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Numerical Solution of a Nonlinear Two-Point Boundary—Value Problem by an Optimization Technique*)

by

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A simple numerical method for solving nonlinear two-point boundary value problem by using an optimization technique is proposed. It is shown that under some assumptions the solution of nonlinear problem can be computed by solving an auxiliary optimization problem. Several numerical examples are presented.

1. Introduction

In this paper we propose a simple numerical method for solving nonlinear two-point boundary- value problems by using an optimization technique.

Optimization technique was used for example in [1] and in [4] in order to solve some nonlinear differential equations.

In section 2 we formulate a simple nonlinear two-point boundary-value problem. We prove under some assumptions that the solution of nonlinear problem can be obtained by solving an auxiliary optimization problem (P). We define a family of finite dimensional optimization problem (P_h) where h is a parameter of approxima-

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tions. For a fixed parameter $h>0$ the problem (P_h) can be solved numerically. We prove convergence of the sequence of solutions of problems (P_h) to the solution of the problem (P).

In section 3 we present several numerical examples. In order to solve nonlinear, two-point, boundary-value problems numerically we use two approaches.

First we formulate auxiliary optimization problem, and for a fixed parameter h we solve the resulting optimization problem by using the Davidon—Fletcher—Powell algorithm of nonlinear programing.

The second approach consists in solving directly the nonlinear boundary-value problem by using Newton's method.

2. Statement of the problem and approximation

Let us consider nonlinear two-point boundary value problem

$$-\frac{d}{dt}\left[a\left(t, y(t), \frac{dy}{dt}(t)\right)\frac{dy}{dt}(t)\right]+qy(t)=f(t) \qquad t\in(0,1) \tag{2.1}$$

$$y(0)=y(1)=0 \tag{2.2}$$

where q is a non negative constant and $f(\cdot)\in L^2(0,1)$. We assume that there exists a unique weak solution $\bar{y}(\cdot)\in H_0^1(0,1)$ to (2.1) (2.2). For sufficient conditions of existence and uniqueness of the solution, we refer to [2].

It there exist constants α and M such that

$$0<\alpha\leqslant a\left(t, \bar{y}(t), \frac{d\bar{y}}{dt}(t)\right)\leqslant M \quad \text{for almost all } t\in(0,1) \tag{2.3}$$

then the solution $\bar{y}(\cdot)$ can be computed by solving the following optimization problem:

(P) minimize the cost functional

$$J(u)=\frac{1}{2}\int_0^1\left[a\left(t, w(t), \frac{dw}{dt}(t)\right)-u(t)\right]^2 dt \tag{2.4}$$

over the set $U_{\text{ad}}=\{u(\cdot)|\alpha\leqslant u(t)\leqslant M \text{ a.e. in } (0,1)\}$

where the function $w(\cdot)\in H_0(0,1)$ satisfies the following linear two-point boundary-value problem:

$$-\frac{d}{dt}\left[u(t)\frac{dw}{dt}(t)\right]+q\,w(t)=f(t) \qquad t\in(0,1) \tag{2.5}$$

$$w(0)=w(1)=0 \tag{2.6}$$

Indeed, it is easy, to see that the optimal solution $\hat{u}(\cdot)$ to (P) has the following form

$$\hat{u}(t)=a\left(t, \bar{y}, (t), \frac{d\bar{y}}{dt}(t)\right) \quad \text{a.e. in } (0,1) \tag{2.7}$$

On the other hand if $u^*(\cdot)$ is an optimal solution to (P) then $J(u^*)=0$ and

$$u^*(t)=a\left(t, w^*(t), \frac{dw^*}{dt}(t)\right), \quad t\in(0,1) \tag{2.8}$$

where w^* is the solution to (2.5), (2.6) corresponding to u^*, Hence w^* is a solution to (2.1), (2.2). We assume that the function $a(.,.,.)$ satisfies the following conditions:

(A1) function $a(., \eta, \xi): R\to R$ is measurable on the interval (0, 1) for every pair $(\eta, \xi)\in R^2$.

(A2) function $a(t, .,.): R\times R\to R$ is continuous for a.e. $t\in(0,1)$.

In order to solve the problem (P) numerically, we define a family of finite dimensional problems $\{P_h\}$, $h=1/n$, $n=1, 2, \ldots$

It can be shown that under some conditions the sequence of optimal solutions $\{\hat{u}_h\}$ of problems (P_h) converges in some sense to the solution $\hat{u}$ of the problem (P).

