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Necessary Conditions for a Minmax Control Problem

by

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The paper concerns an optimal control problem with uncertainity in the state equations. Necessary conditions for a control to be a minmax open-loop control are presented.

1. Introduction

In the paper we consider an optimal control problem where uncertain, time — varying parameters appear in the state equations. We assume that the nature or an opposer may choose values of the uncertain parameters to maximize the cost which the controller is attempting to minimize. There is sought a control which achives the best guarenteed performance; that is, a minmax control. We adopt the open—loop model; that is, we assume that control functions depend only on time, no feedbacks are allowed.

Necessary conditions for a minmax control presented below reduce in the case with no uncertainly to the local maximum principle. Such and similar minmax control problems have been considered among others in [1], [2], [6]-[12].

2. Problem Formulation

Consider the following control system:

$$\begin{cases} \dot{x}(t)=f(t, x(t), u(t), v(t)) \\ x(0)=x_0 \end{cases} \tag{2.1}$$

where $t \in [0, 1]$, the state $x(t) \in R^m$, the control $u(t) \in U \subset R^p$, the uncertain parameters $v(t) \in V \subset R^q$.

We assume that control functions $u(t)$ and uncertain functions $v(t)$ are measurable functions defined on the interval $[0, 1]$ and the sets U and V in which they are bound to take values are given compact subsets of euclidean spaces. Assume moreover that U is convex. Denote by $\mathcal{M}$ and $\mathcal{N}$, respectively, the sets of all control functions and uncertain functions.

Assume that the right — hand side f: $[0, 1]\times R^m\times U\times V\to R^m$ of the state equation satisfies the following conditions;

(A_1) $f(\cdot, x, u, v)$ is measurable in t for each fixed (x, u, v) $f(t, \cdot, \cdot, \cdot)$ is continuous in (x, u, v) for each fixed t.

(A_2) The derivatives $f_x(t, x, u, v)$, $f_u(t, x, u, v)$ exist and are continuous in (x, u, v) for each fixed t.

(A_3) For every $r>0$ there exists a function $k_r(t)$, $k_r(t)\in L^1$ [0, 1], such that:

$$|f(t, x_1, u, v)-f(t, x_2, u, v)|\leqslant k_r(t)\,|x_1-x_2|$$

for all $t\in[0, 1]$, $u\in U$, $v\in V$ and x_1, x_2 from the ball $B(0, r)=\{x\in R^m \,|\, |x|\leqslant r\}$.

(A_4) There exist functions $a(t)$, $b(t)$ both belonging to L^1 [0, 1] such that:

$$|f(t, x, u, v)|\leqslant a(t)\,|x|+b(t)$$

for all $x\in R^m$, $u\in U$, $v\in V$.

Conditions (A_1)–(A_4) imply that all trajectories of (2.1) are uniformly bounded i.e. lie in a ball $B(0, \bar{r})$ in R^m. Assume moreover that:

(A_5) There exists a function $l(t)\in L^1$ [0, 1] such that:

$$|f_x(t, x, u, v)|\leqslant l(t), \qquad |f_u(t, x, u, v)|\leqslant l(t)$$

for all $x\in B(0, \bar{r})$, $u\in U$, $v\in V$, $t\in[0, 1]$.

By L^1 [0, 1] above we denote the space of integrable real functions on [0, 1].

Let h: $R^m\to R$ be a given locally Lipschitz function. Consider the following problem with the terminal cost function h:

$$\min_{u(\cdot)\in\mathcal{M}}\ \sup_{v(\cdot)\in\mathcal{N}}\ h(x_{u(\cdot), v(\cdot)}(1))$$

where $x_{u(\cdot), v(\cdot)}(\cdot)$ denotes the trajectory of (2.1). Corresponding to the pair $u(\cdot)$, $v(\cdot)$. We seek for a necessary condition for a control $u_*(t)$ to be a minmax control; that is to satisfy the following relation:

$$\sup_{v(\cdot)\in\mathcal{N}}\ h(x^*_{u(\cdot)) v(\cdot)}(1))=\min_{u(\cdot)\in\mathcal{M}}\ \sup_{v(\cdot)\in\mathcal{N}}\ h(x_{u(\cdot), v(\cdot)}(1)). \tag{2.2}$$

In other words $u_*(t)$ minimizes over the set $\mathcal{M}$ the functional $I(u(\cdot))$ defined for all $u(\cdot)\in\mathcal{M}$ as follows:

$$I(u(\cdot))=\sup_{v(\cdot)\in\mathcal{N}}\ h(x_{u(\cdot), v(\cdot)}(1)). \tag{2.3}$$

