\}$. We show that there exists an interval $\Delta_l^\varepsilon \subset [0, T]$ such that meas $\Delta_l^\varepsilon \leqslant \delta$ and

$$\int_{\Delta_l^\varepsilon} l^* \varphi(T, t)\, B_0(t)\, \Delta u_l(t)\, dt \geqslant 3\varepsilon. \tag{8}$$

Let $\Delta_1, \ldots, \Delta_p$ be a covering of $[0, T]$ such that meas $\Delta_i \leqslant \delta$, $p = [T/\delta] + 1$. If (8) does not hold for each Δ_i then

$$\int_0^T l^* \varphi(T, t)\, B_0(t)\, \Delta u_l(t)\, dt < 3p\varepsilon \leqslant 3\left(\frac{T}{\delta} + 1\right)\varepsilon \leqslant \frac{3(T+1)\,\varepsilon}{\delta\alpha}\,\alpha \leqslant \alpha,$$

which contradicts (7). Introduce the control

$$u_l^\varepsilon(t) = \begin{cases} u_\varepsilon'(t) & \text{for } t \in [0, T] \setminus \Delta_l^\varepsilon, \\ u_l(t) & \text{for } t \in \Delta_l^\varepsilon. \end{cases}$$

Obviously $u_l^\varepsilon(\cdot)\in U$ and the corresponding final state x_l^ε from (5a) satisfies

$$l^*(x_l^\varepsilon - x_\varepsilon')\geqslant 3\varepsilon. \tag{9}$$

Define the set

$$\Omega_\varepsilon = co\,\{x_l^\varepsilon \in R^m,\ l\in S^m\}.$$

From (9) it follows that for each $l\in S^m$

$$\sup_{x\in\Omega_\varepsilon} l^*(x-x_\varepsilon')\geqslant 3\varepsilon. \tag{10}$$

We shall prove that

$$B_\varepsilon=\{x\in R^m,\ |x-x_\varepsilon'|<2\varepsilon\}\subset\Omega_\varepsilon. \tag{11}$$

Assume that there exists $\bar{x}\in B_\varepsilon$ such that $\bar{x}\notin\Omega_\varepsilon$. Since Ω_ε is convex, by the separation theorem there exists $\bar{l}\in S^m$ such that

$$\bar{l}^*x\leqslant\bar{l}^*\bar{x},\quad \forall x\in\Omega_\varepsilon.$$

Then

$$\sup_{x\in\Omega_\varepsilon}\bar{l}^*(x-x_\varepsilon')\leqslant\bar{l}^*(\bar{x}-x_\varepsilon')$$

and, taking advantage of (10), we obtain

$$3\varepsilon\leqslant\bar{l}^*(\bar{x}-x_\varepsilon')\leqslant|\bar{x}-x_\varepsilon|<2\varepsilon.$$

This contradiction gives us (11). Thus, since $x_\varepsilon''\in B_\varepsilon$, then $x_\varepsilon''\in\Omega_\varepsilon$. By the representation theorem x_ε'' can be reached by means of a control $u_\varepsilon''(\cdot)$, being a convex combination of $m+1$ controls from the set $\{u_l^\varepsilon(\cdot),\ l\in S^m\}$. The function $\Delta u(\cdot)=$ $=u_\varepsilon''(\cdot)-u_\varepsilon'(\cdot)$ differs from zero on a union of no more than $m+1$ intervals Δ_l^ε with full measure less than $(m+1)\,\delta\leqslant 3\varepsilon\,(m+1)\,(T+1)/\alpha$. On each interval Δ_l^ε the variation of $\Delta u(\cdot)$ is bounded by a constant, which depends on ε_0 only. By the known estimate $\omega\,(g,\gamma)_{L_1(0,\beta)}\leqslant\gamma V_0^\beta g(\cdot)$ for $\gamma\in[0,\beta]$, we conclude that there exists a constant K such that $\omega\,(\Delta u,\delta)_1\leqslant k\varepsilon$. Taking $c_0=\max\{k,\ 3\,(m+1)\,(T+1)/\alpha\}$ we complete the proof.

