Control
and Cybernetics

VOL. 12 (1983) No. 1-2

Enumeration of efficient solutions for bicriteria integer programs by number theoretic approach

by

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This paper presents an algorithm for enumeration of all efficient solutions for equality constrained linear integer bicriteria problems.

The idea is based on a reduction of a bicriteria problem to the problem of solving a system of parametric linear diophantine equations. For such a system the general solution is found and the Fourier-Motzkin elimination scheme is applied to verify whether among all solutions there exist nonnegative ones.

Using this idea an iterative algorithm is proposed and a numerical example is given.

1. Introduction

In recent years many papers have been published on the multiple criteria continuous programming. There is natural interest to extend this results for integer programming problems. Some steps in direction have been already done. Bitran [3] investigated multiple criteria zero-one programs. Zionts [10] extended his and Wallenius idea of interactive approach to multiple criteria integer programming. Similar ideas were exploited by Villarreal and Karwan [9]. The main aim of this paper is to demonstrate an algorithimic tool to enumerate all efficient solutions for a special class of these problems, namely for integer bicriteria programs.

The integer multiple criteria problem is formulated as

$$\max\{Cx \mid x \in S\} \tag{MP}$$

where $S=D \cap I$, $D=\{x \in R^n \mid Ax=b,\ x \geqslant 0\}$, $I=\{x \in R^n \mid x_j\text{-integer},\ j=1, \ldots, n\}$, C is a $p \times n$ matrix, A is a $m \times n$ matrix, b is a $m \times 1$ vector. We assume that C, A and b have integer elements, S is bounded and non-empty. Here the sign „max" is an operator of finding all efficient solutions which are defined below. For a bicriteria problem (BP) $p=2$.

Integer vectors $x \in R^n$ and $y \in R^p$ form a decision space and a criteria space respectively.

An example of a practical problem which can be formulated as a MP problem is space (plane) filling with finite number of object types under conflicting goals.

To formulate any multiple criteria problem a relation should be introduced by which solutions can be compared in a criteria space. The most frequently applied is Pareto partial ordering $x \geq y$ which means $x_j \geqslant y_j$, $j \in J$, with at least one strict inequality. A solution $x_0 \in S$ is said to be efficient for a MP program if there is no other solution $x \in S$ such that $Cx \geq Cx_0$. Solutions $x \in S$ for which $Cx_0 \geq Cx$ are said to be dominated by x_0, or equivalently, x_0 dominates them.

It has been shown (see e.g. [5]) that for convex multiple objective programs every efficient solution maximizes a linear functional λCx, $\lambda \in R^p$, $\lambda \geq 0$. This is not the case for integer multiple objective programming as it is shown by the following example.

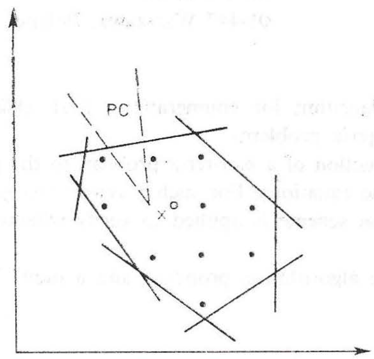

Fig. 1

The preference cone is defined as $PC = \{p \in R^p | Cp \geq 0\}$. An alternative definition of efficient solution is: given a set X, $x_0 \in X$ is efficient if $X \cap (\{x_0\} + PC) = \{x_0\}$. On the fig. 1 a feasible solution set S of an integer programming problem and a preference cone at a feasible solution x^0 is presented. Clearly the solution x^0 is efficient but as it lies inside the conv (S) (convex hull of S) there is no λ such that x^0 maximizes the functional λCx over conv (S).

In this paper an algorithm for enumeration of all efficient solutions for BP problems basing on direct application of the efficient solution definition is proposed. A slight modification of this algorithm can be also applied to enumerate all optimal solutions of single objective integer programs and to perform sensitivity analysis for changes of r.h.s. vector elements. In the next paragraph the BP problem is formulated in terms of parametric diophantine equations. A number-theoretic method for solving systems of diophantine equations with one parameter is described in paragraph three. In paragraph four this method is extended for enumeration of all efficient solutions for BP problems and an algorithm is given. A numerical example and some possible improvements in the Fourier-Motzkin elimination method are presented in appendices.

2. Efficient solutions enumeration problem in number theoretic setting

Efficient solutions for BP problems can be found by a direct application of the efficiency definition. Let $a_j = \max\{c^j x | x \in S\}$, $j=1, 2$ and $b_1 = \max\{c^1 x | x \in S, c^2 x = a_2\}$. The following scheme enumerates efficient solutions for BP problems.