We construct the problems (P_h). Let $n\geqslant 1$ be a given integer. Let V_h be the space of continuous piecewise linear functions on every subinterval $[ih, (i+1)h]$, $i=0, \ldots, n$. It is clear that a set of functions $\{\varphi_i\}_{i=0}^n$ is a basis in V_h, where

$$\varphi_0(t)=\begin{cases}-\dfrac{1}{h}t+1, & t\in[0,h]\\ 0, \text{ otherwise}\end{cases}$$

$$\varphi_i(t)=\begin{cases}\dfrac{1}{h}t+1-i, & t\in((i-1)h, ih]\\ -\dfrac{1}{h}t+1+i, & t\in(ih, (i+1)h]\\ 0, \text{ otherwise}\end{cases}$$

$$i=1, \ldots, n-1$$

$$\varphi_n(t)=\begin{cases}\dfrac{1}{h}t+1-n, & t\in((n-1)h, nh]\\ 0, \text{ otherwise}\end{cases}$$

The problem (P_h) is defined in the following way:

(P_h): minimize the cost functional

$$J_h(u_h)=\frac{1}{2}\int_0^1\left[a\left((t, w_h(t)\frac{dw_h}{dt}(t)\right)-u_h(t)\right]^2 dt \tag{2.9}$$

over the set

$$\{u_h(\cdot)\in V_h \mid \alpha\leqslant u_h(t)\leqslant M, \quad t\in(0,1)\} \tag{2.10}$$

where $w_h(\cdot)$ satisfies

$$w_h(\cdot)\in V_h\cap H_0^1(0,1)$$

$$\int_0^1\left[u_h(t)\frac{dw_h}{dt}(t)\frac{d\varphi_i}{dt}(t)+qw_h(t)\varphi_i(t)\right]dt=\int_0^1 f(t)\varphi_i(t)\,dt,\ \forall\varphi_i,$$

$$i=1,\dots,n-1 \tag{2.11}$$

Denote by $\hat{u}_h$ an optimal solution to the problem (P_h). We have the following result concerning convergence of the sequence $\{\hat{u}_h\}$.

LEMMA 1: *If there is a convergent subsequence $\{\hat{u}_{h'}\}\subset\{\hat{u}_h\}$ such that*

$$(H1)\qquad \hat{u}_{h'}(t)\to\tilde{u}(t)\quad \text{a.e. in } (0,1)$$

then the corresponding sequence $\{\hat{w}_{h'}\}$ of solutions to (2.11) *converges to the solution of* (2.1), (2.2) i.e.

$$\hat{w}_{h'}\to\bar{y}\quad \text{in the norm of } H_0^1(0,1). \tag{2.12}$$

Proof: We denote by $\hat{w}_h\in V_h\cap H_0^1(0,1)$ the unique solution to (2.11) corresponding to $\hat{u}_h(t)$. It can be shown that assumption $(H1)$ implies [5] the convergence of sequence $\{\hat{w}_h\}$ of solutions to (2.11) i.e.:

$$\hat{w}_h\to w\quad \text{in } H_0^1(0,1) \text{ strongly} \tag{2.13}$$

where $w\in H_0^1(0,1)$ satisfies the equation

$$-\frac{d}{dt}\left[\tilde{u}(t)\frac{dw}{dt}(t)\right]+qw(t)=f(t)\quad \text{a.e. in } (0,1) \tag{2.14}$$

Let $\bar{y}(\cdot)$ be the solution to (2.1), (2.2) and denote by $\hat{u}(t)$ an optimal solution to (P) i.e.:

$$\hat{u}(t)=a\left(t,\bar{y}(t),\frac{d\bar{y}}{dt}(t)\right)\ \text{for a.e. } t\in(0,1) \tag{2.15}$$

Let $\{v_h\}$ be an arbitrary sequence such that

$$v_h\in V_h,\quad \alpha\leqslant v_h(t)\leqslant M\quad \text{a.e. in } (0,1)$$

and for $h\downarrow 0$:

$$v_h(t)\to\hat{u}(t)\quad \text{a.e. in } (0,1) \tag{2.16}$$

It can be shown that

$$0=\lim_{h'\downarrow 0} J_h(v_h)\leqslant\limsup_{h'\downarrow 0} J_h(u_{h'})\geqslant\liminf_{h'\downarrow 0} J_{h'}(\hat{u}_{h'})\geqslant 0 \tag{2.17}$$

hence

$$\lim_{h'\downarrow 0} J(\hat{u}_{h'})=0$$

i.e.:

$$a\left(t, w_{h'}(t), \frac{dw_{h'}}{dt}(t)\right) - \hat{u}_{h'}(t) \to 0 \quad \text{a.e. in } (0, 1) \tag{2.18}$$

Combining (2.18) with (2.13) and ($H1$) we obtain

$$a\left(t, w(t), \frac{dw}{dt}(t)\right) = \tilde{u}(t) \quad \text{a.e. in } (0, 1)$$

Hence by (2.14) it follows that $w(\cdot)$ satisfies (2.1) (2.2). Let us consider the particular case of the problem (2.1) (2.2) i.e.:

$$-\frac{d}{dt}\left[a(t, y(t)) \frac{dy}{dt}(t)\right] + qy(t) = f(t) \quad \text{a.e. in } (0, 1) \tag{2.1$'$}$$

$$y(0) = y(1) = 0 \tag{2.2}$$

We define the family $\{P_h'\}$ of finite dimensional optimization problems in the same way as before:
(P_h'): minimize the cost functional

$$I_h(u_h) = \frac{1}{2} \int_0^1 [a(t, w_h(t)) - u_h(t)]^2 \, dt \tag{2.19}$$

over the set (2.10), where $w_h(\cdot)$ is the solution to (2.11).