Observe that the above result is valid without any controllability conditions, i.e. the interior of the attainable set P can be replaced by its relative interior. Moreover, the boundedness of V is not used.

Denote by $\hat{x}_0(\cdot)$ the optimal state and by $\hat{J}_0$ the optimal performance for the reduced problem. Let $\hat{y}_0(\cdot)$ be determined by $\hat{x}_0(\cdot)$ and $\hat{u}_0(\cdot)$ from (5b). The optimal solution of the perturbed problem is denoted by $(\hat{x}_\lambda(\cdot),\hat{y}_\lambda(\cdot),\hat{u}_\lambda(\cdot))$ and $\hat{J}_\lambda$.

THEOREM 1.

$$\lim_{\lambda\to 0}\hat{J}_\lambda=\hat{J}_0$$

Proof: Let the sequence $\{\lambda_k\}$, $\lim\limits_{k\to+\infty}\lambda_k=0$, be arbitrarily chosen. Define the set

$$K=\{(x,y)\in R^{m+n},\ x\in P,\ y\in -A_4^{-1}(T)\,A_3(T)\,x+R\}.$$

From (iv) it follows that $(x^1, y^1) \in \operatorname{Int} K$, hence for $\varepsilon > 0$ sufficiently small there exists a simplex $G = \{(x^1_\varepsilon, y^1_\varepsilon), \ldots, (x^p_\varepsilon, y^p_\varepsilon)\}$, $p = m + n + 1$, such that $G \subset K$, $|x^i_\varepsilon - x^1| < \varepsilon$, $i = 1, \ldots, p$, and for some $d > 0$ $\{(x, y) \in R^{m+n}, |x - x^1| \leqslant d\varepsilon, |y - y^1| \leqslant d\varepsilon\} \subset G$. Using Lemma 1 we conclude that there exist controls $u^i_\varepsilon(\cdot) \in V$, $i = 1, \ldots, p$, such that the corresponding states of (5a) satisfy $x(T) = x^i_\varepsilon$, $i = 1, \ldots, p$, and $u^i_\varepsilon(t) = \hat{u}_0(t)$ for a.e. $t \in [0, T] \setminus M^i_\varepsilon$, where meas $M^i_\varepsilon \leqslant c_0 \varepsilon$, $i = 1, \ldots, p$. On the other hand there exist functions $v^i_\varepsilon(\cdot)$, $v^i_\varepsilon(t) \in V$ for a.e. $t \in [0, +\infty)$, $i = 1, \ldots, p$, such that

$$y^i_\varepsilon = -A_4^{-1}(T) A_3(T) x^i_\varepsilon + \int_0^{+\infty} \exp(A_4(T) s) B_2(T) v^i_\varepsilon(s) \, ds$$

for $i = 1, \ldots, p$. Let the sequence $\{\varphi_k\}$ satisfy $\lim\limits_{k \to +\infty} \varphi_k = 0$, $\lim\limits_{k \to +\infty} \varphi_k / \lambda_k = +\infty$, $\varphi_k > 0$, $k = 1, 2, \ldots$, Define the control

$$\tilde{u}^i_k(t) = \begin{cases} u^i_\varepsilon(t) & \text{for } t \in [0, T - \varphi_k), \\ v^i_\varepsilon\left(\dfrac{T - t}{\lambda_k}\right) & \text{for } t \in [T - \varphi_k, T], \end{cases}$$

and let $(\tilde{x}^i_k(\cdot), \tilde{y}^i_k(\cdot))$ be the corresponding state in (2) for λ_k. In Lemma 1 of [6] it is proved that $\lim\limits_{k \to +\infty} \tilde{x}^i_k(T) = x^i_\varepsilon$ and $\lim\limits_{k \to +\infty} \tilde{y}^i_k(T) = y^i_\varepsilon$. Then for k sufficiently large