Scheme E

In consecutive iterations indexed by i, $i \in I = \{0, 1, \ldots, i^*\}$, i^* to be determined, find all pairs of integers (s_i, r_i), such that

i) $s_{i-1} > s_i$, $i \in I$, $s_0 = a_1$, $s_{i*} = b_1$, $s_{i-1} - s_i$ is as small as possible,

ii) $r_{i-1} < r_i$, $i \in I$, $r_{i*} = a_2$,

iii) the set of diophantine equations

$$\begin{aligned} c^1 x &= s_i \\ c^2 x &= r_i \\ Ax &= b \end{aligned} \tag{*}$$

has nonnegative solutions,

iv) for chosen s_i, r_i is as large as possible,

and for all such pairs (s_i, r_i) determine all solutions satisfying (*).

LEMMA. *The Scheme E enumerates all efficient solutions.*

Proof. There is no efficient solution for $s \notin [a_1, b_1]$. The solutions corresponding to (s_0, r_0) are efficient. Suppose the first k pairs (s_i, r_i) have been enumerated and all of them correspond to efficient solutions. Since $s_0 > \ldots s_k > s_{k+1}$ and $r_0 < \ldots < r_k < r_{k+1}$ no solution corresponding to the $(k+1)$-th pair dominate any solution corresponding to the first k pairs and vice versa. To preserve efficiency of solutions corresponding to the $(k+1)$-th pair (s_{k+1}, r_{k+1}), r_{k+1} must be as large as possible. By the condition *i)* no nondominated pair (s, r) will be omitted. Q.E.D. ■

Let us observe that if we would start the Scheme E with (s_{i*}, r_{i*}) and proceed with decreasing values of i it may cause generation of some non-efficient points.

3. Solving sets of diophantine equations in nonnegative numbers

Suppose that a set of diophantine equations $Ax=b$ is given. The solvability conditions for a set of diophantine equations are given by the following theorem.

THEOREM [8]. *The system of m equations*

$$a_{r_1} x_1 + \ldots + a_{r_n} x_n = b_r \quad (r = 1, \ldots, m)$$

is solvable in integers $x_1, \ldots, x_n$ if and only if the greatest common divisor of all m-row determinants of the matrix of coefficients a_{rs} is equal to the greatest common divisor of all m-row determinants of the matrix of coefficients a_{rs} and b_r.

In the single equation case the theorem specifies the well known condition: a diphantine equation is solvable iff the greatest common divisor (g.c.d) of coefficients divides the r.h.s element.

To check whether the condition given in the theorem holds is not easy. On the other hand there exist efficient algorithms for computing the g.c.d of n numbers. So the solvability of a system of diophantine equations may be verified and general solutions may be found by sequential application of a g.c.d algorithm as follows.

A general solution of a single diophantine equation with n variables has the form $x=x^*+Fy$, where x^* is any particular solution, F — a fundamental (integer) matrix of $n\times(n-1)$ size, y — any integer $(n-1)$ vector. Any algorithm for g.c.d constructs implicitly or explicitly x^* and F.

A set of m diophantine equations with n variables and a constraint matrix of the rank m has its general solution (if exists) in the same form: $x=x^*+Fy$, where F is of $n\times(n-m)$ size, y is of $(n-m)$ size.

It has been proved [7] that if $x=x^*+Fy$ is a general solution of the first i diophantine equations then the set of the first $i+1$ diophantine equations is solvable iff the single diophantine equation $a^{i+1}x^*+a^{i+1}Fy=b_{i+1}$ is solvable. The observation of this fact constitutes the basis for the Rubin's sequential algorithm for solving sets of diophantine equations, which for a given set of equations builds a particular solution x^* and a fundamental matrix F [7].

The sequential algorithm has been adapted by Richmond and Ravindran [6] for solving integer programming problems. Assuming that an (integer) upper bound z_0 for the value of an objective function is known every linear integer program with rational coefficients and with an objective function to be maximized can be formulated as follows.

IP: find minimal integer $k^*\geqslant 0$

for which the set of diophantine equations

$$\begin{aligned} cx&=z_0-k \\ Ax&=b \end{aligned} \tag{1}$$

has nonnegative solutions x, where c is an integer vector of objective function coefficients.

The set of diophantine equations (1), if solvable, has solutions $x(k)=x^*(k)+Fy$, where $x^*(k)$ is a particular solution which depends on k.

It has been proved in [6] that $x^*(k)$ can be expressed as $x^*(k)=x^*(k_0)-(k-k_0)f$, where k_0 is the smallest value of k for which (1) has solutions (not necessarily nonnegative), $x^*(k_0)$ a particular solution for k_0, f a constant integer vector. The vector f can be determined when the second value $k_1>k_0$, for which (1) has solution, and $x^*(k_1)$ are known. Then $f=(x^*(k_1)-x^*(k_0))/(k_0-k_1)$. Thus after substituting the expression for $x^*(k)$ to the formula for solutions of

(1) we get $x(k)=x^*(k_0)-(k-k_0)f+Fy$. The nonnegativity conditions imposed on x variables imply $x^*(k_0)-(k-k_0)f+Fy\geqslant 0$. Now an integer programming problem IP can be reformulated as follows

IP1: find minimal integer $k^*\geqslant 0$

for which the set of diophantine inequalities

$$x^*(k_0)-(k-k_0)f+Fy\geqslant 0 \tag{2}$$

is solvable.