Let $\hat{u}_h(\cdot)$ be an optimal solution to the problem (P_h') and let $\{\hat{w}_h\}$ be the corresponding solution to (2.11). It can be shown that assumption ($H1$) is satisfied.

LEMMA 2: *Assume that the function* $a(t, \eta)$, $t \in (0, 1)$, $\eta \in R$ *satisfies the same assumptions* ($A1$), ($A2$) *and that there exists the unique solution* $\bar{y}(\cdot)$. *to the equation* (2.1) (2.2). *Then when* h *tends to zero, it follows that*

$$\hat{u}_h(t) \to a(t, \bar{y}(t)) \quad \text{a.e. in } (0.1) \tag{2.20}$$

Furthermore

$$\hat{w}_h \to \bar{y} \quad \text{in } H_0^1(0, 1) \text{ strongly} \tag{2.21}$$

Proof: There exists a subsequence $\{u_{h'}\}$, $h' \downarrow 0$ and an element $\varphi(\cdot) \in L^\infty(0, 1)$ such that [6]

$$\alpha \leqslant \varphi(t) \leqslant M \quad \text{a.e.} \quad \text{in } (0, 1)$$

and

$$1/u_h \to 1/\varphi \quad (*)\text{-weakly in } L^\infty(0, 1) \tag{2.22}$$

Let us recall that the element $\hat{w}_h \in V_h \cap H_0^1(0, 1)$ satisfies the integral identity:

$$\int_0^1 \left[\hat{u}_h(t) \frac{d\hat{w}_h}{dt}(t) \frac{d\varphi_h}{dt}(t) + q\hat{w}_h(t)\, \varphi_h(t)\right] dt =$$

$$= \int_0^1 f(t)\, \varphi_h(t)\, dt, \quad \forall \varphi_h \in V_h \cap H_0^1(0, 1) \tag{2.23}$$

Hence we have

$$\|\hat{w}_h\|_{H_0^1(0,1)} \leq C\|f\|_{L^2(0,1)} \quad \forall h>0 \tag{2.24}$$

There exists an element $w \in H_0^1(0, 1)$ such that for some subsequence $\{\hat{w}_{h'}\}$ we obtain

$$\hat{w}_{h'} \rightharpoonup w \text{ weakly in } H_0^1(0, 1) \tag{2.25}$$

We define a sequence $\{f_h\} \subset H^{-1}(0, 1)$ where the element f_h is the projection of the element $f \in L^2(0, 1) \subset H^{-1}(0, 1)$ onto a finite dimensional subspace of $H^{-1}(0, 1)$, i.e.:

$$\langle f_h, \varphi_h \rangle = \int_0^1 f(t)\, \varphi_h(t)\, dt, \quad \forall \varphi_h \in V_h \cap H_0^1(0, 1) \tag{2.26}$$

$$\langle f_h, \psi \rangle = 0, \quad \forall \psi \in [V_h \cap H_0^1(0, 1)]^{\perp} \tag{2.27}$$

By the property of projection we obtain

$$f_h \to f \quad \text{strongly in } H^{-1}(0, 1) \tag{2.28}$$

and by (2.23) we have

$$f_h(t) = -\frac{d}{dt}\left[\hat{u}_h(t) \frac{d\hat{w}_h}{dt}(t)\right] + q\, w_h(t) \tag{2.29}$$

in the sense of distributions.

Using (2.22), (2.25), (2.28) and the similar argument as Murat [3] we obtain

$$-\frac{d}{dt}\left[\varphi(t) \frac{dw}{dt}(t)\right] + qw(t) = f(t) \quad \text{a.e. in } (0, 1) \tag{2.30}$$

On the other hand, by (2.25) we have

$$\hat{w}_{h''}(t) \to w(t) \quad \text{a.e. in } (0, 1) \tag{2.31}$$

for some subsequence $\{h''\}$ of the sequence $\{h'\}$. Therefore

$$a(t, w_{h''}(t)) \to a(t, w(t)) \quad \text{a.e. in } (0, 1) \tag{2.32}$$

Using the same argument as in the proof of Lemma 1, it can be shown that

$$a(t, \hat{w}_h(t)) - u_h(t) \to 0 \quad \text{a.e. in } (0, 1) \tag{2.33}$$

Combining (2.33) with (2.32) we obtain

$$\hat{u}_{h''}(t) \to a(t, w(t)) \quad \text{a.e. in } (0, 1) \tag{2.34}$$

Hence

$$\varphi(t) = a(t, w(t)) \quad \text{a.e. in } (0, 1)$$

and $w = \bar{y}$ is the unique solution of (2.1)′, (2.2). Thus all the sequences $\{\hat{u}_h\}$, $\{\hat{w}_h\}$ converge and we obtain (2.20) instead of (2.34). It can be shown that (2.20) implies (2.21).

3. Numerical examples

Using the finite dimensional approximations proposed in section 2, we solved numerically following four examples. In these examples we set $n=20$, i.e., $h=0.05$.