$$(x^1, y^1) \in co\,\{(\tilde{x}^i_k(T), \tilde{y}^i_k(T))\}_i.$$

Hence there exist numbers $\alpha^1_k, \ldots, \alpha^p_k \geqslant 0$, $\sum\limits_{i=1}^{p} \alpha^i_k = 1$, such that if $\bar{u}^\varepsilon_k(\cdot) = \sum\limits_{i=1}^{p} \alpha^i_k \tilde{u}^i_k(\cdot)$ and $(\bar{x}^\varepsilon_k(\cdot), \bar{y}^\varepsilon_k(\cdot))$ correspond to $\bar{u}^\varepsilon_k(\cdot)$ and λ_k in (2), then, for k sufficiently large

$$\bar{x}^\varepsilon_k(T) = x^1, \quad \bar{y}^\varepsilon_k(T) = y^1.$$

Without loss of generality, let $\lim\limits_{k \to +\infty} \alpha^i_k = \alpha^i$, $\sum\limits_{i=1}^{p} \alpha^i = 1$. Define $\bar{u}^\varepsilon_0(\cdot) = \sum\limits_{i=1}^{p} \alpha^i u^i_\varepsilon(\cdot)$.

It is clear that $\bar{u}^\varepsilon_0(t) = \hat{u}_0(t)$ for a.e. $t \in [0, T] \setminus \bigcup\limits_{i=0}^{p} M^i_\varepsilon$ and $\bar{u}^\varepsilon_k(\cdot)$ is pointwise convergent to $\bar{u}^\varepsilon_0(\cdot)$. Let $\bar{x}^\varepsilon_0(\cdot)$, $\bar{y}^\varepsilon_0(\cdot)$ correspond to $\bar{u}^\varepsilon_0(\cdot)$ from (5a, b). Applying Lemma 2.1 (iv) from [7] we get that $\bar{x}^\varepsilon_k(\cdot)$ converges to $\bar{x}^\varepsilon_0(\cdot)$ uniformly in $[0, T]$ and $\bar{y}^\varepsilon_k(\cdot)$ is strongly L_2 convergent to $\bar{y}^\varepsilon_0(\cdot)$. Without loss of generality one can consider $\{\bar{y}^\varepsilon_k(\cdot)\}$ as a pointwise convergent sequence. Since $\bar{u}^\varepsilon_k(\cdot)$ and $\bar{y}^\varepsilon_k(\cdot)$ are uniformly bounded, by Lebesque's theorem we obtain

$$\lim_{k \to +\infty} \sup \hat{J}_{\lambda_k} \leqslant \lim_{k \to +\infty} J(\bar{x}^\varepsilon_k(\cdot), \bar{y}^\varepsilon_k(\cdot), \bar{u}^\varepsilon_k(\cdot)) = J(\bar{x}^\varepsilon_0(\cdot), \bar{y}^\varepsilon_0(\cdot), \bar{u}^\varepsilon_0(\cdot)). \tag{12}$$

Now choose a sequence $\{\varepsilon_i\}$, $\lim\limits_{i \to +\infty} \varepsilon_i = 0$. Then $\bar{u}^{\varepsilon_i}_0(\cdot)$ is convergent in measure to $\hat{u}_0(\cdot)$, hence it can be identified with a pointwise convergent sequence. The trajectory $\bar{x}^{\varepsilon_i}_0(\cdot)$ is uniformly convergent in $[0, T]$ to $\hat{x}_0(\cdot)$ and $\bar{y}^{\varepsilon_i}_0(\cdot)$ is pointwise convergent to $\hat{y}_0(\cdot)$. Hence

$$\lim_{i \to +\infty} J(\bar{x}^{\varepsilon_i}_0(\cdot), \bar{y}^{\varepsilon_i}_0(\cdot), \bar{u}^{\varepsilon_i}_0(\cdot)) = \hat{J}_0. \tag{13}$$