To solve (2) the Fourier-Motzkin (F-M) elimination method can be applied. We sketch this method briefly here following the notation used in [6].

In a system of linear inequalities

$$Fy\geqslant p \tag{3}$$

where F is a matrix of $n\times r$ size and p has n elements we choose any variable, say y_1, and we partition the set of n inequalities into three groups:

$$h_1=\{s|f_{s,1}>0\},$$

$$h_2=\{t|f_{t,1}<0\},$$

$$h_3=\{u|f_{u,1}=0\}.$$

We write inequalities belonging to the first group as

$$p_s/f_{s,1}-\sum_{j=2}^{r}(f_{s,j}/f_{s,1})\,y_j\leqslant y_1 \quad \text{for} \quad s\in h_1, \tag{4}$$

inequalities belonging to the second group as

$$y_1\leqslant p_t/f_{t,1}-\sum_{j=2}^{r}(f_{t,j}/f_{t,1})\,y_j \quad \text{for} \quad t\in h_2, \tag{5}$$

and inequalities belonging to the third group as

$$\sum_{j=2}^{r}f_{u,j}\,y_j\geqslant p_u \quad \text{for} \quad u\in h_3.$$

We eliminate the variable y_1 by forming for each pair of indices $s\in h_1$, $t\in h_2$ the set of inequalities

$$p_s/f_{s,1}-\sum_{j=2}^{r}(f_{s,j}/f_{s,1})\,y_j\leqslant p_t/f_{t,1}-\sum_{j=2}^{r}(f_{t,j}/f_{t,1})\,y_j$$

This set together with inequalities indexed by $u\in h_3$ form the new set of inequalities with one unknown less and (4) and (5) establish upper and lower bounds on y_1. The second variable is then eliminated, its upper and lower bounds established and so on until bounds for the last eliminated variable are determined. Then going in the order reverse to the order of variable elimination we choose any feasible value of the last eliminated variable and substitute it to bounding expressions for the last but one eliminated variable, choose any of its feasible value etc. However choosing integer values of variables it might happen that for some particular values taken by a part of variables, feasibility ranges for other variables are empty. Therefore the F-M method when applied for solving diophantine inequalities, must be combined with a simple backtrack procedure for determination of integer solution vectors (if any). This procedure works as follows. Let the order of variable elimination be $1, 2, \ldots, r$. We denote L_i and U_i lower and upper bounds on variables resulted from the *F-M* elimination, $\lfloor a \rfloor$ — the greatest integer $\leqslant a$, $\lceil a \rceil = \lfloor a \rfloor + 1$.

1. If $\lceil L_r \rceil > \lfloor U_r \rfloor$ then (3) has no integer solution.
   If $\lceil L_r \rceil \leqslant \lceil U_r \rceil$ set $y_r := \lceil L_r \rceil$.
2. Substitute the value of y_r to the set of inequalities containing y_r and y_{r-1} only. This gives conditional bounds $\lceil L_{r-1} \rceil$ and $\lfloor U_{r-1} \rfloor$ on y_{r-1}. We continue this process until one of the following cases ocur.
   a) If for every i, $1 \leqslant i \leqslant r$, $\lceil L_i \rceil > \lfloor U_i \rfloor$ does not hold then $y_i = \lceil L_i \rceil$, $1 \leqslant i \leqslant r$ is integer solution for (4).
   b) If for any i, $\lceil L_i \rceil > \lfloor U_i \rfloor$ then if $\lceil L_{i+1} \rceil + 1 \leqslant \lfloor U_{i+1} \rfloor$ then $\lceil L_{i+1} \rceil := \lceil L_{i+1} \rceil + 1$ and the process of backward substitution must be repeated, otherwise if $\lceil L_{i+2} \rceil + 1 \leqslant \lfloor U_{i+2} \rfloor$ then $\lceil L_{i+2} \rceil := \lceil L_{i+2} \rceil + 1$ and the process of backward substitution must be repeated, ..., otherwise if $\lceil L_r \rceil + 1 \leqslant \lfloor U_r \rfloor$ then $\lceil L_r \rceil := \lceil L_r \rceil + 1$ and the process of backward substitution must be repeated, otherwise $\lceil L_r \rceil + 1 > \lfloor U_r \rfloor$ implies that (3) has no integer solution.

Now the method for solving integer programs can be summarized as follows

1. Reduce an integer program to the IP formulation.
2. Applying sequential algorithm solve a system of diophantine equations (1) and find k_0, k_1, $x^*(k_0)$, $x^*(k_1)$ and f.
3. Solve a system of linear diophantine inequalities (2) by the variant of *F-M* method described above, where $r = n - m - 1$, $y_{r+1} = k$ and $\lceil L_{r+1} \rceil = \max(0, \lceil L_{r+1} \rceil)$. If there is a solution $\tilde{k} = \lceil L_{r+1} \rceil$, $\tilde{y}_i = \lceil L_i \rceil$ then $x(\tilde{k}) = x^*(k_0) - (\tilde{k} - k_0) f + F\tilde{y}$ is the optimal solution of P with the objective function value $z_{\text{opt.}} = cx(\tilde{k}) = z_0 - \tilde{k}$.