Control problems with uncertainities were considered in many papers, some of them are quoted at the end. In [1], [6] uncertainity appears in the initial state. This case can be transformed to the form (2.1) with the additional condition that functions $v(t)$ are constant. Such a problem is essentially easier as uncertainity is finite dimensional. The papers [1], [6] contain necessary conditions for minmax controls in this case. Finite dimensional uncertainity is treated also in [2], [12]. Uncertanities in the state equation of the form (2.1) are considered in [9] where sufficient conditions for minmax controls are given. Necessary conditions for such problems are presented in [8] for a linear case and in [7], [10] for nonlinear. The conditions in [7], [10] corespond to the maximum principle and are obtained under additional assumptions. Namely, in [10] there is assumed that a minmax control $u(\cdot)$ under consideration remains optimal when the problem is convexified in u what for minmax problems often does not hold, while in [7], roughly speaking, there is assumed that for, u^* the solution v^* of the corresponding maximization problem is unique. The cost function h is usually assumed to be C^1, while we require it to be locally Lipschitz only.

3. The main theorem

We start with relaxing the problem in the parameter v. We follow the procedure described by Warga in [11].

Denote by $C(V)$ the Banach space of continuous real functions defined on V and its norm by $|\cdot|_{\sup}$.

Let $\mathscr{B}=\mathscr{B}([0,1], V; R)$ denotes the vector space of equivalence classes of functions $\varphi\colon [0,1]\times V\to R$ such that $\varphi(\cdot, v)$ is Lebesgue measurable on $[0, 1]$ for each $v\in V$ $\varphi(t, \cdot)\in C(V)$ for $t\in[0,1]$ and moreover there exists a function $\psi_\varphi(\cdot)\in$ $\in L^1[0,1]$ for which the following inequality holds:

$$|\varphi(t, \cdot)|_{\sup}\leqslant\psi_\varphi(t) \quad \text{for} \quad t\in[0,1].$$

Two elements φ_1, φ_2 are identified if $\varphi_1(t,\cdot)=\varphi_2(t,\cdot)$ for almost all $t\in[0,1]$. Then for each $\varphi\in\mathscr{B}$ the function $t\to|\varphi(t,\cdot)|_{\sup}$ is integrable on $[0, 1]$ and

$$|\varphi|_{\mathscr{B}}=\int_0^1 |\varphi(t,\cdot)|_{\sup}\, dt$$

is a norm on $\mathscr{B}$. It can be shown that $(\mathscr{B}, |\cdot|_{\mathscr{B}})$ is a separable Banach space.

Next, let frm (V) denotes the space of Radon measures on V i.e. dual of $C(V)$. For each element $\nu\in\text{frm}(V)$ we denote by $|\nu|$ the variation of ν over the set V. Take the space $\mathscr{L}$ of equivalence classes of functions $\mu\colon[0,1]\to\text{frm}(V)$ such that:

$$\operatorname*{ess\,sup}_{t\in[0,1]} |\mu(t)|>+\infty$$

and moreover for each element $c\in C(V)$ the function $t\to\int_V c(v)\,\mu(t)\,(\mathrm{d}v)$ is measurable. We identify two elements μ_1, μ_2 if $\mu_1(t)=\mu_2(t)$ a.e. $t\in[0,1]$. The essential supremum is a norm in $\mathscr{L}$. It can be proved that for each $\varphi\in\mathscr{B}$, $\mu\in\mathscr{L}$ the function

$$t\to\int_V \varphi(t,v)\,\mu(t)\,(\mathrm{d}v)$$

is integrable

The space $\mathscr{L}$ (*cf* [11] Th. IV 1.8.) can be identified with the dual $\mathscr{B}^*$ where the duality is given by:

$$\langle \mu, \varphi \rangle = \int_0^1 dt \int_V \varphi(t, v)\, \mu(t)\, dv \quad \text{for} \quad \mu \in \mathscr{L},\ \varphi \in \mathscr{B} \tag{3.1}$$

and

$$\operatorname*{ess\,sup}_{t \in [0,1]} |\mu(t)| = |\mu|_{\mathscr{B}^*} = \sup_{|\varphi|_{\mathscr{B}} \le 1} \langle \mu, \varphi \rangle.$$