On the other hand one can assume that the sequence of the optimal controls $\{\hat{u}_{\lambda_k}(\cdot)\}$ is weakly L_2 convergent to some $u_0(\cdot)\in U$. From Lemma A in [6] we conclude that the optimal state $\hat{x}_{\lambda_k}(\cdot)$ is uniformly convergent to $x_0(\cdot)$, which satisfies (3a) and (5a) for $u_0(\cdot)$, and $\hat{y}_{\lambda_k}(\cdot)$ is weakly L_2 convergent to $y_0(\cdot)$, which satisfies (5b) for $x_0(\cdot)$ and $u_0(\cdot)$. Hence, by (iii)

$$\hat{J}_0 \leqslant J(x_0(\cdot), y_0(\cdot), u_0(\cdot)) \leqslant \lim_{k\to+\infty} \inf \hat{J}_{\lambda_k}. \tag{14}$$

Combining (12) and (13) and taking into account (14) we conclude that from every sequence $\{\lambda_k\}$ one can extract a subsequence $\{\lambda_l\}$ for which

$$\lim_{l\to+\infty} \hat{J}_{\lambda_l} = \hat{J}_0 .$$

This completes the proof.

3. Estimates

In this section we develop the result from the previous section obtaining an estimate of the performance convergence rate. We assume that all the conditions (i) theough (iv) hold and, moreover.

(v) The components of the matrices A_2, A_3, A_4, B_2 are in $C_1[0, T]$. The function f is Lipschitz continuos with respect to (x, y) uniformly in $(u, t)\in V\times[0, T]$ for bounded (x, y).

LEMMA 2. *Let $u'_\lambda(\cdot), u''_\lambda(\cdot)$ be feasible controls, $(x'_\lambda(\cdot), y'_\lambda(\cdot))$ solves (2) for $u'_\lambda(\cdot)$, and $(x''_\lambda(\cdot), y''_\lambda(\cdot))$ is the solution of (5) for $u''_\lambda(\cdot)$. There exists a constant c_1, which does not depend on $u'_\lambda(\cdot)$ and $u''_\lambda(\cdot)$ such that*

$$\|x'_\lambda(\cdot)-x''_\lambda(\cdot)\|_C \leqslant c_1(\lambda+\|u'_\lambda(\cdot)-u''_\lambda(\cdot)\|_{L_\infty}),$$

$$\|y'_\lambda(\cdot)-y''_\lambda(\cdot)\|_{L_1} \leqslant c_1(\lambda+\|u'_\lambda(\cdot)-u''_\lambda(\cdot)\|_{L_\infty}+\omega(u''_\lambda, \lambda)_1).$$

Proof. Let $Y(t, \tau, \lambda)$ be the fundamental matrix solution of the equation $\lambda\dot{y}=A_4(t)\,y$. It is known that there exist constants $\sigma_0, \sigma>0$ such that $|Y(t, \tau, \lambda)|\leqslant \sigma_0 \exp(-\sigma(t-\tau)/\lambda)$ for all $t, \tau\in[0, T]$, $t\geqslant\tau$. Let $A_4(t)\,y''_\lambda(t)=g_\lambda(t)$, $g_\lambda(t)=0$ for $t<0$. Then

$$\frac{1}{\lambda}\int_0^T\left|\int_0^t Y(t, \tau, \lambda)(g_\lambda(\tau)-g_\lambda(t))\,d\tau\right|dt \leqslant \frac{\sigma_0}{\lambda}\int_0^T\int_0^t \exp\left(-\sigma\frac{t-\tau}{\lambda}\right)\times$$

$$\times|g_\lambda(\tau)-g_\lambda(t)|\,d\tau\,dt \leqslant \sigma_0\int_0^T\int_0^{t/\lambda} e^{-\sigma s}|g_\lambda(t-\lambda s)-g_\lambda(t)|\,ds\,dt \leqslant \sigma_0\times$$

$$\times\int_0^{t/\lambda} e^{-\sigma s}\,\omega(g_\lambda, \lambda s)\,ds+O(\lambda) \leqslant \sigma_0\sum_{k=0}^{[T/\lambda]+1} e^{-\sigma k}\,\omega(g_\lambda, (k+1)\lambda)_1+O(\lambda)\leqslant$$

$$\leqslant \sigma_0\,\omega(g_\lambda, \lambda)_1\sum_{k=0}^{+\infty} e^{-\sigma k}(k+1)+O(\lambda) \leqslant c_2(\omega(g_\lambda, \lambda)_1+\lambda).$$