Let $k_0, k_1, k_2, \ldots$, where $k_i < k_{i+1}$ be the consecutive values of k for which (1) has solutions. It has been proved in [6] that $k_{i+1} - k_i = \delta$ for all nonnegative i, where δ—integer constant. Thus each time $[L_{n+1}]$ is increased it can be increased by $\delta \geqslant 1$ instead of 1.

4. The Algorithm

The method for solving integer programming problems as described in the preceeding paragraph can be easily adapted to implement the Scheme E.
We start with computation of a_1. To do this we form from (*) the following problem.
P_1: find minimal integer $t^* \geqslant 0$

for which the set of diophantine equations*)

$$Ax=b$$
$$c^2x-l=0$$
$$c^1x=z_0^1-t$$

has nonnegative solution vectors x, where z_0^1 is an integer upper bound for c^1x. Then $a_1=z_0^1-t^*$. $l(t^*)$ establishes a lower bound for $r_i's$ in (*).

Analogously we form P_2 problem.

P_2: find minimal integer $k^*\geqslant 0$

for which the set of diophantine equations

$$Ax=b$$
$$c^1x-s=0$$
$$c^2x=z_0^2-k$$

has nonnegative solution vectors x, where z_0^2 is an integer upper bound for c^2x. Then $a_2=z_0^2-k^*$. Determine m.

Let us observe that by extra computations of the described backtrack procedure other optimal solutions (if any) can be enumerated. Thus we can determine b_1 as a maximal value of s over all optimal solutions of P_2.

The general solution of the diophantine equation set of the P_2 problem is

$$\begin{bmatrix} x \\ s \end{bmatrix}=\begin{bmatrix} x^*(k_0) \\ s^*(k_0) \end{bmatrix}-(k-k_0)f+Fy \tag{6}$$

Now we can take advantage of conditional bounds on k and y established during the elimination phase of the F-M method to check whether for some specific values of s there could be $x\geqslant 0$ in (6). This time F has $(n+1)\times r$ size, where $r=n+1-m-2$. Assume that the last row of F has only one non zero element, namely the element $f_{n+1,r}$ (if it is not the case F can be always reduced to this form). Then the last row of (6) has the form

$$s=s^*(k_0)-(k-k_0)f_{n+1}+f_{n+1,r}\,y_r \tag{7}$$

To implement the Scheme E we must determine consecutive values of $s\in[b_1, a_1]$ which satisfy the requirement i) of the scheme and guarantee $x\geqslant 0$ in (6). We need to take into account only those s for which (7) has solutions i.e. those s for which g.c.d. $(f_{n+1}, f_{n+1,r})$ divides $(s-s^*(k_0)-k_0 f_{n+1})$ and to satisfy the requirement iv) we must have k as small as possible. The Scheme E is equivalent to the following algorithm.

Algorithm E

i) Solve P_1 and determine a_1 and $l(t^*)$.
ii) Solve P_2 and determine a_2 and b_1. During the Fourier-Motzkin elimination phase, eliminate k as the last variable and y_r as the last but one variable. Set $p:=0$, $h:=z_0^2-l(t^*)$.

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*) We write now constraints at the top because of numerical convenience. See App. I.

Step p. Take consecutive admissible s. Check wheather the equation $s=s^*(k_0)+-(k-k_0)f_{n+1}+f_{n+1,r}y_r$ has solutions. If not, take consecutive values of s until the equation has solutions. Start the Fourier-Motzkin substitution phase by setting $\lfloor L_{r+1}\rfloor:=\max(\lfloor L_{r+1}\rfloor, 0)$, $\lfloor U_{r+1}\rfloor:=\min(\lfloor U_{r+1}\rfloor, h)$, $k:=\lfloor L_{r+1}\rfloor$.

Step (p, s, k). Having k and s determine y.

If there is no feasible y satisfying (6) and such that $s=s^*(k_0)-(k-k_0)f_{n+1}++f_{n+1,r}y_r$, set $k:=\lceil L_{r+1}\rceil+\delta$ and repeat Step (p, s, k). If there is no feasible y satisfying (6) for all admissible k, repeat Step p. If a feasible (k, y) is determined set $s_p:=s$, $r_p:=z_0^2-k$, $h:=k-\delta$ and save $(s_p, r_p)^T$. Find all solutions x corresponding to $(s_p, r_p)^T$. If $s_p=b_1$ then $i^*:=p$ and STOP.

Set $p=p+1$ and repeat Step p.

In Appendix I a complete numerical example illustrates various steps of the algorithm.