Example 1

$$-\frac{d}{dt}\left[(y^2+0.1)\frac{dy}{dt}\right]+y=10\,t^4-20\,t^3+11\,t^2-t+0.2=f(t) \tag{3.1}$$

$$y(0)=y(1)=0$$

For this example we know the exact solution

$$y(t)=-t^2+t \tag{3.2}$$

From (2.11) we obtain a linear system of equations

$$A(v)\,\alpha=F \tag{3.3}$$

Here $\alpha=(\alpha_1, \alpha_2, ..., \alpha_{19})^T$, $\alpha_i=w_h(ih)$, $v=(v_0, v_1, ..., v_{20})^T$, $v_i^2=u_h(ih)-0.1$ $F=(f_1, f_2, ..., f_{19})^T$, $f_i=\int_0^1 f(t)\,\varphi_i(t)\,dt$, and $A(v)$ is a tridiagonal matrix $[a_{ij}(v)]$ $i, j=1, 2, ..., 19$, whose nonzero elements are given by

$$\begin{aligned} a_{ii}(v)&=10\,(v_{i-1}^2+v_i^2+v_{i+1}^2)+(12.1/3)\\ a_{i,i+1}(v)&=a_{i+1,i}(v)=-10\,(v_i^2+v_{i+1}^2)-(11.95/6) \end{aligned} \tag{3.4}$$

Thus the optimization problem is to seek a value of v which minimizes the cost $\tilde{\mathcal{J}}$

$$\tilde{\mathcal{J}}=c\mathcal{J}=\frac{1}{2}ch\sum_{i=0}^{20}(\alpha_i-v_i)^2 \tag{3.5}$$

under the equality constraint (3.3). In (3.5) $\alpha_0=\alpha_{20}=0$ and c is appropriately chosen positive constant. To minimize the cost $\tilde{\mathcal{J}}$ we used the Davidon–Fletcher–Powell method.

On the other hand, if we apply to (3.1) the same approximation scheme as stated in section 2, we obtain a system of nonlinear equations.

$$\theta(\alpha)=A(\tilde{\alpha})\,\alpha-F=0, \quad \tilde{\alpha}=(0, \alpha^T, 0)^T. \tag{3.6}$$

We used Newton's method to solve the nonlinear equation (3.6). Let us define the equation error E by (3.7)

$$E=\max\{|\theta_1(\alpha)|, |\theta_2(\alpha)|, ..., |\theta_{19}(\alpha)|\} \tag{3.7}$$

where $\theta_i(\alpha)$ denotes the i-th component of $\theta(\alpha)$.

Both the solutions of the optimization problem and the nonlinear equation 3.6 coincided with the exact solution (3.2).

Table 1 shows the values of the cost $\tilde{\mathcal{J}}$ defined by (3.5) and the equation error E defined by (3.7) at each iteration and the CPU time for two sets of initial value.

Table 1

Computational results for Example 1.

iteration	case 1	$v_i=20$, v_i	case 2	$v_i=0$, i
		E		E
0	0.2100E+03	0.2865E—01	0.1945E—01	0.1141E—01
1	0.3593E—02	0.1141E—01	0.2704E—04	0.7724E—03
2	0.2037E—05	0.7724E—03	0.7231E—06	0.2234E—05
3	0.3093E—08	0.2232E—05	0.2660E—08	0.6659E—07
4	—	0.6566E—07	—	—
CPU time (sec)	0.83	0.86	0.76	0.69

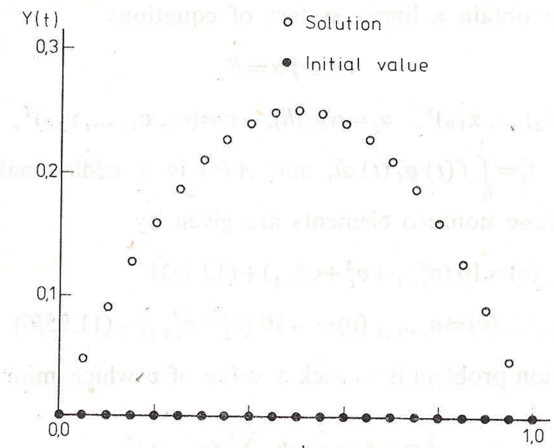

Figure 1. The initial value and the numerical solution for Example 1.

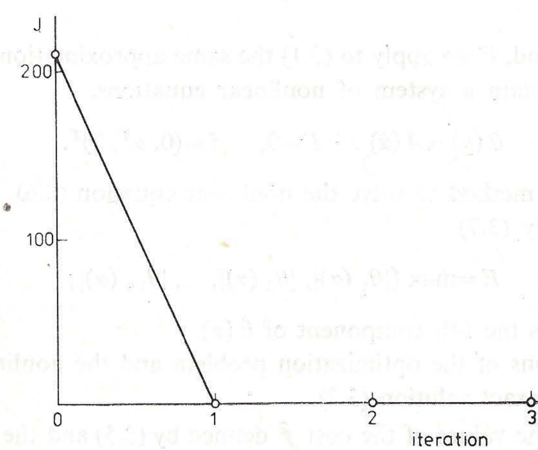

Figure 2. The values of cost versus iteration for Example 1.