Integrating by parts we have

$$\frac{1}{\lambda}\int_0^T \left| \int_0^t y(t,\tau,\lambda)\, A_4(\tau)\, y''(\tau)\, d\tau - y_\lambda''(t) \right| dt \leqslant c_3\, (\omega(u_\lambda'', \lambda)_1 + \lambda).$$

Further the proof repeats the arguments in the proof of Lemma 2.1 (iv) from [7].

THEOREM 2: *There exists a constant c such that*

$$-c\,(\lambda+\omega(\hat{u}_\lambda, \lambda)_1) \leqslant \hat{J}_\lambda - \hat{J}_0 \leqslant c\,(\lambda+\omega(\hat{u}_0, \lambda)_1). \tag{15}$$

Proof. Although the proof, in principle, goes paralelly to the proof of Theorem 1, it differs from that in several important details. Therefore we present here the complete proof.

Choose vectors $\xi_i \in R^m$, $|\xi_i|=1$, $i=0, \dots, m$, such that $0 \in \operatorname{Int} \operatorname{co}(\xi_i)_i$. There exists $\alpha>0$ such that for every $\bar{\xi}_i \in R^m$, $i=0, \dots, m$, $|\bar{\xi}_i-\xi_i| \leqslant \alpha$, one has $0 \in \operatorname{co}(\bar{\xi}_i)_i$. Let $x_i^\varepsilon = x^1 + \varepsilon\xi_i$, $i=0, \dots, m$, where $\varepsilon>0$. From the choice of ξ_i it follows that if $|\bar{x}_i^\varepsilon - x_i^\varepsilon| \leqslant \alpha\varepsilon$ then $x^1 \in \operatorname{co}(\bar{x}_i^\varepsilon)_i$. Analogically, let $\eta_j \in R^n$, $|\eta_j|=1, j=0, \dots, m$, $0 \in \operatorname{Int} \operatorname{co}(\eta_j)_j$, and $y_j = y^1 + d\eta_j$. For sufficiently small ε and d we have $(x_i^\varepsilon, y_j) \in \operatorname{Int} K$. There exists $\beta>0$ such that if $|\bar{\eta}_j - \eta_j| \leqslant \beta$, $j=0, \dots, n$, then $0 \in \operatorname{co}(\bar{\eta}_j)$. Thus, if $|\bar{y}_j - y_j| \leqslant \beta d$, then $y^1 \in \operatorname{co}(\bar{y}_j)_j$. Using Lemma 1, for sufficiently small ε we choose a control $u_i^\varepsilon(\cdot)$, which drives the state of (5a) from x^0 to x_i^ε, and which differs from $\hat{u}_0(\cdot)$ on a set of measure $c_0\,\varepsilon$. Since $y_j \in -A_4^{-1}(T)\, A_3(T)\, x_i^\varepsilon + R$, there exists a measurable function $v_{ij}^\varepsilon(\cdot)$, $v_{ij}^\varepsilon(t) \in V$, $t \in [0, +\infty)$ such that

$$y_j = -A_4^{-1}(T)\, A_3(T)\, x_i^\varepsilon + \int_0^{+\infty} \exp(A_4(T)\, s)\, B_2(T)\, v_{ij}^\varepsilon(s)\, ds.$$