5. Conclusions

The main disadvantage of the Fourier-Motzkin elimination method is an increase of the inequality number in the elimination phase. In the Appendix II we discuss the possibility to avoid this phenomenon.

The approach presented in this paper can be applied also to inequality constrained problems by transforming inequalities to equalities. It can be also used to perform the sensitivity analysis. We can take advantage of the fact that any changes of r.h.s. vectors do not affect fundamental matrices but they affect particular solutions only. A slight modification of sequential algorithm will provide a particular solution for any r.h.s. vector. We observe further that the process of building of new inequalities in the F-M elimination method is controlled by coefficient values of fundamental matrices but not by constant values. It implies that as long as fundamental matrix is unchanged the number and structure of inequalities built during elimination of variables does not vary. Once this structure is found we may use it for any values of r.h.s. vectors. The changes of r.h.s. vectors change constant values in inequalities (e.g. particular solutions of diophantine equations sets) and at last they change lower and upper bound L_i and U_i. Keeping track of h_1, h_2, h_3 on each step of the F-M elimination, each time the r.h.s. is changed new bounds L_i and U_i can be easily computed. In this way we can easily find stability ranges for given optimal vector x^0.

Appendix I. An Example

Consider the BP as follows

$$x_1+2x_2+x_3+2x_4=2$$
$$4x_1+5x_2+7x_3+2x_4=w_1 \to \max$$
$$-2x_1+7x_2-x_3+8x_4=w_2 \to \max$$

$x_1, x_2, x_3, x_4 \geqslant 0$ and integer.

Applying the Algorithm E we enumerate all efficient solutions of this problem.

Algorithm E

i) Assume $z_0^1=14$.

P_1: find minimal integer $t^*\geqslant 0$ for which the set of diophantine equations

$$x_1+2x_2+x_3+2x_4=2$$

$$-2x_1+7x_2-x_3+8x_4-l=0$$

$$4x_1+5x_2+7x_3+2x_4=14-t$$

has a nonnegative solution vector x.

Solution of P_1.

$t=0$. The r.h.s. vector $(2, 0, 14)^T$ and the general solution of this set is

$$\begin{bmatrix} x \\ l \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ -2 \end{bmatrix} + \begin{bmatrix} -3 & -4 \\ 1 & 0 \\ 1 & 2 \\ 0 & 1 \\ 12 & 14 \end{bmatrix} y$$

By substituting $y=0$ we get solution vector $x\geqslant 0$, hence $t^*=0$, $a_1=14$, $l\,(t^*)=-2$.

ii) Assume $z_0^2=8$.

P_2: find minimal integer $k^*\geqslant 0$ for which the set of diophantine equations

$$x_1+2x_2+x_3+2x_4=2$$

$$4x_1+5x_2+7x_3+2x_4-s=0$$

$$-2x_1+7x_2-x_3+8x_4=8-k$$

has a nonnegative solution vector x.

Solution of P_2.

$k=0$. The r.h.s vector $(2, 0, 8)^T$. The general solution of this set is

$$\begin{bmatrix} x \\ s \end{bmatrix} = \begin{bmatrix} -10 \\ 0 \\ 12 \\ 0 \\ 44 \end{bmatrix} + \begin{bmatrix} 9 & 10 \\ 1 & 0 \\ -11 & -12 \\ 0 & 1 \\ -36 & -42 \end{bmatrix} y$$

$k=1$. The r.h.s. vector $(2, 0, 7)^T$. The particular solution exists, namely $(x^*, s^*)^T= =(-9, 0, 11, 0, 41)^T$. (Note that the fundamental matrix is the same as for $k=0$). The difference vector $f=(-1, 0, 1, 0, 3)^T$, $\delta=1$. Now

$$\begin{bmatrix} x \\ s \end{bmatrix} = \begin{bmatrix} -10 \\ 0 \\ 12 \\ 0 \\ 44 \end{bmatrix} - k \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 9 & 10 \\ 1 & 0 \\ -11 & -12 \\ 0 & 1 \\ -36 & -42 \end{bmatrix} y$$

Condition $x \geqslant 0$ implies the following set of diophantine inequalities

$$\begin{aligned} 9y_1+10y_2+k &\geqslant 10 \\ y_1 &\geqslant 0 \\ -11y_1-12y_2-k &\geqslant -12 \\ y_2 &\geqslant 0 \end{aligned}$$

By unimodular transformations we reduce F to the required form (only one non-zero element in the last row of F).
Then the above set reduces to

$$\begin{aligned} 3z_1+z_2+k &\geqslant 10 \\ 7z_1-z_2 &\geqslant 0 \\ -5z_1-z_2-k &\geqslant -12 \\ -6z_1+z_2 &\geqslant 0 \end{aligned}$$

and $s=44-6z_2-3k$
We apply now the $F-M$ elimination method.