For the optimization problem the initial value was given as $v_i=20$, $i=0, 1, \ldots, 20$ for the case 1 and $v_i=0$, $i=0, 1, \ldots, 20$ for the case 2.
The initial value α for Newton's method was given by solving (3.3)

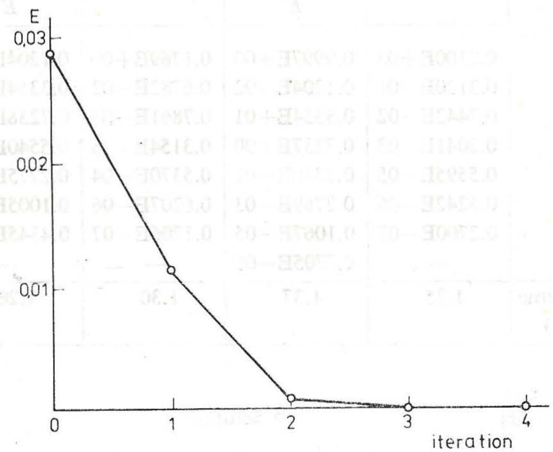

Figure 3. The equation errors versus iteration for Example 1.

Figure 1 represents the numerical solution of (3.1) and the initial value of α for the case 1 of Table 1. Figure 2 represents the graph of the cost $\mathcal{J}$ versus the number of iteration. Figure 3 represents the graph of the equation error E versus the number of iteration. Figures 2 and 3 correspond to the case 1 of Table 1.

All computations were done by using ACOS System 900 of Osaka University.

Example 2

$$-\frac{d}{dt}\left[(y^2+0.1)\frac{dy}{dt}\right]+y=\delta\left(t-\frac{1}{4}\right)=f(t) \qquad y(0)=y(1)=0 \tag{3.8}$$

For this example we don't know the exact solution. The solutions of the optimization problem and the nonlinear equation (3.6) for this example coincided each other.

Table 2 shows the values of the cost $\mathcal{J}$ and the equation error E at each iteration and the CPU time for two sets of initial value as in Example 1.

Figure 4 represents the numerical solution of 3.8 and the initial value of α for the case 1 of Table 2. Figure 5 represents the of the cost $\mathcal{J}$ versus the number of iteration. Figure 6 represents the graph of the equation error E versus the number of iteration. Figures 5 and 6 correspond to the case 1 of Table 2.

Example 3

$$-\frac{d}{dt}\left[\left(\left(\frac{dy}{dt}\right)^2+0.1\right)\frac{dy}{dt}\right]+y=23\,t^2-23\,t+6.2=f(t) \qquad y(0)=y(1)=0 \tag{3.9}$$

For this example we have the same exact solution as (3.2). However we used a method of approximation different from the method described in section 2. Namely we

Table 2.

Computational results for Example 2.

iteration	case 1 $v_i=20,\ i$		case 2 $v_i=0,\ i$	
		E		E
0	0.2100E+03	0.9997E+00	0.1769E+00	0.1204E+02
1	0.3120E−01	0.1204E+02	0.6782E−02	0.3354E+01
2	0.7442E−02	0.3354E+01	0.7861E−03	0.7238E+00
3	0.2041E−03	0.7237E+00	0.3154E−03	0.5540E−01
4	0.5595E−05	0.5540E−01	0.5370E−04	0.2775E−03
5	0.3242E−06	0.2769E−03	0.6207E−06	0.1005E−05
6	0.2700E−07	0.1067E−05	0.1706E−07	0.4345E−06
7	—	0.7705E−06	—	—
CPU time (sec)	1.25	1.37	1.30	1.20

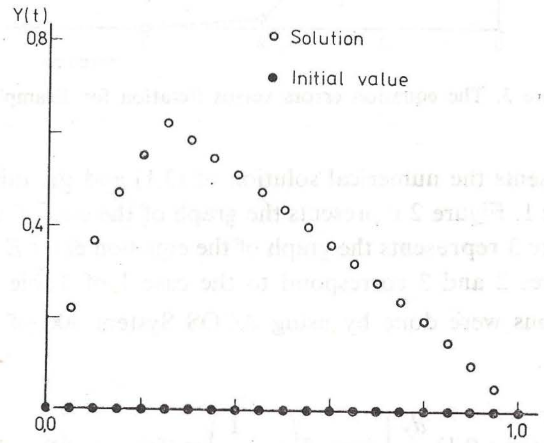

Figure 4. The initial value and the numerical solution for Example 2.