The space $\mathscr{B}$ is separable therefore the unit ball in $\mathscr{B}^* \simeq \mathscr{L}$ is sequentially compact with respect to the weak star topology. Define the following subset of the unit ball which we shall take as our set of relaxed disturbarces:

$$\mathscr{S} = \{\mu \in \mathscr{L} \mid \mu(t) \in \text{rpm}(v) \text{ a.e. } t \in [0, 1]\}$$

where rpm (V) denotes the set of Radon probability measures on V. We can identify the set $\mathscr{N}$ with the following subset of $\mathscr{S}$:

$$\mathscr{S}_{\mathscr{N}} = \{\mu \in \mathscr{L} \mid \mu(t) = \delta_{v(t)} \text{ a.e. for some } v(\cdot) \in \mathscr{N}\}$$

where $\delta_{v(t)}$ denotes the Dirac measure at $v(t)$.

The set $\mathscr{S}$ is weakly sequentially closed hence weakly sequentially compact and $\mathscr{S}_{\mathscr{N}}$ is dense in $\mathscr{S}$ with respect to the weak star topology (cf [11]).

Take $\mu \in \mathscr{S}$, a control $u(\cdot) \in \mathscr{M}$ and define the trajectory $x(\cdot) = x_{\mu,u}(\cdot)$ corresponding to the pair (μ, u) as a solution of the following equation:

$$\begin{cases} \dot{x}(t) = \int_V f(t, x(t), u(t), v)\, \mu(t)(dv) \\ x(0) = x_0 \end{cases} \tag{3.2}$$

One can check that for each $\mu \in \mathscr{S}$ the function $f^\mu(t, x, u) = \int_V f(t, x, u, v)\, \mu(t)(dv)$ satisfies all asumptions (A_1)–(A_5). Moreover we have:

$$f_x^\mu(t, x, u) = \int_V f_x(t, x, u, v)\, \mu(t)(dv), \quad f_u^\mu(t, x, u) = \int_V f_u(t, x, u, v)\, \mu(t)(dv)$$

for $x \in R^m$, $u \in \mathscr{M}$, $t \in [0, 1]$. Hence there exists a unique solution of (3.2) on [0, 1] and all trajectories of (3.2) are uniformly bounded. We can assume that they lie in the same ball $B(O, \bar{r})$.

If follows from Th.V.6.1. of [11] (as well as from Lemma 1 of the next section) that if we take a weakly convergent sequence $\mu_n \xrightarrow{w^*} \mu_0$ of elements from $\mathscr{S}$ then for each fixed $u(\cdot) \in \mathscr{M}$ the corresponding trajectories $x_{\mu_n,u}(\cdot)$ converge uniformly on [0, 1] to the trajectory $x_{\mu_0,u}(\cdot)$ This together with the density of $\mathscr{S}_{\mathscr{N}}$ in $\mathscr{S}$ implies that for each $u(\cdot) \in \mathscr{M}$ the following equality holds:

$$I(u) = \sup_{v \in \mathscr{N}} h(x_{u,v}(1)) = \max_{\mu \in \mathscr{S}} h(x_{\mu,u}(1)) \tag{3.3}$$

Define for each $u \in \mathscr{M}$ the set:

$$M(u) = \{\mu \in \mathscr{S} \mid h(x_{\mu,u}(1)) = I(u)\}.$$

The cost function h is locally lipschitz hence it can be differentiated in the Clarke's sens. The simbol $\partial h(x)$ below refers to Clarke's generalized gradient (see [3], [4]). By $\langle \cdot , \cdot \rangle$ we denote below the usual scalar product, superscript T denotes the transposed matrix.

We can state now our necesary condition for a minmax control.

THEOREM 1. *Let $u_* \in \mathcal{M}$ satisfies (2.2). Then the following condition holds:*

$$\min_{u(\ )\in \mathcal{M}} \max_{\mu \in \mathcal{M}(u_*)} \max_{\zeta \in \partial h(x_{u_*,\mu}(1))} \int_0^1 < \int_V f_u^T(t, u_*(t), x_{u_*,\mu}(t), v) \tag{3.4}$$

$$\mu(t)(\mathrm{dv})\, \psi_{\mu,\zeta}(t), u(t)-u_*(t) < dt = 0$$

where $\psi_{\mu,\zeta}$ is the solution of the following equation:

$$\begin{cases} \dot{\psi}_{\mu,\zeta}(t) = -\int_V f_x^T(t, u_*(t), x_{u_*,\mu}(t), v)\, \mu(t)(\mathrm{dv})\, \psi_{\mu,\zeta}(t) \\ \psi_{\mu,\zeta}(1) = \zeta \end{cases} \tag{3.5}$$

It is easy to see that in the case when $f(t, x, u, v)$ does not depend on v and h is C^1 then (3.3), (3.4) reduce to the well-known local maximum principle for the optimal control u_*.