Elimination phase

Iteration I.

$$\begin{aligned} z_1 &\geqslant 10/3-1/3z_2-1/3k \\ z_1 &\geqslant 1/7z_2 \\ z_1 &\leqslant 12/5-1/5z_2-1/5k \\ z_1 &\leqslant 1/6z_2 \end{aligned}$$

Iteration II.

$$\begin{aligned} z_2 &\geqslant 7-k \\ z_2 &\leqslant 7-7/12k \\ z_2 &\geqslant 20/3-2/3k \\ z_2 &\geqslant 0 \end{aligned}$$

Iteration III.

$$\begin{aligned} k &\geqslant 0 \\ k &\geqslant -4 \\ k &\leqslant 12 \end{aligned}$$

Substitution & backtrack phase

$$\begin{aligned} k&=0 \\ 7 \leqslant z_2 &\leqslant 7 \\ 1 \leqslant z_1 &\leqslant 1 \end{aligned}$$

Hence $k^*=0$ and $a_2=r_{i*}=8$. Since there is only one nonnegative solution for $k=0$, namely this given by $z_2=7$, $z_1=1$, the value b_1 is determined by $x^T=$ $=(-10, 0, 12, 0)^T+(3, 7, -5, -6)^T+(7, -7, -7, 7)^T=(0, 0, 0, 1)^T$.

Substituting this to the first objective function we get $b_1=s_{i*}=2$.

$p=0$, $h=10$.

Step 0. $s=14$. The equation (7) takes the form $14=44-6z_2-3k$. This equation has solutions (for g.c.d. (6, 3) divides 30). $\lceil U_{r+1}\rceil:=\min(12, 10)$, $[L_{r+1}]:=\max(0, 0)$.

Step (0, 14, 0). $k=0$, $z_2=5$ impossible (conditional bounds imply $k=0$, $z_2=7$).

Step (0, 14, 1). $k=1$, $6z_2=27$ no solution.

Step (0, 14, 2). $k=2$, $z_2=4$ impossible (conditional bounds imply $k=2$, $z_2\geqslant 5$).

Step (0, 14, 3). $k=3$, $6z_2=21$ no solution.

Step (0, 14, 4). $k=4$, $z_2=3$ impossible.

Step (0, 14, 5). $k=5$, $6z_2=15$ no solution.

Step (0, 14, 6). $k=6$, $z_2=2$ impossible.

Step (0, 14, 7). $k=7$, $6z_2=9$ no solution.

Step (0, 14, 8). $k=8$, $z_2=1$ impossible.

Step (0, 14, 9). $k=9$, $6z_2=3$ no solution.

Step (0, 14, 10). $k=10$, $z_2=0$ possible (conditional bounds imply $k=10$, $z_2=0$, $z_1=0$).

$s_0=14$, $r_0=-2$, $h=9$. The only solution corresponding to (s_0, r_0) is $x=(0, 0, 2, 0)^T$.

Step 1. $s=13$. The equation (7) takes the form $13=44-6z_2-3k$. This equation has no solution (for g.c.d. (6, 3) does not divide 31).

$s=12$. The equation (7) takes the form $12=44-6z_2-3k$. This equation has no solution.

$s=11$. The equation (7) takes the form $11=44-6z_2-3k$. This equation has solutions. $\lfloor U_{r+1}\rfloor:=\min(12, 9)$, $\lceil L_{r+1}\rceil:=0$.

Step (1, 11, 0). $k=0$, $6z_2=33$ no solution.

Step (1, 11, 1). $k=1$, $z_2=5$ impossible.

Step (1, 11, 2). $k=2$, $6z_2=27$ no solution.

Step (1, 11, 3). $k=3$, $z_2=4$ impossible.

Step (1, 11, 4). $k=4$, $6z_2=21$ no solution.

Step (1, 11, 5). $k=5$, $z_2=3$ impossible.

Step (1, 11, 6). $k=6$, $6z_2=15$ no solution.

Step (1, 11, 7). $k=7$, $z_2=2$ impossible.

Step (1, 11, 8). $k=8$, $6z_2=9$ no solution.

Step (1, 11, 9). $k=9$, $z_2=11$ impossible.

$s=10$. The equation (7) takes the form $10=44-6z_2-3k$. This equation has no solution.

$s=9$. The equation (7) takes form $9=44-6z_2-3k$. This equation has no solution.

$s=8$. The equation (7) takes form $8=44-6z_2-3k$. This equation has solutions. $\lceil U_{r+1}\rceil:=\min(12, 9)$, $\lceil L_{r+1}\rceil:=0$.

Step (1, 8, 0). $k=0$, $z_2=6$ impossible.
Step (1, 8, 1). $k=1$, $6z_2=33$ no solution.
Step (1, 8, 2). $k=2$, $z_2=5$ impossible.
Step (1, 8, 3). $k=3$, $6z_2=27$ no solution.
Step (1, 8, 4). $k=4$, $z_2=4$ impossible.
Step (1, 8, 5). $k=5$, $6z_2=21$ no solution.
Step (1, 8, 6). $k=6$, $z_2=3$ impossible.
Step (1, 8, 7). $k=7$, $6z_2=15$ no solution.
Step (1, 8, 8). $k=8$, $z_2=2$ impossible.
Step (1, 8, 9). $k=9$, $6z_2=9$ no solution.