For small λ define the control

$$\tilde{u}_{ij}^\lambda(t) = \begin{cases} u_i^\varepsilon(t) & \text{for } t \in [0, T-\theta\lambda) \\ v_{ij}^\varepsilon\left(\dfrac{T-t}{\lambda}\right) & \text{for } t \in [T-\theta\lambda, T], \end{cases}$$

where $\theta \geqslant 1$ is arbitrarily chosen. If $(\tilde{x}_{ij}^\lambda(\cdot), \tilde{y}_{ij}^\lambda(\cdot))$ is the corresponding trajectory of (3) then, applying Lemma 2 we have

$$|\tilde{x}_{ij}^\lambda(T) - x_i^\varepsilon| \leqslant c_1\,(\theta+1)\,\lambda,$$

where the constant c_1 does not depend on ε, θ and λ. Using this estimate we get

$$\frac{1}{\lambda}\int_0^T \exp\left(A_4(T)\frac{T-t}{\lambda}\right) A_3(T)\, \tilde{x}_{ij}^\lambda(t)\, dt = -A_4^{-1}(T)\, A_3(T)\, \tilde{x}_{ij}^\lambda(T) +$$

$$+A_4^{-1}(T) \exp\left(A_4(T)\frac{T}{\lambda}\right) A_3(T)\, \tilde{x}_{ij}^\lambda(T) - \frac{1}{\lambda}\int_0^T\int_0^t \exp\left(A_4(T)\frac{T-s}{\lambda}\right) ds \times$$

$$\times A_3(T)\, \dot{\tilde{x}}_{ij}^\lambda(t)\, dt = -A_4^{-1}(T)\, A_3(T)\, x_i^\varepsilon + h(\theta, \lambda), \tag{16}$$

where $h(\theta, \lambda) \leqslant c_4\,(\theta+1)\,\lambda$. Denote by $\bar{y}_{ij}^\lambda(\cdot)$ the solution of the equation

$$\lambda\dot{y} = A_3(T)\, \tilde{x}_{ij}^\lambda(t) + A_4(T)\, y + B_2(T)\, \tilde{u}_{ij}^\lambda(t),\; y(0) = y^0.$$

As in Lemma 1 from [6] one can prove that $|\tilde{y}^{\lambda}_{ij}(T)-\bar{y}^{\lambda}_{ij}(T)|\leqslant\delta_\lambda$, where δ_λ does not depend on $\tilde{u}^{\lambda}_{ij}(\cdot)$ and $\lim\limits_{\lambda\to 0}\delta_\lambda=0$. Using (16) we have

$$|\tilde{y}^{\lambda}_{ij}(T)-y_j|\leqslant\delta_\lambda+|\bar{y}^{\lambda}_{ij}(T)-y_j|\leqslant\delta_\lambda+\sigma_0\exp\left(-\sigma\frac{T}{\lambda}\right)|y^0|+$$
$$+\left|-y_j-A_4^{-1}(T)A_3(T)x_i^{\varepsilon}+h(\theta,\lambda)+\frac{1}{\lambda}\int\limits_{T-\theta\lambda}^{T}\exp\left(A_4(T)\frac{T-t}{\lambda}\right)\times\right.$$
$$\left.\times B_2(T)v^{\varepsilon}_{ij}\left(\frac{T-t}{\lambda}\right)dt\right|+\left|\frac{1}{\lambda}\int\limits_0^{T-\theta\lambda}\exp\left(A_4(T)\frac{T-t}{\lambda}\right)B_2(T)u_i^{\varepsilon}(t)\,dt\right|\leqslant$$
$$\leqslant\delta_\lambda+c_5(\lambda+\theta\lambda+e^{-\sigma\theta})\leqslant\gamma(\theta)+\theta\varphi(\lambda),$$

where $\lim\limits_{\theta\to+\infty}\gamma(\theta)=0$, $\lim\limits_{\lambda\to+\infty}\varphi(\lambda)=0$, $\gamma(\cdot)$ does not depend on ε and λ and $\varphi(\cdot)$ does not depend on ε and θ. Fix θ such that $\gamma(\theta)<\beta d/2$. Then for each ε and λ satisfying

$$c_1(\theta+1)\lambda<\alpha\varepsilon,\qquad \varphi(\lambda)<\beta d/2\theta, \tag{17}$$