4. Proof of Theorem 1

Denote by $L^\infty([0, 1]; R^P)$ the Banach space of essentially bounded functions on $[0, 1]$ taking values in R^P. Let $u \in \mathcal{M}$, $\mu \in \mathcal{S}$, $\bar{u} \in L^\infty([0, 1]; R^P)$. Denote by $\delta x^{\bar{u}}_{u,\mu}(\cdot)$ the solution of the following linear equation:

$$\begin{cases} \delta \dot{x}^{\bar{u}}_{\mu,\mu}(t) = \int_V f_x(t, x_{u,\mu}(t), u(t), v)\, \mu(t)(\mathrm{dv})\, \delta x^{\bar{u}}_{u,\mu}(t) + \\ \qquad + \int_V f_u(t, x_{u,\mu}(t), u(t), v)\, \mu(t)(\mathrm{dv})\, \bar{u}(t) \\ \delta x^{\bar{u}}_{u,\mu}(0) = 0 \end{cases} \tag{4.1}$$

We start with the following

LEMMA 1. *Let $u_n \in \mathcal{M}$ $n=1, 2, \ldots$ be a sequence of controls convergent a.e. in $[0, 1]$ to a control u_0, $\mu_n \in \mathcal{S}$ $n=1, 2, \ldots$ a weakly convergent sequence $\mu_n \xrightarrow{w^*} \mu_0$. Then for each $\bar{u} \in L^\infty([0, 1]; R^P)$ we have:*

$$x_{u_n,\mu_n}(t) \to x_{u_0,\mu_0}(t)$$

$$\delta x^{\bar{u}}_{u_n,\mu}(t) \to \delta x^{\bar{u}}_{u_0,\mu_0}(t)$$

uniformly for $t \in [0, 1]$.

Proof: Denote $x_n(\cdot)=x_{u_n,\mu_n}(\cdot)$, $x_0(\cdot)=x_{u_0,\mu_0}(\cdot)$. We have from (2.1) and assumptions (A_1)-(A_4):

$$|x_n(t)-x_0(t)|=\Big|\int_0^t\Big(\int_V f(\tau, x_n(\tau), u_n(\tau), v)\,\mu_n(\tau)(dv)-$$

$$-\int_V f(\tau, x_0(\tau), u_0(\tau), v)\,\mu_0(\tau)(dv)\Big)d\tau\Big|\leqslant\Big|\int_0^t\Big(\int_V f(\tau, x_n(\tau)u_n(\tau), v)\,\mu_n(\tau)(dv)-$$

$$-\int_V f(\tau, x_0(\tau), u_n(\tau), v)\,\mu_n(\tau)(dv)\Big)d\tau\Big|+\Big|\int_0^t\Big(\int_V f(\tau, x_0(\tau), u_n(\ ), v)\,\mu_n(\tau)(dv)-$$

$$-\int_V f(\tau, x_0(\tau), u_0(\tau)v)\,\mu_0(\tau)(dv)\Big)d\tau\Big|\leqslant\int_0^t k_{\bar r}(\tau)|x_n(\tau)-x_0(\tau)|+|l_n(t)|+|r_n(t)|$$

where

$$l_n(t)=\int_0^t\int_V f(\tau, x_0(\tau), u_n(\tau), v)\,\mu^n(\tau)(dv)-\int_V f(\tau, x_0(\tau), u_0(\tau), v)\,\mu_n(\tau)(dv)\,d\tau$$

$$r_n(t)=\int_0^t\int_V f(\tau, x_0, (\tau), u_0(\tau), v)\,\mu_n(\tau)(dv)-\int_V f(\tau, x_0(\tau), u_0(\tau), v)\,\mu_0(\tau)(dv)\,d\tau.$$

We have that $r_n(t)\to 0$ for every $t\in[0, 1]$ because of the weak convergence $\mu_n\xrightarrow{w_*}\mu_0$ (see (3.1)).