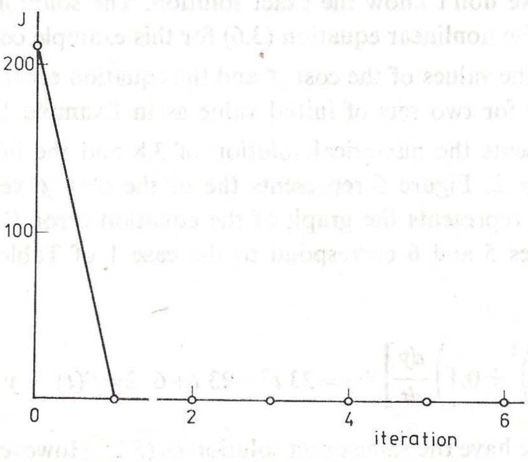

Figure 5. The values of cost versus iteration for Example 2.

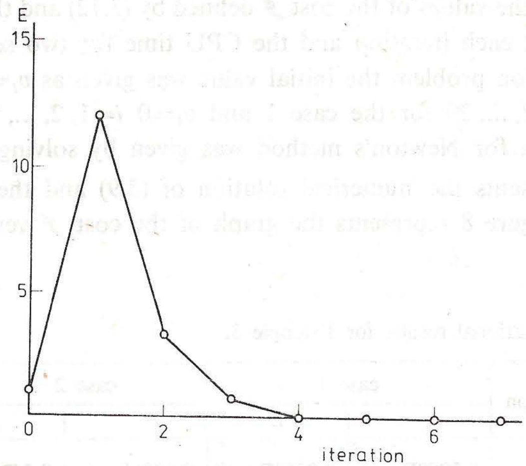

Figure 6. The equation errors versus iteration for Example 2.

approximated control $u(\cdot)$ by a piecewise constant function instead of a piecewise linear function. Using the approximation, we obtain the system of linear equations.

$$C(v)\,\alpha=F \tag{3.10}$$

which correspond to (2.11). Here $C(v)$ is a tridiagonal matrix $[C_{ij}(v)]$, $i,j=1,2,\dots,19$, whose nonzero elements are given by

$$\left.\begin{aligned} C_{ii}(v)&=20\,(v_i^2+v_{i+1}^2)+(12.1/3)\\ C_{i,i+1}(v)&=C_{i+1,i}(v)=-20\,v_{i+1}^2-(11.95/6)\end{aligned}\right\} \tag{3.11}$$

$$v=(v_1,v_2,\dots,v_{20})^T,\quad v_i^2=u_h(ih-0)-0.1,$$

$$\alpha=(\alpha_1,\alpha_2,\dots,\alpha_{19})^T$$

$$\alpha_i=w(ih),\quad F=(f_1,f_2,\dots,f_{19})^T,\ \text{and}\quad f_i=\int_0^1 f(t)\,\varphi_i(t)\,dt$$

The optimization problem is to seek a value of v which minimizes the cost

$$\tilde{\mathcal{J}}=c\mathcal{J}=\frac{1}{2}ch\sum_{i=1}^{20}\left[v_i-\frac{1}{h}(\alpha_i-\alpha_{i-1})\right]^2 \tag{3.12}$$

under the equality constraint (3.10), where $\alpha_0=\alpha_{20}=0$. To minimize the cost $\tilde{\mathcal{J}}$ we also used the Davidon–Fletcher–Powell method.

In the same way as before, applying to (3.9) the same approximation scheme, we obtain

$$\theta(\alpha)=C(\bar{\alpha})\,\alpha-F=0 \tag{3.13}$$

where $\bar{\alpha}=(\bar{\alpha}_1,\bar{\alpha}_2,\dots,\bar{\alpha}_{20})^T$ and $\bar{\alpha}_i=(\alpha_i-\alpha_{i-1})/h$. We also used Newton's method to solve (3.13). Both the solution of the optimization problem and the nonlinear equation (3.13) coincided with the exact solution (3.2).

Table 3 shows the values of the cost $\mathcal{J}$ defined by (3.12) and the equation error E defined by (3.7) at each iteration and the CPU time for two sets of initial value. For the optimization problem the initial value was given as $v_i=20$, $i=1, 2, ..., 10$, $v_i=-20$, $i=11, 12, ..., 20$ for the case 1 and $v_i=0$ $i=1, 2, ..., 20$ for the case 2. The initial value α for Newton's method was given by solving (3.10).

Figure 7 represents the numerical solution of (3.9) and the initial value of α for the case 1 Figure 8 represents the graph of the cost $\mathcal{J}$ versus the number of

Table 3.

Computational results for Example 3.

iteration	case 1		case 2	
		E		E
0	0.2000E+03	0.2558E+00	0.3996E+01	0.2424E+03
1	0.3997E+01	0.2422E+03	0.3720E+00	0.7177E+02
2	0.7782E+00	0.7172E+02	0.2529E+00	0.2121E+02
3	0.3033E+00	0.2120E+02	0.1453E+00	0.6222E+01
4	0.1112E+00	0.6218E+01	0.6338E−01	0.1768E+01
5	0.3466E−01	0.1770E+01	0.3112E−01	0.4301E+00
6	0.1409E−01	0.4297E+00	0.8469E−02	0.5402E−02
7	0.5814E−02	0.5396E−01	0.2293E−02	0.1047E−02
8	0.2041E−02	0.1045E−02	0.7402E−03	0.6109E−06
9	0.8338E−04	0.7078E−06	0.1133E−03	—
10	0.2416E−04	—	0.2482E−04	
11	0.2742E−05		0.1521E−05	
12	0.6974E−06		0.1238E−06	
13	0.1247E−06		0.2220E−07	
14	0.2317E−07		—	
CPU time (sec)	2.67	1.79	2.66	1.61