$s=7$. The equation (7) takes the form $7=44-6z_2-3k$. This equation has no solution.

$s=6$. The equation (7) takes the form $6=44-6z_2-3k$. This equation has no solution.

$s=5$. The equation (7) takes the form $5=44-6z_2-3k$. This equation has solutions. $\lfloor U_{r+1}\rfloor:=\min(12, 9)$, $\lceil L_{r+1}\rceil:=0$.

Step (1, 5, 0). $k=0$, $6z_2=39$ no solution.
$k=1$, $z_2=6$ possible (conditional bounds imply $k=1$, $z_2=6$, $z_1=1$).

$s_1=5$, $r_1=7$, $h=0$. The only solution corresponding to (s_1, r_1) is $x=(0, 1, 0, 0)^T$.

Step 2. $s=4$. The equation (7) takes the form $4=44-6z_2-3k$. This equation has no solution.

$s=3$. The equation (7) takes the form $3=44-6z_2-3k$. This equation has no solution.

$s=2$. $s=b$ hence $(s_2, r_2)=(2, 8)$. STOP. The vector x corresponding to (s_2, r_2) has been already determined.

Appendix II. Implementing the Fourier-Motzkin elimination method

Consider the following set of linear inequalities

$$Fx \geqslant p$$

The main drawback of the Fourier-Motzkin elimination method is the fast grow of the inequalities number during the elimination phase. Bradley has shown [4] that if F is $(n+1)\times n$ matrix or $(n+2)\times n$ matrix then its triangularization and the proper choice of an eliminated variable guarantee that the number of inequalities on each iteration decreases. This result, presented originally in the form of two theorems, we restate now using our notation in the Lemma 1. For explanatory purpose we present also here the proof of it.

LEMMA 1. *Let F be a $m\times n$ real matrix of full rank, $m\leqslant n+2$. Then by the triangularization of the matrix and the proper choice of an eliminated variable the number of inequalities to be dealt with on each iteration of the Fourier-Motzkin elimination method decreases.*

Proof.

Case A) $m \leqslant n$

By a transformation $FKK^{-1}y \geqslant p$, where K is an unimodular matrix, the set of inequalities can be reduced to $(T, 0)\, z \geqslant p$, where $z = K^{-1}y$, T-lower triangular $m \times m$ marix of the rank m. Then we can eliminate from the first m inequalities the variable z_m what results in one extra inequality of the form $z_m \geqslant \frac{p_m}{t_{m,m}} - \frac{t_{m,1}}{t_{m,m}} z_1 - \ldots - \frac{t_{m,m-1}}{t_{m,m}} z_{m-1}$ if $t_{m,m} \in h_1$ or $z_m \leqslant \frac{p_m}{t_{m,m}} - \frac{t_{m,1}}{t_{m,m}} z_1 - \ldots - \frac{t_{m,m-1}}{t_{m,m}} z_{m-1}$ if $t_{m,m} \in h_2$ (see paragraph 3 for definitions of h_i). The case $t_{m,m} \in h_3$ is impossible for the rank $(T) = m$. Analogously we can eliminate $z_{m-1}, z_{m-2}, \ldots, z_1$. On each iteration the number of inequalities to be dealt with decreases.

Case B) $m = n+1$

This time by an unimodular transformation the matrix F can be reduced to $\left[\frac{T}{f}\right]$, where T — lower triangular $n \times n$ matrix of the rank n. We start the elimination process with z_n.

If $t_{n,n}$ and $f_{m,n}$ belong both to h_1 or both to h_2 we get two inequalities on z_n with the same inequality sign. Thus, these inequalities do not restrict variables $z_1, \ldots, z_{n-1}$, hence we can eliminate $z_{n-1}, z_{n-2}, \ldots, z_1$ from the first $n-1$ inequalities of T what reduces to the case A.

If $t_{n,m}$ and $f_{m,n}$ belong to different $h_i^{\prime} s$, $i = \{1, 2\}$ then they generate two inequalities on z_n with different inequality signs. Thus, these inequalities generate a new inequality in variables $z_1, \ldots, z_{n-1}$. The first $n-1$ inequalities plus the new inequality constitute the new inequality set in $n-1$ variables which again corresponds to the Case B.

If $f_{m,n}$ belongs to h_3 then we get one inequality on z_n. Further we can eliminate $z_{n-1}, z_{n-2}, \ldots, z_1$ from the first $n-1$ inequalities plus the last m-th inequality. The inequalities constitute the new inequality set in $n-1$ variables which again corresponds to the Case B.

Case C) $m = n+2$

By an unimodular transformation the matrix F can be reduced to $\begin{bmatrix} T \\ f \\ g \end{bmatrix}$, where T — lower triangular $n \times n$ matrix of the rank n. We start the elimination process with z_n.