All derivatives $\dot r_n(t)$ $n=1, 2, \ldots$ are uniformly bounded on [0, 1] by an integrable function $2a(t)\bar r+b(t)$ (see (A_4)). Therefore $r_n(t)$ are uniformly bounded and equicontinuous functions on [0, 1] thus $r_n(t)\to 0$ uniformly for $t\in[0, 1]$. In order to estimate $l_n(t)$ notice that for each n the function $\varphi_n(\tau)=\sup_{v\in V}|f(\tau, x_0(\tau), u_n(\tau), v)-$ $-f(\tau, x_0(\tau), u_0(\tau), v)|$ is measurable since $f(\tau, x, u, \cdot)$ is continuous and integrable since (A_4). Moreover:

$$|l_n(t)|\leqslant\int_0^t\varphi_n(\tau)\,d\tau.$$

Take $\bar\tau$ such that $\bar\tau_n(u)\to u_0(\bar\tau)$ then from uniform continuity of $f(\tau, x, \cdot, \cdot)$ on $U\times V$ we obtain that $\varphi_n(\bar\tau)\to 0$. Thus $\varphi_n(\tau)\to 0$ a.e. for $\tau\in[0, 1]$. All functions $\varphi_n(\tau)$ are uniformly bounded by an integrable function since (A_4). Therefore $l_n(t)\to 0$ for $t\in[0, 1]$ and by the same argument as before we obtain that $l_n(t)\to 0$ uniformly for $t\in[0, 1]$. Denote $C_n=\sup_{t\in[0,1]}(|r_n(t)|+|l_n(t)|)$. We have from Gronwall's inequality that:

$$|x_n(t)-x_0(t)|\leqslant C_n+e^{\int_0^1 k_{\bar r}(\tau)\,d\tau}\int_0^1 C_n\,k_{\bar r}(\tau)\,d\tau \quad \text{for each } t\in[0, 1],$$

therefore $x_n(t)\to x_0(t)$ uniformly for $t\in[0, 1]$.

In order to prove the second part we put $W=B(0,\bar{r})\times$ instead of U, $w=(x,u)$ instead of u and consider the function:

$$g(t,y,w,v)=f_x(t,w,v)\,y+f_u(t,w,v)\,\bar{u}(t)$$

instead of $f(t,x,u,v)$. From the first part we have

$$w_n(t)=(x_n(t),u_n(t))\to w_0(t)=(x_0(t),z_0(t))\qquad \text{a.e.}\quad t\in[0,1]$$

The function g satisfies assumptions analogous to (A_1)–(A_4) and we come to the conclusion as before.

We proceed to prove the Theorem. Take an arbitrary element $u(\cdot)\in\mathcal{M}$ and denote $\bar{u}(t)=u(t)-u_*(t)$. For each $s\in[0,1]$ the function $u_s(t)=u(t)+s\bar{u}(t)$ belongs to $\mathcal{M}$ since U is convex. The control u_* satisfies (2.2) so for each $s\in[0,1]$ we have:

$$\frac{\max\limits_{\mu\in s} h(x_{u_s,\mu}(1))-\max\limits_{\mu\in s} h(x_{u_*,\mu}(1))}{s}\geqslant 0 \tag{4.2}$$

Take a sequence $s_n\to 0$. Choose $\mu_n\in M(u_{s_n})$. We can assume taking eventually a subsequence that $\mu_n\xrightarrow{w*}\mu_0$, $\mu_0\in\mathcal{S}$. From Lemma 1 we easily obtain that $\mu_0\in M(u^*)$. From (4.2) we deduce the following inequality:

$$\frac{h(x_{u_{s_n},\mu_n}(1))-h(x_{u_*,\mu_n}(1))}{s_n}\geqslant 0\qquad\text{for each } n=1,2,\ldots.$$

Applying the mean–value theorem for generalized gradients (see [4]) we obtain that there exist for every $n=1,2,\ldots$ a number $\lambda_n\in(0,1)$ and an element

$$\zeta^n\in\partial h\left(\lambda_n x_{u_{s_n},\mu_n}(1)+(1-\lambda_n)\,x_{u_*,\mu_n}(1)\right)$$

such that:

$$\frac{\langle\zeta^n, x_{u_{s_n},\mu_n}(1)-x_{u_*,\mu_n}(1)\rangle}{s_n}\geqslant 0\qquad\text{for each } n=1,2,\ldots. \tag{4.3}$$

Fix an arbitrary $\mu\in\mathcal{S}$, $\zeta\in R^m$ and consider the following function:

$$\varphi_{\mu,\zeta}(s)=\langle\zeta, x_{u_s,\mu}(1)\rangle\qquad s\in[0,1].$$

The function is differentiable for $s\in(0,1)$ and its right–hand side derivative at $s=0$ exists and:

$$\varphi'_{\mu,\zeta}(s)=\langle\zeta,\delta x^{\bar{u}}_{u_s,\mu}(1)\rangle\qquad\text{for}\quad s\in[0,1]$$

where $\delta x^{\bar{u}}_{u_s,\mu}$ is the solution of (4.1).