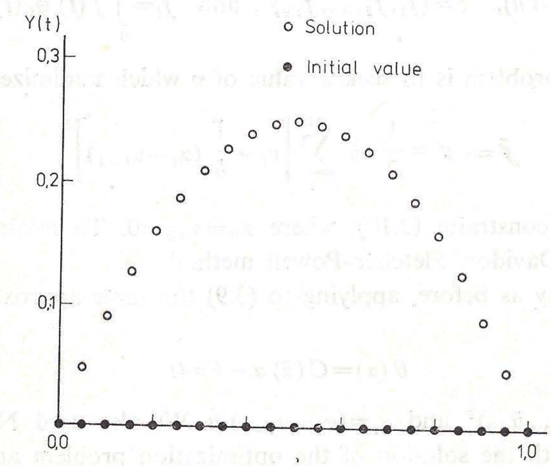

Figure 7. The initial value and the numerical solution for Example 3.

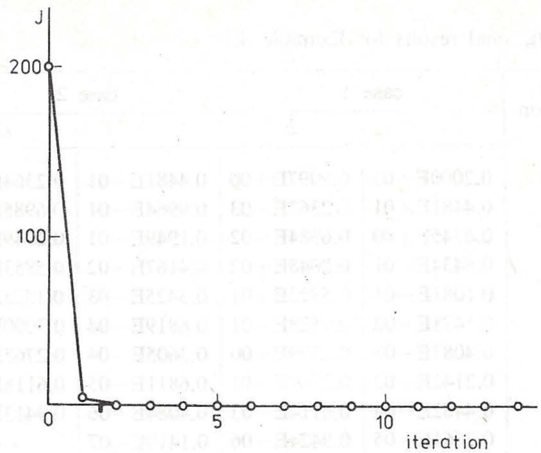

Figure 8. The values of cost versus iteration for Example 3.

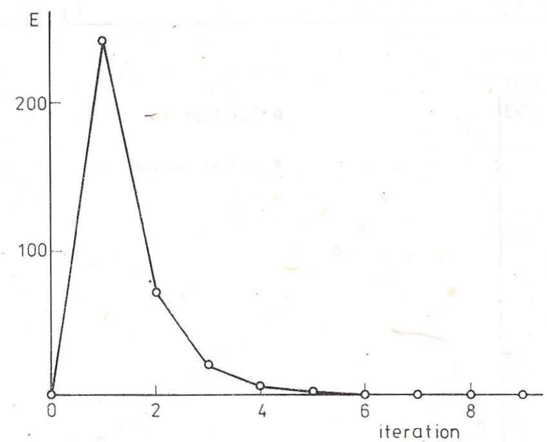

Figure 9. The equation errors versus iteration for Example 3.

iteration. Figure 9 represents the graph of the equation error E versus the number of iteration. Figures 8 and 9 correspond to the case 1 of Table 3.

Example 4

$$-\frac{d}{dt}\left[\left(\left(\frac{dy}{dt}\right)^2+0.1\right)\frac{dy}{dt}\right]+y=\delta\left(t-\frac{1}{4}\right)=f(t) \quad y(0)=y(1)=0 \qquad (3.14)$$

The solutions of the optimization problem and the nonlinear equation (3.13) foı this example coincided each other.

Table 4 shows the values of the cost J and the equation error E at each iteration and the CPU time for two sets of initial value as in Example 3.

Figure 10 represents the numerical solution of (3.14) and the initial value of α for the case 1 of Table 4. Figure 11 represents the graph of the cost $\mathcal{J}$ versus the

Table 4.

Computational results for Example 4.

iteration	case 1		case 2	
		E		E
0	0.2000E+03	0.9997E+00	0.4481E+01	0.2364E+03
1	0.4481E+01	0.2363E+03	0.9964E−01	0.6985E+02
2	0.6345E+00	0.6984E+02	0.1949E−01	0.2049E+02
3	0.6434E−01	0.2048E+02	0.4167E−02	0.5853E+01
4	0.1081E−01	0.5852E+01	0.3425E−03	0.1528E+01
5	0.1478E−02	0.1528E+01	0.6819E−04	0.3000E+00
6	0.4083E−03	0.2999E+00	0.3605E−04	0.2768E−01
7	0.2142E−03	0.2767E−01	0.6811E−05	0.6118E−03
8	0.4482E−04	0.6114E−03	0.4084E−06	0.9413E−06
9	0.2661E−05	0.9424E−06	0.1419E−07	—
10	0.1936E−06	—	—	
11	0.9610E−08			
CPU time (sec)	2.16	1.79	1.88	1.62