If $t_{n,n}, f_{n+1,n}, g_{m,n}$ belong all three to h_1 or all three to h_2 then we get three inequalities on z_n with the same inequality sign. Thus, these inequalities do not restrict variables $z_1, \ldots, z_{n-1}$, hence we can eliminate $z_{n-1}, z_{n-2}, \ldots, z_1$ from the first $n-1$ inequalities what reduces to the Case A.

If $t_{n,n}, f_{n+1,n}, g_{m,n}$ do not belong all three to h_1 and none of them belongs to h_3 then they generate three inequalities on z_n with different inequality signs. Thus, these inequalities generate two new inequalities in variables $z_1, \ldots, \ldots, z_{n-1}$. The first $n-1$ inequalities plus the two new inequalities constitute the new inequality set in $n-1$ variables which again corresponds to the Case C.

If both $f_{r+1,n}$, $g_{m,n}$ belong to h_3 we eliminate z_n from the n-th inequality and we get again the Case C with $n-1$ variables.

If only one of $f_{n+1,n}$, $g_{m,n}$ belongs to h_3 assume that this is $f_{n+1,n}$.

If $g_{m,n}$ and $t_{n,n}$ belong both to h_1 or both to h_2 then we get two inequalities on z_n with the same inequality sign.

Thus these inequalities fo not restrict variables $z_1, \ldots, z_{n-1}$, hence we can eliminate $z_{n-1}, z_{n-2}, \ldots, z_1$ from the first $n-1$ inequalities plus $(n+1)$ th inequality what reduces to the Case B.

If $g_{m,n}$ and $t_{n,n}$ belong to different h_i' s, $i=1$, 2 then they generate two inequalities on z_n with different inequality signs. Thus, these inequalities generate a new inequality in variables $z_1, \ldots, z_{n-1}$. The first $n-1$ inequalities plus $(n+1)$ inequality plus the new inequality constitute the new inequality set in $n-1$ variables which again corresponds to the Case C. Q.E.D. ■

The Lemma 1 implies the following fact.

LEMMA 2. *The maximal number of inequalities in the Fourier-Motzkin elimination method is*

a) m if $m \leqslant n$

b) $2n$ if $m=n+1$

c) $3n$ if $m=n+2$

Proof.

a) This is clear from the proof of the Lemma 1, Case A.

b) and c) At the worst case the number of inequalities grows as follows.

number of variables	b: number of inequalities for elimination	b: number of inequalities for substitution	c: number of inequalities for elimination	c: number of inequalities for substitution
n	$n+1$	0	$n+2$	0
$n-1$	n	2	$n+1$	3
$n-2$	$n-1$	4	n	6
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
$n-i$	$n-i+1$	$2i$	$n-i+2$	$3i$
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.

Let t=maximal number of inequalities

b) $t = \max_{i=0,1,2,\ldots,n-1} (n-i+1+2i)=2n$

c) $t = \max_{i=0,1,2,\ldots,n-1} (n-i+2+3i)=3n$ Q.E.D.

■

For cases others then those described in Lemma 1 and 2 to avoid an excessive grow of the inequality number different kinds of heuristics can be applied. One possible heuristic is such a choice of variables to be eliminated which minimizes cardinality of the carthesian product $h_1 \times h_2$. The unimodular transformations applied in the proof of Lemma 1 might be very helpful in minimizing product cardinalities.

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Received, September 1982.

Wyznaczanie rozwiązań efektywnych dla dwukryteriowych zadań programowania całkowitoliczbowego metodami teorii liczb

W pracy zaprezentowany został algorytm wyznaczania wszystkich rozwiązań efektywnych dla dwukryteriowego zadania programowania liniowego całkowitoliczbowego z ograniczeniami równościowymi.

Podstawą działania algorytmu jest idea transformacji zadania dwukryteriowego do problemu znajdowania rozwiązań układu parametrycznych, liniowych równań diofantycznych. Dla takiego

układu znajdowane jest rozwiązanie w postaci ogólnej a następnie za pomocą eliminacji Fouriera-Motzkina badane jest istnienie rozwiązań nieujemnych.

Działanie opartego na tej zasadzie algorytmu iteracyjnego zilustrowane zostało przykładem numerycznym.

Вычисление эффективных решений для двукритериальных задач целочисленного программирования при использовании численно-теоретического подхода

В работе представлен алгоритм для вычисления всех эффективных решений двукритериальной задачи целочисленного программирования с ограничениями в виде равенств. Идея алгоритма состоит в сведении двукритериальной задачи к задаче решения системы параметрических диофантовых уравнений. Для этой системы находится общее решение, а затем используется метод исключения Фурье-Мотзкина для нахождения среди всех решений таких, которые являются неотрицательными.

На основе этой идеи представлена итеративная схема решения исходной задачи. В заключение приведен численный пример.