Applying the mean–value theorem for $\varphi_{\mu_n,\zeta}\,n(s)$, $s\in[0,s_n]$ we obtain from (4.3) that there exists $\theta_n\in(0,s_n)$ such that:

$$\langle\zeta^n,\delta x^{\bar{u}}_{u_{\theta_n},\mu_n}(1)\rangle>\geqslant 0. \tag{4.4}$$

From Lemma 1 we have that $x_{u_{s_n}, \mu_n}(1) \to x_{u_*, \mu_0}(1)$ and $\delta x^{\bar{u}}_{u_{\theta_n}, \mu_n}(1) \to \delta x^{\bar{u}}_{u_*, \mu_0}(1)$ as $n \to +\infty$. The sequence ζ^n is bounded since all gradients $\partial h(x)$ are uniformly bounded in a neighbourhood of $x_{u_* \mu_0}(1)$ so we can assume that $\zeta^n \to \zeta_0$. By the upper semicontinuity of the generalized gradient we deduee that $\zeta_0 \in \partial h(x_{u_*, \mu_0}(1))$. Therefore (4.4) gives that:

$$\langle \zeta_0, \delta x^{u}_{u_*, \mu_0}(1) \rangle > \geqslant 0 \tag{4.5}$$

and $\zeta_0 \in \partial h(x_{u_*, \mu_0}(1))$, $\mu_0 \in M(u_*)$.

Take $\psi_{\mu_0, \zeta_0}(t)$ which satisfies the following equation:

$$\begin{cases} \dot{\psi}_{\mu_0, \zeta_0}(t) = -\int\limits_V f_x^T(t, u_*(t), x_{u_*, \mu_0}(t), v)\, \mu_0(t)(\mathrm{dv})\, \psi_{\mu_0, \zeta_0}(t) \\ \psi_{\mu_0, \zeta_0}(1) = \zeta_0 \end{cases} \tag{4.6}$$

It follows from (4.5) that:

$$> \psi_{\mu_0, \zeta_0}(1), \delta x^{\bar{u}}_{u_* \mu_0}(1) \rangle = \int\limits_0^1 \frac{d}{dt} \langle \psi_{\mu_0, \zeta_0}(t), \delta x^{\bar{u}}_{\mu_0, u_*}(t) \rangle > \geqslant 0$$

The latter inequality together with (4.6), (4.1) imply that:

$$\int\limits_0^1 \Big\langle \int\limits_V f_u^T(t, u_*(t), x_{u_*, \mu_0}(t), v)\, \mu_0(t)(\mathrm{dv})\, \psi_{\mu_0, \zeta_0}(t), \bar{u}(t) \Big\rangle > dt \geqslant 0$$

what completes the proof of Theorem 1.

5. Suboptimal solutions

Ckearly a minmax control u^* for the problem under consideration may not exist. The following Theorem 2 concerns the case when there is no minmax solutions. Theorem 2 states that close to any suboptimal solution always exists another suboptimal solution which approximately satisfies the necessary condition of the previous section. Let $d = 2 \max\{|u|: u \in U\}$.

THEOREM 2. *Let $w_\varepsilon \in \mathcal{M}$ be such that*

$$I(w_\varepsilon) \leqslant \inf_{u \in \mathcal{M}} I(u) + \varepsilon$$

Then there exists $u_\varepsilon \in \mathcal{M}$ such that

$$I(u_\varepsilon) \leqslant \inf_{u \in \mathcal{M}} I(u) + \varepsilon \tag{5.1}$$

$$\operatorname*{ess\,sup}_{t \in [0,1]} |u_\varepsilon(t) - w_\varepsilon(t)| \leqslant \sqrt{\varepsilon}$$

and moreover u_ε satisfies the following condition:

$$\max_{\mu\in\mathcal{M}(u_\varepsilon)} \max_{\zeta\in\partial h(x_{u_\varepsilon,\mu}(1))} \int_0^1 \Big\langle \int_V f_u^T(t, x_{u_\varepsilon,\mu}(t), u_\varepsilon(t), v)\times \mu(t)(dv)\, \psi_{\mu,\zeta}(t), u(t)-u_\varepsilon(t)\Big\rangle \geqslant dt \geqslant -\sqrt{\varepsilon}\cdot d$$

for each $u\in\mathcal{M}$ where $\psi_{\mu,\zeta}(t)$ is defined by (3.5) with u_* replaced by u_ε.