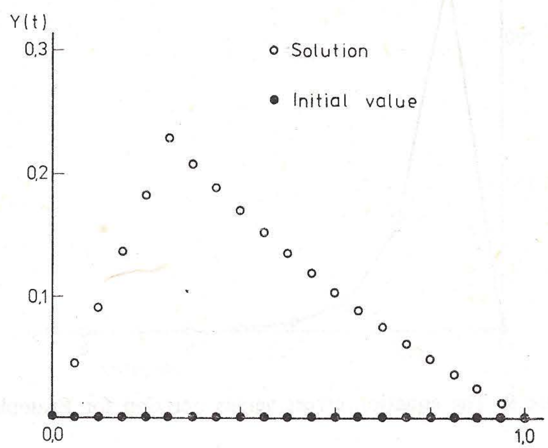

Figure 10. The initial value and the numerical solution for Example 4.

number of iteration. Figure 12 represents the graph of the equation error E versus the number of iteration. Figures 11 and 12 correspond to the case 1 of Table 4.

Remark 1. Stopping conditions were given by $\mathscr{J}\leqslant 10^{-7}$ and $E\leqslant 10^{-6}$, respectively.

Remark 2. The proper choice of the constant c in (3.5) and (3.12) prompts the convergence. The value of c was chosen as 5 to 30 for the above examples.

Remark 3. In Example 3, if the initial value $v_i=20$ $i=1, 2, ..., 20$ is given, the Davidon–Fletcher–Powell method gave an local minimum solution. However, if we use the simple gradient method for the first several iterations and the Davidon–Flecher–Powell method for later iterations, then we obtained the exact minimum solution.

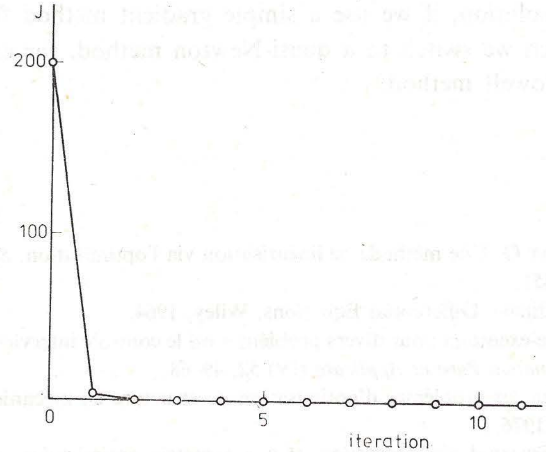

Figure 11. The values of cost versus iteration for Example 4.

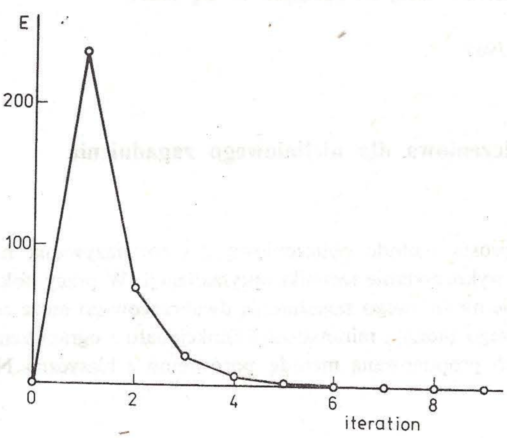

Figure 12. The equation errors versus iteration for Example 4.

4. Concluding remarks

We have proposed a simple numerical method for solving nonlinear two-point boundary-value problems by using an optimization technique. Corresponding finite-dimensional optimization problem is proposed, and the convergence of the sequence of solutions of the finite-dimensional optimization problems to the solution of the original two-point boundary-value problem is proved.

Computational results about four examples are shown. The numerical solutions by using the optimization technique and by using Newton's method are compared.

Both the methods gave satisfactory results. In Newton's method, if an initial value is far from the solution, it happens that the iterative computations do not converge. In the optimization method, however, even if an initial value is far away from the solution, the iterative computations can reduce the value of the cost function and converge to the solution, if we use a simple gradient method for the first several iterations and then we switch to a quasi-Newton method, for example to the Davidon–Fletcher–Powell method.

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Pewna metoda obliczeniowa dla nieliniowego zagadnienia dwubrzegowego.

Zaproponowano prostą metodę obliczeniową dla rozwiązywania nieliniowego zagadnienia dwubrzegowego przez wykorzystanie techniki optymalizacji. W pracy pokazano, że przy pewnych założeniach rozwiązanie nieliniowego zagadnienia dwubrzegowego może zostać wyznaczone drogą rozwiązania pomocniczego zadania minimalizacji funkcjonału z ograniczeniami. Podano przykłady obliczeniowe w których proponowaną metodę porównano z klasyczną Newtona.

Некоторый численный метод для нелинейной двуграничной задачи

Предложен простой численный метод для решения нелинейной двуграничной задачи посредством использования оптимизационных методов. В работе показано, что при некоторых предпосылках решение нелинейной двуграничной задачи может быть найдено путем решения вспомогательной задачи минимизации функционала с ограничениями. Приведены численные примеры в которых предлагаемый метод сравнивается с классическим методом Ньютона.