we have $x^1\in\mathrm{co}\,(\tilde{x}^{\lambda}_{ij}(T))_i$ for each $j=0,\dots,m$. This means that there exist $\alpha_{ij}\geqslant 0$, $\sum\limits_{i=0}^{m}\alpha_{ij}=1$, such that $x^1=\sum\limits_{i=0}^{m}\alpha_{ij}\tilde{x}^{\lambda}_j(T)$. If $\bar{\bar{y}}_j=\sum\limits_{i=0}^{m}\alpha_{ij}\tilde{y}^{\lambda}_{ij}(T)$, then $|\bar{\bar{y}}_j-y_j|\leqslant\beta d$, hence one can choose $\beta_j\geqslant 0$, $\sum\limits_{j=0}^{n}\beta_j=1$, so that $y^1=\sum\limits_{j=0}^{n}\beta_j\bar{\bar{y}}_j$. Finally, we get that $(x^1,y^1)=\sum\limits_{i=0}^{m}\sum\limits_{j=0}^{n}\alpha_{ij}\beta_j(\tilde{x}^{\lambda}_{ij}(T),\tilde{y}^{\lambda}_{ij}(T))$. Then there exists a control $u^{\varepsilon}_{\lambda}(\cdot)$, which is a convex combination of the controls $\tilde{u}^{\lambda}_{ij}(\cdot)$, and which drives the state of (3) to (x^1,y^1). Moreover, $u^{\varepsilon}_{\lambda}(\cdot)$ differs from $\hat{u}_0(\cdot)$ on a set of measure $O(\lambda+\varepsilon)$. Let ε and λ be chosen such that $\varepsilon=2c_1(\theta+1)\lambda/\alpha$. Then (17) holds and $\|u^{\varepsilon}_{\lambda}(\cdot)-\hat{u}_0(\cdot)\|_{L_\infty}=$ $=O(\lambda)$. Using Lemma 2 we obtain that the corresponding trajectory $(x_\lambda(\cdot),y_\lambda(\cdot))$ satisfies $\|x_\lambda(\cdot)-\hat{x}_0(\cdot)\|_c=O(\lambda)$ and $\|y_\lambda(\cdot)-\hat{y}_0(\cdot)\|_{L_1}=O(\lambda+\omega(\hat{u}_0,\lambda)_1)$. Since $\hat{J}_\lambda\leqslant J(x_\lambda(\cdot),y_\lambda(\cdot),u^{\varepsilon}_{\lambda}(\cdot))$, taking into account the assumption (v) we obtain the upper estimate in (15).

Now, let $x^{\lambda}_0(\cdot)$ be the solution of (5a) for $\hat{u}_\lambda(\cdot)$. By Lemma 2 we have $|x^{\lambda}_0(T)-x^1|=O(\lambda)$. Applying Lemma 1 we select a feasible control $\bar{u}_\lambda(\cdot)$ driving the system (5a) to x^1, such that $\|\bar{u}_\lambda(\cdot)-\hat{u}_\lambda(\cdot)\|_{L_\infty}=O(\lambda)$ and $\omega(\bar{u}_\lambda,\lambda)_1\leqslant$ $\leqslant\omega(\hat{u}_\lambda,\lambda)_1+c_5\lambda$. If $(\bar{x}_\lambda(\cdot),\bar{y}_\lambda(\cdot))$ solves (3) for $\bar{u}_\lambda(\cdot)$, then, by Lemma 2 we have $\|\bar{x}_\lambda(\cdot)-\hat{x}_\lambda(\cdot)\|_C=O(\lambda)$, $\|\bar{y}_\lambda(\cdot)-\hat{y}_\lambda(\cdot)\|_{L_1}=O(\lambda+\omega(\bar{u}_\lambda,\lambda)_1)=O(\lambda+\omega(\hat{u}_\lambda,\lambda)_1)$. Since $\hat{J}_0\leqslant J(\bar{x}_\lambda(\cdot),\bar{y}_\lambda(\cdot),\bar{u}_\lambda(\cdot))$, the lower estimate in (15) holds. The proof is complete.