Proof. The set $\mathcal{M}$ is a closed subset of $L^\infty([0, 1]; R^P)$ therefore it is a complete metric space with the norm of $L^\infty\cdot I(u)$ is continuous on $\mathcal{M}$ since Lemma 1 and weak sequential compactess of $\mathcal{S}$ so Ekeland's theorem of [5] may be applied. We deduce that there exists $u_\varepsilon\in\mathcal{M}$ satisfying (5.1) and such that:

$$I(u)+\sqrt{\varepsilon}\ \text{ess sup}\ |u(t)-u_\varepsilon(t)|\geqslant I(u_\varepsilon)$$

for each $u\in\mathcal{M}$. Take an arbitrary $u\in\mathcal{M}$, define $\bar{u}=u-u_\varepsilon$ as before, and $u=u_\varepsilon+s\bar{u}$ for $s\in[0, 1]$. The latter inequality implies that:

$$\frac{I(u_s.)-I(u_\varepsilon)}{s}\geqslant-\sqrt{\varepsilon}d \quad \text{for} \quad s\in(0, 1)].$$

We proceed next as in the proof of Theorem 1. ■

Theorem 1 (as well as Theorem 2) can be restated in the case when h is C^1 without using relaxed functions in the following way. Let $\mathrm{Sub}_\delta(u)$ denotes for each $u\in\mathcal{M}$, $\delta\leqslant 0$ the subset of $\mathcal{N}$ defined as follows:

$$\mathrm{Sub}_\delta(u)=\{v\in\mathcal{N}: h(x_{u,v}(1))\leqslant I(u)+\delta\}$$

Assume additionally that the cost function h is of the class C^1. Then Theorem 1 Lemma 1 and the density of $\mathcal{N}$ in $\mathcal{S}$ imply easily the following.

COROLLARY 1: *Let u_* satisfies (2.2). Then for every $\delta>0$ the following condition holds:*

$$\min_{u\in\mathcal{M}} \sup_{v\in \mathrm{Sub}_\delta(u_*)} \int_0^1 \langle f_u^T(t, u_*(t), x_{u_*,v}(t), v(t))\,\psi_v(t), u(t)-u_*(t)\rangle\, dt\geqslant 0$$

where

$$\begin{cases}\dot{\psi}_v(t)=-f_x^T(t, x_{u_*,v}(t), u_*(t), v(t))\,\psi_v(t)\\ \psi(1)=h'(x_{u_*,v}(1)).\end{cases}$$

Similarly in Theorem 2 the set $M(u_\varepsilon)$ can be replaced by Sub $\delta(u_\varepsilon)$ for every $\delta>0$.

6. Examples

Consider the following system on the plane, $x=(x_1, x_2)\in R^2$:

$$\begin{cases}\dot{x}_1(t)=x_2(t)+u(t)v(t)\\ \dot{x}_2(t)=u(t)+v(t) \qquad U=V=[-1, 1], \qquad t\in[0, 1]\\ x(0)=0\end{cases}$$

$$h(x)=x_1-x_2.$$

First we notice that since V is convex and the parameter v appears in the dynamics linearly passing to the set $\mathcal{S}$ does not give anything new and we can consider $\mathcal{N}$ instead of $\mathcal{S}$. Every of the conditions under consideration which holds for an element $\mu \in \mathcal{S}$ holds for the coresponding $v(t) = \int_V v\mu(t)(dv) \in \mathcal{N}$ as well.

Take and fix $u \in \mathcal{M}$. We shall determine the set $M(u)$ with the aid of the maximum principle.

The odjoint equation takes the following form:

$$\begin{cases} \dot{\psi}_1(t) = 0 \\ \dot{\psi}_2(t) = -\psi_1(t), \quad \psi(1) = (1, -1) \end{cases}$$

Therefore for each pair $u \in \mathcal{M}$, $v \in \mathcal{N}$ the adjoint function is the following $\psi(t) = (1, -t)$. Therefore the maximum principle gives that each element of the set $M(u)$ is of the following form:

$$v(t) = \begin{cases} 1 & \text{if} \quad u(t) - t > 0 \\ -1 & \text{if} \quad u(t) - t < 0 \\ \text{arbitraty if } u(t) = t \end{cases} \tag{6.1}$$

Theorem 1 gives that if $u_* \in \mathcal{M}$ is a minmax control then for each $u \in \mathcal{M}$:

$$\max_{v \in M(u)} \int_0^1 (v(t) - t)(u(t) - u_*(t))\, dt \geq 0 \tag{6.2}$$

In particular (6.1) has to hold for $u(t) \equiv t$. Denote

$$A_1 = \{t : u_*(t) - t > 0\}, \quad A_2 = \{t : u_*(t) - t < 0\}$$

The condition (6.2) for $u(t) \equiv t$ gives that:

$$\int_{A_1} (1 - t)(t - u_*(t))\, dt + \int_{A_2} (-1 - t)(t - u_*(t))\, dt \geq 0$$

But the latter inequality holds if and only if both set A_1 and A_2 are of the Lebesque measure zero. Therefore the only control which can satisfy the condition of Theorem 1 is $u_*(t) \equiv t$ and satisfies indeed as for such $u_*(t)$ the set $M(u_*) = \mathcal{N}$ and (6.2) holds as we can put $v(t) \equiv t$. The control $u_*(t) \equiv t$ is a minmax control really what can be checked by the direct analysis of the following expression:

$$h(x_{u,v}(1)) = \int_0^1 \int_0^\tau u(s) + v(s)\, ds\, d\tau + \int_0^1 u(\tau) v(\tau)\, d\tau - \int_0^1 u(\tau) + v(\tau)\, d\tau$$

Integrating by parts the double integral we obtain:

$$h(x_{u,v}(1)) = \int_0^1 (u(t) - t) v(t) - tu(t)\, dt$$

So we see that $M(u)$ consists of the elements satisfying (6.1) and $u_*(t)\equiv t$ is the only minmax control as:

$$I(u)=\int_0^1 |u(t)-t|-tu(t)=\int_0^1 |u(t)-t|-(u(t)-t)\,t-t^2\geqslant$$

$$\leqslant\int_0^1 |u(t)-t|(1-t)-t^2\geqslant-\int_0^1 t^2=I(u_*)$$

an the latter inequality becomes equality iff $u(t)=u_*(t)$.

The second example shows that the most natural generalization of the strong maximum principle does not hold in general for minmax problems. Consider the following system:

$$\begin{cases}\dot{x}=(u-v)^2\\ x(0)=0 \quad u,v\in[0,1], \quad x\in R, \quad t\in[0,1]\end{cases}$$
$$h(x)=-x$$

The condition of Theorem 1 holds here for every $u\in\mathcal{M}$ since $f_u=2(u-v)$ and $v(t)\equiv u(t)$ always belongs to $M(u)$. In fact all $u\in\mathcal{M}$ are minmax controls. Take $u_*\equiv 0$ It is easy to see that $M(u_*)=\{v_*\}$, $v_*\equiv 0$, $\psi_*(t)=\psi^{u^*}_{v_*}(t)\equiv-1$, where $\zeta=h'(x)=-1$, $x_{u_*,v_*}(t)=x_*(t)\equiv 0$,

$$\psi_*(t)f(t,x_*(t),u,v_*(t))=-1u^2=-u^2$$

So we see that the control $u_*=0$ maximizes the hamiltonian along $x_*(t), v_*(t)$ instead of minimizing. In the latter example u^* satisfies along $\psi_*(t)$, $x_*(t)$ corresponding to $u_*, v_*\in M(u^*)$ the following equality:

$$\max_{v\in V}\psi_*(t)f(t,x_*(t),u_*v)=\min_{u\in U}\max_{a\in V}\psi_*(t)f(t,x_*(t),u,v).$$

This does not hold in general either as shows the following example

$$\begin{cases}\dot{x}=u-v\\ x(0)=0 \quad x\in R,\ u,v\in[-1,1],\ h(x)=|x|,\ u_*\equiv 0.\end{cases}$$

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Warunki konieczne dla minimaksowego zadania sterowania

Praca dotyczy minimaksowego zadania sterowania z niepewnością występującą w równaniu stanu. Podane są warunki konieczne na to aby sterowanie było minimaksowe w klasie sterowań typu open-loop.

Необходимые условия для минимаксной задачи управления

В работе рассматривается минимаксна задача управления с неопределенностью в уравнении состояния. Представляется необходимые условия для того чтобы управление являлось минимаксным в классе программных управлении.