COROLLARY 1. *Let the function f be differentiable with respect to (x, y, u) and its derivatives be continuous: the function $f_0(u,t)=f\left(\hat{x}_0(t),-A_4^{-1}(t)\left(A_3(t)\hat{x}_0(t)+\right.\right.$ $\left.\left.+B_2(t)u\right),u,t\right)$ be strongly convex with respect to u uniformly in $[0, T]$; the variations of the matrix $B_0(t)$ and the function $\frac{\partial}{\partial u}f_0(u,t)$ on $[0, T]$ be bounded uniformly in $u\in V$. Then $\hat{J}_\lambda-\hat{J}_0\leqslant c_6\lambda$.*

Proof. Using the strong convexity of the Hamiltonian and the maximum principle as in [8] one can show that the variation of $\hat{u}_0(\cdot)$ is bounded. Then the desired estimate follows from (15).

Corollary 2: *Let $f(x, y, u, t)=f_1(x, u, t)+f_2(x, t)\,y$, where $f_1(x, u, t)$ has continuous derivatives $\frac{\partial}{\partial x} f_1$, $\frac{\partial}{\partial u} f_1$, $\frac{\partial^2}{\partial x\,\partial u} f_1$, and it is strongly convex with respect to u uniformly in (x, t). The function $f_2(x, t)$ is continuously differentiable, $A_1(\cdot) \in C_1[0, T]$ and $B_1(\cdot)$ is of bounded variation. Then $\hat{J}_\lambda - \hat{J}_0 \geqslant c_8 \lambda \ln \lambda$. Moreover, if $B_2=0$ then $\hat{J}_\lambda - \hat{J}_0 \geqslant -c_9 \lambda$.*

Proof. Since for small λ the point (x^1, y^1) belongs to the interior of the attainable set of the perturbed system (e.g.cf. [6]), then the vectors of the boundary values for the adjoint variables in the maximum principle are bounded for $\lambda \to 0$. The estimation of the adjoint variables leads to the equation

$$\lambda \dot{z} = A_4(t)\, z + p_\lambda(t), \qquad z(T) = q_\lambda/\lambda,$$

where $\limsup_{\lambda \to 0} \left(|q_\lambda| + \max_{0 \leqslant t \leqslant T+\lambda \ln \lambda} |\dot{p}_\lambda(t)|\right) < +\infty$. Its solution has bounded variation on the interval $[0, T+\lambda \ln \lambda]$. Then, using the strong convexity of the hamiltonian in the maximum principle one can get

$$\omega(\hat{u}_\lambda, \lambda)_1 \leqslant \omega(\hat{u}_\lambda, \lambda)_{L_1(0,\, T+\lambda \ln \lambda)} - 2\lambda \ln \lambda \|\hat{u}_\lambda(\cdot)\|_{L_\infty} \leqslant -c_{10}\, \lambda \ln \lambda .$$

We note that there are examples for which the variation of $\hat{u}_\lambda(\cdot)$ is unbounded for $\lambda \to 0$. If $B_2=0$, the adjoint variables which reduce, are not involved in the maximum principle. In this case the estimate $O(\lambda)$ is exact for the class of problems considered.

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Pewne zadanie sterowania optymalnego dla układu z osobliwością przy ograniczeniach na sterowanie i przy ustalonym stanie końcowym

Rozważane jest zadanie sterowania optymalnego dla układu liniowego z małym parametrem przy pochodnych i z całkowym funkcjonałem jakości. Stan początkowy i stan końcowy są ustalone, sterowania dopuszczalne są elementami wypukłego zbioru zwartego. Badana jest zbieżność optymalnej wartości funkcjonału jakości w funkcji małego parametru.

Некоторая задача оптимального управления для систем с особенностью при ограничениях по управлению и зафиксированном конечном состоянии

Рассматривается задача оптимального управления для линейной системы с малым параметром для производных и интегральным функционалом качества. Начальное и конечное состояния зафиксированы, допустимые управления являются элементами выпуклого компактного множества. Исследуется сходимость оптимального значения функционала качества в функции малого параметра.